SOLVABILITY OF EXTENDED GENERAL STRONGLY MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper, a new class of extended general strongly mixed variational inequalities is introduced and studied in Hilbert spaces. An existence theorem of solution is established and using resolvent operator technique, a new iterative algorithm for solving the extended general strongly mixed variational inequality is suggested. A convergence result for the iterative sequence generated by the new algorithm is also established.

1. Introduction and Preliminaries

Variational inequality theory, which was introduced by Stampacchia [24] in 1964, has had a great impact and influence in the development of several branches on pure and applied sciences. A useful and important generalization of variational inequality is the general mixed variational inequality containing a nonlinear term \( \phi \). Finding fixed points of a nonlinear mapping is an equally important problem in the functional analysis. Equivalent fixed point formulation of a variational inequality problem, has given a new dimension to the study of solution of variational inequality problems.

In many problems of analysis, one encounters operators who may be split in the form \( S = A \pm T \), where \( A \) and \( T \) satisfies some conditions, and \( S \) itself has neither of these properties. An early theorem of this type was given by Krasnoselskii [12], where a complicated operator is split into the sum of two simpler operators. There is another setting arises from perturbation theory. Here the operator equation \( Tx \pm Ax = x \) is considered as a perturbation of \( Tx = x \) (or \( Ax = x \)), and one would like to assert that the original unperturbed equation has a solution. In such a situation, there is, in general, no continuous dependence of solutions on the perturbations. For various results in this direction, please see [4, 7, 8, 11, 22, 26]. Another argument is concerned with the approximate solution of the problem: For \( f \in H \), find \( x \in H \) such that \( Tx \pm Ax = f \). Here \( T, A : H \to H \) are given operators. Many boundary value problems for quasi linear partial differential equations arising in physics, fluid mechanics and other areas of applications can be formulated as the equation \( Tx \pm Ax = f \), see, e.g. Zeidler [28]. Combettes and Hirstoaga [5] showed that the finding of zeros of sum of two operators can be solved via the variational inequality involving sum of two operators. Several authors study this
type of situations, see, e.g. [6, 21] and references therein. Motivated by these facts, in this paper we study a variational inequality problem involving operator of the form $T - A$.

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function. Let $T : H \to H$ be a nonlinear operator and $g, h : H \to H$ are any mappings. We consider the problem of finding $x^* \in H$ such that

\begin{equation}
\langle T(x^*) - A(x^*), h(y^*) - g(x^*) \rangle + \varphi(h(y^*)) - \varphi(g(x^*)) \geq 0, \quad \forall y^* \in H,
\end{equation}

where $A$ is a nonlinear continuous mapping on $H$ and $\partial \varphi$ denotes the subdifferential of $\varphi$. We call inequality (1) as extended general strongly mixed variational inequality.

We now consider some special cases of the problem (1):

1. If $A \equiv 0$, then the problem (1) reduces to the extended general mixed variational inequality problem considered in [20].
2. If $h$ is an identity mapping on $H$, then the problem (1) reduces to the problem studied by [10].
3. If $A \equiv 0$ and $h \equiv g$, then the problem (1) reduces to the general mixed variational inequality problem considered in [2, 17, 18, 19].
4. If $h, g$ be identity mappings on $H$, then the problem (1) reduces to a class of variational inequality studied by [25].
5. If $A \equiv 0$ and $h, g$ be identity mappings on $H$, then the problem (1) reduces to the mixed variational inequality or variational inequality of second kind see [1, 9, 15, 16].

For a multivalued operator $T : H \to H$, we denote by

\[ D(T) = \{ u \in H : T(u) \neq \emptyset \}, \]

the domain of $T$,

\[ R(T) = \bigcup_{u \in H} T(u), \]

the range of $T$,

\[ \text{Graph}(T) = \{ (u, u^*) \in H \times H : u \in D(T) \text{ and } u^* \in T(u) \}, \]

the graph of $T$.

