IMPLICIT SUMMATION FORMULA FOR 2-VARIABLE LAGUERRE-BASED POLY-GENOCCHI POLYNOMIALS

WASEEM A. KHAN∗, IDREES A. KHAN AND MOIN AHMAD

Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India

∗Corresponding author: waseem08.khan@rediffmail.com

Abstract. The main object of this paper is to introduce a new class of Laguerre-based poly-Genocchi polynomials and investigate some properties for these polynomials and related to the Stirling numbers of the second kind. We derive summation formulae and general symmetry identities by using different analytical means and applying generating functions.

1. Introduction

The generalized Bernoulli, Euler and Genocchi polynomials of (real or complex) order α are usually defined by means of the following generating functions (see [1-16]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B^{(\alpha)}_n(x) \frac{t^n}{n!}, \quad (|t| < 2\pi; 1^\alpha = 1), \quad (1.1)$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E^{(\alpha)}_n(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha = 1) \quad (1.2)$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G^{(\alpha)}_n(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha = 1). \quad (1.3)$$

Received 2017-09-19; accepted 2017-12-07; published 2018-11-02.

2010 Mathematics Subject Classification. 33C45, 33C99, 05A10, 05A15.

Key words and phrases. Laguerre polynomials, poly-Genocchi polynomials, Laguerre-based poly-Genocchi polynomials, summation formulae, symmetric identities.
So that obviously

\[ B_n(x) = B_n^1(x), \ E_n(x) = E_n^1(x) \text{ and } G_n(x) = G_n^1(x), \ (n \in \mathbb{N}), \]

where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) (\( \mathbb{N} = 1, 2, 3, \cdots \)).

The classical polylogarithmic function \( \text{Li}_k(z) \) is defined by (see [2], [10]):

\[ \text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \ (k \in \mathbb{Z}). \quad (1.4) \]

The poly-Bernoulli numbers and polynomials are defined by following generating functions (see [7], [8], [9]):

\[ \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \quad (1.5) \]

\[ \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}. \quad (1.6) \]

In the case \( k = 1 \) in (1.5) and (1.6), we have

\[ B_n^{(1)} = B_n, \ B_n^{(1)}(x) = B_n(x). \]

The poly-Genocchi numbers and polynomials are defined by following generating functions (see [14]):

\[ \frac{2\text{Li}_k(1 - e^{-t})}{e^t + 1} = \sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!}, \quad (1.7) \]

\[ \frac{2\text{Li}_k(1 - e^{-t})}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!}. \quad (1.8) \]

In the case \( k = 1 \) in (1.7) and (1.8), we have

\[ G_n^{(1)} = G_n, \ G_n^{(1)}(x) = G_n(x). \]

The 2-variable Laguerre polynomials (2-VLP) \( L_n(x, y) \), which is defined by (see [5]):

\[ \frac{1}{(1-yt)} \exp \left( \frac{-xt}{1-yt} \right) = \sum_{n=0}^{\infty} L_n(x, y) t^n, \ (|yt| < 1) \quad (1.9) \]

It is equivalently given by (see [6]).

\[ \exp(yt) C_0(x) t^n = \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!}, \quad (1.10) \]

where \( C_0(x) \) denotes the 0\(^{th}\) order Tricomi function. The \( n\(^{th}\) order Tricomi functions \( C_n(x) \) are defined as:

\[ C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}, \ (n \in \mathbb{N}_0) \quad (1.11) \]
with the following generating function:

\[
\exp \left( t - \frac{x}{t} \right) = \sum_{n=0}^{\infty} C_n(x)t^n, \quad (1.12)
\]

for \( t \neq 0 \) and for all finite \( x \).

From (1.9) and (1.10), we get

\[
L_n(x, y) = \frac{n!}{\pi} \sum_{s=0}^{n} \frac{(-1)^s x^s y^{n-s}}{(s!)^2 (n-s)!} = y^n L_n(x/y).
\]

(1.13)

Thus, we have

\[
L_n(x, 0) = \frac{(-1)^n x^n}{n!}, L_n(0, y) = y^n, L_n(x, 1) = L_n(x),
\]

(1.14)

where \( L_n(x) \) are the classical Laguerre polynomials (see [1]).