**Definition 1.1.** $T$ is called monotone if and only if for each $u \in D(T)$, $v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$, we have

\[ \langle v^* - u^*, v - u \rangle \geq 0. \]

$T$ is maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

$T^{-1}$ is the operator defined by

\[ v \in T^{-1}(u) \iff u \in T(v). \]

**Definition 1.2** (See [3]). For a maximal monotone operator $T$, the resolvent operator associated with $T$, for any $\sigma > 0$, is defined as

\[ J_T(u) = (I + \sigma T)^{-1}(u), \quad \forall u \in H. \]
It is known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. Furthermore, the resolvent operator is single-valued and nonexpansive i.e. \( |J_T(x) - J_T(y)| \leq \|x - y\| \), \( \forall x,y \in H \). In particular, it is well known that the subdifferential \( \partial \varphi \) of \( \varphi \) is a maximal monotone operator; see [13].

Lemma 1.3. [3] For a given \( z \in H \), \( u \in H \) satisfies the inequality

\[
\langle u - z, x - u \rangle + \lambda \varphi(x) - \lambda \varphi(u) \geq 0, \quad \forall x \in H
\]

if and only if \( u = J_{\varphi}(z) \), where \( J_{\varphi} = (I + \lambda \partial \varphi)^{-1} \) is the resolvent operator and \( \lambda > 0 \) is a constant.

Inequality (1), can be written in an equivalent form as follows:

Find \( x^* \in H \) such that

\[
\langle \rho(T(x^*) - A(x^*)) + g(x^*) - h(x^*), h(y^*) - g(x^*) \rangle + \rho \varphi(h(y^*)) - \rho \varphi(g(x^*)) \geq 0, \quad \text{for all } y^* \in H.
\]

This equivalent formulation plays an important role in the development of iterative methods for solving the mixed variational inequality problem (1).

Using Lemma 1.3, we will establish following important relation:

Lemma 1.4. \( x^* \in H \) is a solution of (2) if and only if \( x^* \) satisfies the following relation

\[
g(x^*) = J_{\varphi}(h(x^*) - \rho(T(x^*) - A(x^*)))
\]

where \( \rho > 0 \) is a constant and \( J_{\varphi} = (I + \rho \partial \varphi)^{-1} \) is the proximal mapping, \( I \) stands for the identity operator on \( H \).

Proof. Let \( x^* \in H \) be a solution of problem (2), then

\[
\langle g(x^*) - (h(x^*) - \rho(T(x^*) - A(x^*))), h(y^*) - g(x^*) \rangle + \rho \varphi(h(y^*)) - \rho \varphi(g(x^*)) \geq 0, \quad \text{for all } y^* \in H.
\]

Applying Lemma 1.3 for \( \lambda = \rho \), inequality (4) is equivalent to

\[
g(x^*) = J_{\varphi}(h(x^*) - \rho(T(x^*) - A(x^*)))
\]

the required result. \( \square \)

Lemma 1.4 implies that the problem (2) is equivalent to the fixed point problem (3). This alternative equivalent formulation provides a natural connection between variational inequality problem (2) and the fixed point theory which will be used to prove existence result. The following lemma is in this sense :

Lemma 1.5. \( x^* \in H \) is a solution of (2) if and only if \( x^* \) is a fixed point of the mapping \( F \) given by

\[
F(u) = u - g(u) + J_{\varphi}(h(u) - \rho(T(u) - A(u))), \quad u \in H.
\]

Proof. Let \( x^* \in H \) be a fixed point of the mapping \( F \). Then

\[
g(x^*) = J_{\varphi}(h(x^*) - \rho(T(x^*) - A(x^*)))
\]

From Lemma 1.4, \( x^* \) is a solution of (2). \( \square \)
We now recall some definitions:

**Definition 1.6.** An operator $T : H \to H$ is said to be:

(i) strongly monotone, if for each $x \in H$, there exists a constant $\nu > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \nu \|x - y\|^2$$

holds, for all $y \in H$;

(ii) $\phi$-cocoercive, if for each $x \in H$, there exists a constant $\phi > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq -\phi \|T(x) - T(y)\|^2$$

holds, for all $y \in H$;

(iii) relaxed $(\phi, \gamma)$-cocoercive or relaxed cocoercive with respect to constant $(\phi, \gamma)$, if for each $x \in H$, there exists constants $\gamma > 0$ and $\phi > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq -\phi \|T(x) - T(y)\|^2 + \gamma \|x - y\|^2$$

holds, for all $y \in H$;

(iv) $\mu$-Lipschitz continuous or Lipschitz with respect to constant $\mu$, if for each $x, y \in H$, there exists a constant $\mu > 0$ such that

$$\|T(x) - T(y)\| \leq \mu \|x - y\|.$$

2. **Main results**

Lemma 1.5, is the main motivation for our next result:

**Theorem 2.1.** Let $H$ be a real Hilbert space and $T, A, g, h : H \to H$ are operators. Suppose that the following assumptions are satisfied:

(i) $T, g, h$ are relaxed cocoercive with constants $(\phi_T, \gamma_T), (\phi_g, \gamma_g), (\phi_h, \gamma_h)$ respectively,

(ii) $T, A, g, h$ are Lipschitz mappings with constants $\mu_T, \mu_A, \mu_g, \mu_h$ respectively.

If

$$1 + \mu_T^2(1 + 2\phi_T) > 2\gamma_T, \quad 1 + \mu_A^2(1 + 2\phi_A) > 2\mu_A,$$

and

$$\rho \in \left(\frac{\gamma_T - \phi_T \mu_T^2}{\mu_T^2 + \mu_A^2}, \frac{\gamma_T - \phi_T \mu_T^2}{\mu_T^2 + \mu_A^2} + \sqrt{d}\right),$$

where

$$d := (\phi_T \mu_T^2 - \gamma_T)^2 - \frac{1}{2}(\mu_T^2 + \mu_A^2)(1 + \kappa(2 - \kappa)) > 0,$$

$$\kappa = \sqrt{1 - 2\gamma_T + \mu_T^2(1 + 2\phi_T) + \sqrt{1 - 2\gamma_T + \mu_T^2(1 + 2\phi_T)},}$$

then the problem (2) has a unique solution.

**Proof.** It is enough to show that the mapping $F$ defined by (5) has a fixed point. For $u \in H$, set $p(u) = T(u) - A(u)$. 
For all $x \neq y \in H$, we have
\[
\|F(x) - F(y)\| \leq \|x - y - (g(x) - g(y))\|
+ \|J_{\varphi} (h(x) - \rho (p(x))) - J_{\varphi} (h(y) - \rho (p(y)))\|
\leq \|x - y - (g(x) - g(y))\| + \|h(x) - h(y) - \rho (p(x) - p(y))\|
\leq \|x - y - (g(x) - g(y))\| + \|x - y - (h(x) - h(y))\|
+ \|x - y - \rho (p(x) - p(y))\|.
\]
(7)

Since $g$ is relaxed $(\phi_g, \gamma_g)$-cocoercive and $\mu_g$-Lipschitz mapping, we can compute the following:
\[
\|x - y - (g(x) - g(y))\|^2 = \|x - y\|^2 - 2\langle g(x) - g(y), x - y \rangle + \|g(x) - g(y)\|^2
\leq (1 + \mu^2_g)\|x - y\|^2 + 2\phi_g\|g(x) - g(y)\|^2 - 2\gamma_g \|x - y\|^2
\]
(8)

Similarly,
\[
\|x - y - (h(x) - h(y))\|^2 \leq (1 - 2\gamma_h + \mu^2_h (1 + 2\phi_h)) \|x - y\|^2.
\]
(9)