Now, we recall here the following definition as follows:

The Stirling number of the first kind is given by

\[
(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n, l)x^l, \quad (n \geq 0)
\]

(1.15)

and the Stirling number of the second kind is defined by generating function:

\[
(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}.
\]

(1.16)

2. 2-Variable Laguerre-based poly-Genocchi polynomials

Let \( k \in \mathbb{Z} \), we introduce 2-variable Laguerre-based poly-Genocchi polynomials by the following generating function:

\[
\frac{2Li_k(1 - e^{-t})}{e^t + 1} \exp(yt)C_0(xt) = \sum_{n=0}^{\infty} L^{(k)}_n(x, y) \frac{t^n}{n!},
\]

so that

\[
L^{(k)}_n(x, y) = \sum_{m=0}^{n} \binom{n}{m} G^{(k)}_{n-m} L_m(x, y).
\]

(2.1)

(2.2)

When \( x = y = 0 \), \( L^{(k)}_n(0, 0) \) are called the poly-Genocchi numbers. For \( k = 1 \) in (2.1), we have

\[
\frac{2Li_1(1 - e^{-t})}{e^t + 1} \exp(yt)C_0(xt) = \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!}.
\]

(2.3)

where \( L_n(x, y) \) is Laguerre-based Genocchi polynomials (see [13]).

Thus, we have

\[
L^{(k)}_n(x, y) = L_n(x, y), \quad (n \geq 0).
\]
On setting $x = 0$, (2.1) reduces to the known result of Kim et al. [14., p. Eq.(4)4776]:

$$
\frac{2Li_k(1 - e^{-t})}{e^t + 1} \exp(yt) = \sum_{n=0}^{\infty} G_n^{(k)}(y) \frac{t^n}{n!} \quad (k \in \mathbb{Z}).
$$

(2.4)

**Theorem 2.1.** The following explicit summation formulae for Laguerre-based poly-Genocchi polynomials holds true:

$$
L^{(2)}_G(n, x, y) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{B_m m!}{m + 1} L_{n-m}(x, y).
$$

(2.5)

**Proof.** Using generating function for Laguerre-based poly-Genocchi polynomials (2.1), we have

$$
\sum_{n=0}^{\infty} L_G^{(k)}(y) \frac{t^n}{n!} = 2Li_k(1 - e^{-t}) \exp(yt) C_0(x t)
$$

In particular $k = 2$, we have

$$
\sum_{n=0}^{\infty} L_G^{(2)}(x, y) \frac{t^n}{n!} = \frac{2}{e^t + 1} \exp(yt) C_0(x t) \int_{0}^{t} \frac{z}{e^{z-1}} \left( \int_{0}^{t} \frac{1}{e^{z-1}} \left( \int_{0}^{t} \frac{z}{e^{z-1}} \right) \right) dz.
$$

Replacing $n$ by $n - m$ in the r.h.s of above equation, we have

$$
\sum_{n=0}^{\infty} L_G^{(2)}(x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{B_m m!}{m + 1} L_{n-m}(x, y) \frac{t^n}{n!}.
$$

On equating the coefficients of the like powers of $t$ in both sides, we get (2.5).

**Remark 2.1.** On setting $x = 0$, Theorem (2.1) reduces to the known result of Kim et al. [14., p. 4777, Theorem (2.1)].

**Corollary 2.1.** For $n \geq 0$, we have

$$
G^{(2)}_n(y) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{B_m m!}{m + 1} G_{n-m}(y).
$$

(2.6)

**Theorem 2.2.** For $n \geq 1$, the degree of $L^{(k)}_G(n, x, y)$ is $n-1$. We have

$$
\frac{L^{(k)}_G(n, x, y)}{n} = \sum_{m=0}^{n-1} \left( \begin{array}{c} n - 1 \\ m \end{array} \right) \frac{m+1}{m+1} L_{n-m-1}(x, y).
$$

(2.7)