Also,
\[
\|x - y - \rho (p(x) - p(y))\|^2 = \|x - y - \rho (T(x) - T(y)) + \rho (A(x) - A(y))\|^2
\leq 2\|x - y - \rho (T(x) - T(y))\|^2 + 2\rho^2 \|A(x) - A(y)\|^2
\leq 2\|x - y - \rho (T(x) - T(y))\|^2 + 2\rho^2 \mu^2_A \|x - y\|^2.
\]
(10)

Now, we estimate
\[
\|x - y - \rho (T(x) - T(y))\|^2 \leq \|x - y\|^2 - 2\rho \langle T(x) - T(y), x - y \rangle
+ \rho^2 \|T(x) - T(y)\|^2
\leq (1 - 2\rho \gamma_T + 2\rho \mu^2_T \phi_T + \rho^2 \mu^2_A) \|x - y\|^2.
\]
(11)

Substituting (11) into (10), gives
\[
\|x - y - \rho (p(x) - p(y))\| \leq \sqrt{2(1 - 2\rho \gamma_T + 2\rho \mu^2_T \phi_T + \rho^2 (\mu^2_T + \mu^2_A))} \|x - y\|.
\]
(12)

Substituting (8), (9), (12) into (7), we have
\[
\|F(x) - F(y)\| \leq (\kappa + f(\rho)) \|x - y\|,
\]
where
\[
\kappa = \sqrt{1 - 2\gamma_g + \mu^2_g (1 + 2\phi_g) + \sqrt{1 - 2\gamma_h + \mu^2_h (1 + 2\phi_h)}},
\]
and
\[
f(\rho) = \sqrt{2(1 - 2\rho \gamma_T + 2\rho \mu^2_T \phi_T + \rho^2 (\mu^2_T + \mu^2_A))}.
\]

From (6), we get that $(\kappa + f(\rho)) < 1$, thus $F$ is a contraction mapping and therefore has a unique fixed point in $H$, which is a solution of variational inequality (2). \qed

Remark 2.2. Theorem 2.1, extend and improve Theorem 3.1 of [20].
If $K$ is closed convex set in $H$ and $\varphi(x) = \delta_K(x)$, for all $x \in K$, where $\delta_K$ is the indicator function of $K$ defined by

$$\delta_K(x) = \begin{cases} 0, & \text{if } x \in K; \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (2) reduces to the following variational inequality problem: Consider the problem of finding $x^* \in K$

$$\langle \rho(T(x^*) - A(x^*)) + g(x^*) - h(x^*), h(y^*) - g(x^*) \rangle \geq 0, \quad \forall y^* \in K.$$  \hfill (13)

We immediately obtain following result from Theorem 2.1:

**Corollary 2.3.** Let $H$ be a real Hilbert space, $K$ be a nonempty closed convex subset of $H$ and $T, A : H \to H$ and $g, h : K \to K$ are operators. Suppose that following assumptions are satisfied:

(i) $T, g, h$ are relaxed cocoercive with constants $(\phi_T, \gamma_T)$, $(\phi_g, \gamma_g)$, $(\phi_h, \gamma_h)$ respectively,

(ii) $T, A, g, h$ are Lipschitz mappings with constants $\mu_T, \mu_A, \mu_g, \mu_h$ respectively. If (6) holds, then the problem (13) has a unique solution.

If we take $h$ as identity mapping in (13), we get an inequality, equivalent to the general strongly nonlinear variational inequality studied by Siddiqi and Ansari [23]. Corollary 2.3 partially extends and improves the result of [14, 23].

3. ITERATIVE ALGORITHM AND CONVERGENCE

We rewrite the relation (3) in the following form

$$x^* = x^* - g(x^*) + J_\varphi \left( h(x^*) - \rho(T(x^*) - A(x^*)) \right).$$  \hfill (14)

Using the fixed point formulation (14), we now suggest and analyze the following iterative methods for solving the variational inequality problem (2).