**Proof.** From (2.1), we have

$$
\sum_{n=0}^{\infty} L^{(k)}_G(n, x, y) \frac{t^n}{n!} = \frac{2Li_k(1 - e^{-t})}{1 - e^{-t}} \exp(yt) C_0(x t)
$$
Replacing \( n \) by \( n - m \) in above equation and comparing the coefficients of \( t^n \), we get

\[
L G_n^{(k)}(x, y) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) G_m^{(k)} L_{n-m}(x, y), (n \geq 0).
\]  

(2.8)

From (2.8), we have

\[
\frac{L G_n^{(k)}(x, y)}{n} = \sum_{m=0}^{n-1} \left( \begin{array}{c} n - 1 \\ m \end{array} \right) \frac{G_m^{(k)}}{m+1} n^{m+1} - m-1(x, y), (n \geq 1)
\]  

(2.9)

Therefore by (2.9), we obtain the result (2.7).

**Remark 2.2.** For \( x = 0 \), Theorem (2.2) reduces to the known result of Kim et al. [14, p. 4778, Theorem (2.2)].

**Corollary 2.2.** For \( n \geq 1 \), the degree of \( G_n^{(k)}(x) \) is \( n-1 \). We have

\[
\frac{G_n^{(k)}(y)}{n} = \sum_{m=0}^{n-1} \left( \begin{array}{c} n - 1 \\ m \end{array} \right) \frac{G_m^{(k)}}{m+1} n^{m+1} - m-1.
\]  

(2.10)

**Theorem 2.3.** For \( n \geq 0 \), we have

\[
L G_n^{(k)}(x, y) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l S_2(p, l)}{l^k(p+1)} \left( \begin{array}{c} n \\ p \end{array} \right) L G_{n-p}(x, y).
\]  

(2.11)

**Proof.** By using (2.1), we can be written as

\[
\sum_{n=0}^{\infty} L G_n^{(k)}(x, y) \frac{t^n}{n!} = \left( \frac{\text{Li}_k(1 - e^{-t})}{t} \right) \left( \frac{2t}{e^t + 1} \exp(yt) C_0(x t) \right).
\]  

(2.12)

Now

\[
\frac{1}{l} \text{Li}_k(1 - e^{-t}) = \frac{1}{l} \sum_{l=1}^{\infty} \frac{(1 - e^{-t})^l}{l^k} = \frac{1}{l} \sum_{l=1}^{\infty} \frac{(-1)^{l}}{l^k} (1 - e^{-t})^l
\]

\[
= \frac{1}{l} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} l! \sum_{p=0}^{l} (-1)^p S_2(p, l) \frac{t^p}{p!}
\]

\[
= \frac{1}{l} \sum_{p=1}^{\infty} \sum_{l=1}^{p} \frac{(-1)^{l+p+1}}{l^k} l! S_2(p, l) \frac{t^p}{p!}
\]

\[
= \sum_{p=0}^{\infty} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! S_2(p+1, l) \frac{t^p}{p!}.
\]

(2.13)

From equations (2.12) and (2.13), we get

\[
\sum_{n=0}^{\infty} L G_n^{(k)}(x, y) \frac{t^n}{n!} = \sum_{p=0}^{\infty} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! S_2(p+1, l) \frac{t^p}{p!} \left( \sum_{n=0}^{\infty} L G_n(x, y) \frac{t^n}{n!} \right).
\]
Replacing $n$ by $n - p$ in the r.h.s. of above equation and comparing the coefficients of $t^n$ in both sides, we arrive at the desired result (2.11).

**Remark 2.3.** For $x = 0$, Theorem (2.3) reduces to the known result of Kim et al. [14., p. 4779, Theorem (2.3)].

**Corollary 2.3.** For $n \geq 0$, we have

$$G^{(k)}_n(y) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1, l)}{l^k} \binom{n}{p} G_{n-p}(y). \quad (2.14)$$

**Theorem 2.4.** For $n \geq 1$, we have

$$LG^{(k)}_n(x, y + 1) + LG^{(k)}_n(x, y) = 2 \sum_{p=1}^{n} \sum_{l=1}^{p} \frac{(-1)^{l+p} l! S_2(p, l)}{l^k} \binom{n}{p} L_{n-p}(x, y). \quad (2.15)$$