**Algorithm 1.** For a given $x_0 \in H$, find the approximate solution $x_{n+1}$ by the iterative scheme

$$x_{n+1} = x_n - g(x_n) + J_\varphi \left( h(x_n) - \rho(T(x_n) - A(x_n)) \right), \quad n = 0, 1, 2, \ldots$$

which is called explicit iterative method.

**Algorithm 2.** For a given $x_0 \in H$, find the approximate solution $x_{n+1}$ by the iterative scheme

$$x_{n+1} = x_n - g(x_n) + J_\varphi \left( h(x_{n+1}) - \rho(T(x_{n+1}) - A(x_{n+1})) \right), \quad n = 0, 1, 2, \ldots$$

which is an implicit iterative method.

Now, we use Algorithm 1 as predictor and Algorithm 2 as a corrector to obtain the following predictor-corrector method for solving variational inequality problem (1).

**Algorithm 3.** For a given $x_0 \in H$, find the approximate solution $x_{n+1}$ by the iterative scheme

$$y_n = x_n - g(x_n) + J_\varphi \left( h(x_n) - \rho(Tx_n - Ax_n) \right)$$

$$x_{n+1} = x_n - g(x_n) + J_\varphi \left( h(y_n) - \rho(Ty_n - Ay_n) \right), \quad n = 0, 1, 2, \ldots.$$
Using Algorithm 3, we can suggest following:

**Algorithm 4.** For a given \( x_0 \in H \), find the approximate solution \( x_{n+1} \) by the iterative scheme

\[
y_n = x_n - g(x_n) + J_\varphi(h(x_n) - \rho(Tx_n - Ax_n))
\]

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(x_n - g(x_n) + J_\varphi(h(y_n) - \rho(Ty_n - Ay_n)))
\]

where \( n = 0, 1, 2, \ldots \), \( \{\alpha_n\} \) is sequences in \([0, 1]\), satisfying certain conditions.

Now, we define a more general predictor-corrector iterative method for approximate solvability of variational inequality problem (1).

**Algorithm 5.** For a given \( x_0 \in H \), find the approximate solution \( x_{n+1} \) by the iterative scheme

\[
y_n = (1 - \beta_n)x_n + \beta_n(x_n - g(x_n) + J_\varphi(h(x_n) - \rho(Tx_n - Ax_n)))
\]

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(x_n - g(x_n) + J_\varphi(h(y_n) - \rho(Ty_n - Ay_n)))
\]

where \( n = 0, 1, 2, \ldots \), \( \{\alpha_n\}, \{\beta_n\} \) are sequences in \([0, 1]\), satisfying certain conditions.

We need following result to prove the next result:

**Lemma 3.1.** [27] Let \( \{a_n\} \) be a non negative sequence satisfying

\[
a_{n+1} \leq (1 - c_n)a_n + b_n,
\]

with \( c_n \in [0, 1] \), \( \sum_{n=0}^{\infty} c_n = \infty \), \( b_n = o(c_n) \). Then \( \lim_{n \to \infty} a_n = 0 \).

**Theorem 3.2.** Let \( T, A, g, h \) satisfy all the assumptions of Theorem 2.1, also condition (6) holds and \( \{\alpha_n\}, \{\beta_n\} \) are sequences in \([0, 1]\) for all \( n \geq 0 \) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then the approximate sequence \( \{x_n\} \) constructed by the Algorithm 3 converges strongly to a solution \( x^* \) of (2).

**Proof.** For \( u \in H \), set \( pu = Tu - Au \). Since \( x^* \in H \) is a solution of (1), by (14), we have

\[
x^* = x^* - g(x^*) + J_\varphi(h(x^*) - \rho(T(x^*) - A(x^*)))
\]

Using (15), we have

\[
\left\|x_{n+1} - x^*\right\| \leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n\left(\left\|x_n - x^* - (g(x_n) - g(x^*))\right\| + \alpha_n\left\|J_\varphi(h(y_n) - \rho p(y_n)) - J_\varphi(h(x^*) - \rho p(x^*))\right\|\right)
\]

\[
\leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n\sqrt{1 - 2\gamma + \mu_T^2(1 + 2\phi)}\left\|x_n - x^*\right\|
\]

\[
+ \alpha_n\left\|h(y_n) - h(x^*) - \rho p(y_n) - p(x^*)\right\|
\]

\[
\leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n\sqrt{1 - 2\gamma + \mu_T^2(1 + 2\phi)}\left\|x_n - x^*\right\|
\]

\[
+ \alpha_n\left\|y_n - x^* - (h(y_n) - h(x^*))\right\|
\]

\[
+ \alpha_n\left\|y_n - x^* - \rho p(y_n) - p(x^*)\right\|
\]

\[
\leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n\sqrt{1 - 2\gamma + \mu_T^2(1 + 2\phi)}\left\|x_n - x^*\right\|
\]

\[
+ \alpha_n\sqrt{1 - 2\gamma + \mu_T^2(1 + 2\phi)}\left\|y_n - x^*\right\|
\]

\[
+ \alpha_n\sqrt{2(1 - 2\rho \gamma T + 2\mu_T^2 \phi_T + \rho^2(\mu_T^2 + \mu_A^2))}\left\|y_n - x^*\right\|
\]

\[
= (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n \theta_g \left\|x_n - x^*\right\| + \alpha_n (\theta_h + f(\rho)) \left\|y_n - x^*\right\|.
\]
where $\theta_g = \sqrt{1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)}$, $\theta_h = \sqrt{1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)}$ and $f(\rho) = \sqrt{2(1 - 2\rho\gamma_T + 2\rho\mu_T^2\phi_T + \rho^2(\mu_T^2 + \mu_T^2))}$.

Similarly, we have

$$\|y_n - x^*\| \leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (g(x_n) - g(x^*))\|$$
$$+ \beta_n\|J_{\varphi}(h(x_n) - pp(x_n)) - J_{\varphi}(h(x^*) - pp(x^*))\|$$
$$\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (g(x_n) - g(x^*))\|$$
$$+ \beta_n\|h(x_n) - h(x^*) - \rho(p(x_n) - p(x^*))\|$$
$$\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (h(x_n) - h(x^*))\|$$
$$+ \beta_n\|x_n - x^* - \rho(p(x_n) - p(x^*))\|$$
$$\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (h(x_n) - h(x^*))\|$$
$$+ \beta_n\|x_n - x^* - \rho(p(x_n) - p(x^*))\|$$
$$= (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\|$$
$$\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\|$$
$$= \|x_n - x^*\|.$$

Substituting (17) into (16), yields that

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n(\theta_g + \theta_h + f(\rho))\|x_n - x^*\|$$
$$= (1 - \alpha_n)(1 - (\kappa + f(\rho)))\|x_n - x^*\|.$$

By virtue of Lemma 3.1, we get from (18) that, $\lim_{n \to \infty}\|x_{n+1} - x^*\| = 0$, i.e. $x_n \to x^*$, as $n \to \infty$. This completes the proof.

Remark 3.3. Theorem 3.2, extend and improve Theorem 2.1 of [10] and Theorem 3.2 of [20].

It is well known that, if $\varphi(\cdot)$ is the indicator function of $K$ in $H$, then $J_{\varphi} = P_K$, the projection operator of $H$ onto the closed convex set $K$, and consequently, the following result can be obtain from Theorem 3.2.

Corollary 3.4. Let $T, A, g, h$ satisfy all the assumptions of Corollary 2.3. Let $x_0 \in K$, construct a sequence $\{x_n\}$ in $K$ by

$$y_n = x_n - g(x_n) + P_K(h(x_n) - \rho(Tx_n - Ax_n))$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(x_n - g(x_n) + P_K(h(y_n) - \rho(Ty_n - Ay_n))), \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ for all $n \geq 0$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a solution $x^*$ of (13).

References


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