**Proof.** By using definition (2.1), we have

$$\sum_{n=0}^{\infty} LG^{(k)}_n(x, y + 1) \frac{t^n}{n!} = \sum_{n=0}^{\infty} LG^{(k)}_n(x, y) \frac{t^n}{n!}$$

$$= \frac{2Li_k(1 - e^{-t})}{e^{t} + 1} \exp((y + 1)t)C_0(x t) + \frac{2Li_k(1 - e^{-t})}{e^{t} + 1} \exp(yt)C_0(x t)$$

$$= \sum_{n=0}^{\infty} \left( \frac{2}{n!} \left( \sum_{l=1}^{p} \frac{(-1)^{l+p} l! S_2(p, l)}{l^k} \right) \frac{t^n}{p!} \right) \left( \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!} \right).$$

Replacing $n$ by $n - p$ in the above equation and comparing the coefficients of $t^n$ in both sides, we obtain the result (2.15).

**Remark 2.4.** Taking $x = 0$, Theorem 2.4 reduces to the known result of Kim et al. [14., p. 4780, Theorem (2.4)].

**Corollary 2.4.** For $n \geq 1$, we have

$$G^{(k)}_n(y + 1) + G^{(k)}_n(y) = 2 \sum_{p=1}^{n} \sum_{l=1}^{p} \frac{(-1)^{l+p} l! S_2(p, l)}{l^k} \binom{n}{p} y^{n-p}. \quad (2.16)$$

**Theorem 2.5.** For $d \in \mathbb{N}$ with $d \equiv 1(\text{mod}2)$, we have

$$LG^{(k)}_n(x, y) = \sum_{p=0}^{n} \binom{n}{p} d^{n-p-1} \sum_{l=0}^{p+1} \sum_{a=0}^{d-1} \frac{(-1)^{l+p+1} l! S_2(p+1, l)}{l^k} (-1)^a LG_{n-p}(\frac{a + y d}{d}, x). \quad (2.17)$$
Proof. From equation (2.1), we can be written as
\[
\sum_{n=0}^{\infty} \mathcal{L}G_n^{(k)}(x,y) \frac{t^n}{n!} = \frac{2\text{Li}_k(1-e^{-t})}{e^t+1} \exp(yt)C_0(x) t
\]
\[
= \left( \frac{2\text{Li}_k(1-e^{-t})}{t} \right) \left( \frac{2t}{e^t+1} \sum_{a=0}^{d-1} (-1)^a \exp((a+y)t)C_0(x) \right)
\]
\[
= \left( \sum_{p=0}^{\infty} \left( \sum_{l=1}^{p+1} \frac{(-1)^{l+1}l!S_2(p+1,l)}{l^k} \frac{tp}{p!} \right) \right) \left( \sum_{n=0}^{\infty} \sum_{a=0}^{d-1} (-1)^a L_{G_n}(a+y, -x) \frac{t^n}{n!} \right)
\]
Replacing \( n \) by \( n-p \) in above equation and equating the resulting equation to the above equation, we get (2.17).

Remark 2.5. For \( x = 0 \), Theorem 2.5 reduces to the known result of Kim et al. [14., p. 4780].

Corollary 5. For \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \), we have
\[
G_n^{(k)}(y) = \sum_{p=0}^{n} \binom{n}{p} \frac{d^{n-p-1}}{p!} \sum_{l=0}^{p+1} \frac{(-1)^{l+1}l!S_2(p+1,l)}{l^k} \frac{tp}{p!} (-1)^a G_{n-p}(a+y, -x).
\]

3. Summation formulae for Laguerre-based poly-Genocchi polynomials

In this section, we establish summation formula for Laguerre-based poly-Genocchi polynomials by using series techniques method.

Theorem 3.1. The following implicit summation formulae for Laguerre-based poly-Genocchi polynomials \( \mathcal{L}G_n^{(k)}(x,y) \) holds true:
\[
\mathcal{L}G_{l+p}^{(k)}(x,z) = \sum_{m,n=0}^{l} \binom{l}{m} \binom{p}{n} (z-y)^{m+n} \mathcal{L}G_{l+p-m-n}^{(k)}(x,y).
\]

Proof. Replacing \( t \) by \( t+u \) and rewrite the generating function (2.1) as
\[
\frac{2\text{Li}_k(1-e^{-(t+u)})}{e^{t+u}+1} C_0(x(t+u)) = e^{-y(t+u)} \sum_{l,p=0}^{\infty} \mathcal{L}G_{l+p}^{(k)}(x,y) \frac{t^l u^p}{l! p!}.
\]
Replacing \( y \) by \( z \) in the above equation and equating the resulting equation to the above equation, we get
\[
e^{(z-y)(t+u)} \sum_{m,l=0}^{\infty} \mathcal{L}G_{l+p}^{(k)}(x,y) \frac{t^l u^p}{l! p!} = \sum_{l,p=0}^{\infty} \mathcal{L}G_{l+p}^{(k)}(x,z) \frac{t^l u^p}{l! p!}.
\]
On expanding exponential function (3.3) gives
\[
\sum_{N=0}^{\infty} \frac{[(z-y)(t+u)]^N}{N!} \sum_{l,p=0}^{\infty} \mathcal{L}G_{l+p}^{(k)}(x,y) \frac{t^l u^p}{l! p!} = \sum_{l,p=0}^{\infty} \mathcal{L}G_{l+p}^{(k)}(x,z) \frac{t^l u^p}{l! p!}.
\]
which on using formula [16, p.52(2)]
\[
\sum_{N=0}^{\infty} f(N) \frac{[x+y]^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n y^m}{n! m!},
\]

(3.5)
in the left hand side becomes

\[
\sum_{m,n=0}^{\infty} \frac{(z-x)^{m+n} u^n}{m! n!} \sum_{l,p=0}^{\infty} H_{l+p}^{(k)}(x,y) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} H_{l+p}^{(k)}(z,y) \frac{t^l}{l!} \frac{u^p}{p!}
\] (3.6)

Now replacing \( l \) by \( l - m \), \( p \) by \( p - n \) and using the lemma [16, p.100(1)] in the left hand side of (3.6), we get

\[
\sum_{m,n=0}^{\infty} \frac{(z-x)^{m+n} u^n}{m! n!} L_{l+p}^{(k)}(x,y) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} L_{l+p}^{(k)}(x,z) \frac{t^l}{l!} \frac{u^p}{p!}
\] (3.7)

Finally on equating the coefficients of the like powers of \( t \) and \( u \) in the above equation, we get the required result.

**Remark 3.1.** Taking \( l = 0 \) in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

**Corollary 3.1.** The following summation formula for Laguerre-based poly-Genocchi polynomials \( H_{l+p}^{(k)}(z,y) \) holds true:

\[
L_{p}^{(k)}(x,z) = \sum_{n=0}^{p} \binom{p}{n} (z-y)^n L_{p-n}^{(k)}(x,y).
\] (3.8)

**Remark 3.2.** Replacing \( z \) by \( z + y \) in (3.8), we obtain

\[
L_{p}^{(k)}(x,z+y) = \sum_{n=0}^{p} \binom{p}{n} z^n L_{p-n}^{(k)}(x,y).
\] (3.9)

**Theorem 3.2.** The following summation formula for Laguerre-based poly-Genocchi polynomials \( H_{l+p}^{(k)}(z,y) \) holds true:

\[
L_{n}^{(k)}(x,y+u) = \sum_{j=0}^{n} \binom{n}{j} u^j L_{n-j}^{(k)}(x,y).
\] (3.10)

**Proof.** Using (2.1), we can be written as

\[
\sum_{n=0}^{\infty} L_{n}^{(k)}(x,y+u) \frac{t^n}{n!} = \frac{2L_k(1-e^{-t})}{e^t+1} \exp((y+u)t) C_0(xt) = \left( \sum_{n=0}^{\infty} L_{n}^{(k)}(x,y) \frac{t^n}{n!} \right) \left( \sum_{j=0}^{\infty} u^j \frac{t^j}{j!} \right)
\]

Now replacing \( n \) by \( n - j \) and comparing the coefficients of \( t^n \) in both sides, we obtain (3.10).
**Theorem 3.3.** The following summation formula for Laguerre-based poly-Genocchi polynomials $L^k_n(x,y)$ holds true:

$$L^k_n(x + w, y + u) = \sum_{m=0}^{n} \binom{n}{m} L^k_{n-m}(x,y) L_m(u,w).$$

**Proof.** From (2.1) and (1.10), we have

$$\frac{2\text{Li}_k(1 - (e^{-t})^{y+u})}{e^t + 1} \exp(((x+w)t) C_0((x+w)t)) = \left(\sum_{n=0}^{\infty} L^k_n(x,y) \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} L_m(u,w) \frac{t^n}{m!}\right).$$

Now replacing $n$ by $n - m$ and comparing the coefficients of $t^n$ in both sides, we get (3.11).

**Theorem 3.4.** The following summation formula for Laguerre-based poly-Genocchi polynomials $L^k_n(x,y)$ holds true:

$$L^k_n(x,y + 1) = \sum_{m=0}^{n} \binom{n}{m} L^k_{n-m}(x,y).$$

**Proof.** Using definition (2.1), we have

$$\sum_{n=0}^{\infty} L^k_n(x,y + 1) \frac{t^n}{n!} - \sum_{n=0}^{\infty} L^k_n(x,y) \frac{t^n}{n!} = \frac{2\text{Li}_k(1 - (e^{-t})^{y+u})}{e^t + 1} \exp((yt) C_0((x+w)t))(e^t - 1)$$

$$= \left(\sum_{n=0}^{\infty} L^k_n(x,y) \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} \frac{t^m}{m!}\right) - \sum_{n=0}^{\infty} L^k_n(x,y) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} L^k_{n-m}(x,y) \frac{t^n}{(n-m)!m!} - \sum_{n=0}^{\infty} L^k_n(x,y) \frac{t^n}{n!}.$$ 

Finally, equating the coefficients of the like powers of $t^n$, we get (3.12).

### 4. Identities for 2-Variable Laguerre-Based Poly-Genocchi Polynomials

In this section, we derive general symmetry identities for 2-variable Laguerre-based poly-Genocchi polynomials $L^k_n(x,y)$ by applying the generating function (2.1). Such type of identities have been introduced by several authors (see [11], [12], [13], [15]).

**Theorem 4.1.** Let $a, b > 0$ and $a \neq b, x, y \in \mathbb{R}$, $n \geq 0$, then the following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m L^k_{n-m}(bx, by) L^k_m(au, aw)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} L^k_{n-m}(ax, ay) L^k_m(bu, bw).$$

(4.1)
Proof. Let

$$G(t) = \left( \frac{(2Li_k(1 - e^{-at})(2Li_k(1 - e^{-bt}))}{(e^{at} + 1)(A^{bt} - B^{-bt})} \right) \exp(ab(y + u)t)C_0(abxt)C_0(abwt). \quad (4.2)$$

Since $G(t)$ is symmetric in $a$ and $b$ and $G(t)$ can be written as

$$G(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{L_G^{(k)}(bx, by)(at)^n}{n!} \sum_{m=0}^{\infty} L_G^{(k)}(au, aw)(bt)^m \frac{t^n}{n!}.$$ \quad (4.3)

Similarly, we can show that

$$G(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{L_G^{(k)}(ax, ay)(bt)^n}{n!} \sum_{m=0}^{\infty} L_G^{(k)}(bu, bw)(at)^m \frac{t^n}{n!}.$$ \quad (4.4)

Comparing the coefficients of $\frac{t^n}{n!}$ in (4.3) and (4.4), we arrive at the desired result.

Remark 4.1. On setting $b = 1$ in Theorem 4.1, we get

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} L_G^{(k)}(x, y) L_G^{(k)}(u, w)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^{m} L_G^{(k)}(x, y) L_G^{(k)}(u, w).$$

Theorem 4.2. Let $a, b > 0$ and $a \neq b$, $x, y \in \mathbb{R}$ and $n \geq 0$, then the following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} a^{-1} b^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} L_G^{(k)}(by + \frac{b}{a} i + j, bx) L_G^{(k)}(au, aw) b^{m} a^{n-m}$$

$$= \sum_{m=0}^{n} \binom{n}{m} b^{a-1} a^{-1} \sum_{i=0}^{a} \sum_{j=0}^{b} L_G^{(k)}(ay + \frac{a}{b} i + j, ax) L_G^{(k)}(bu, bw) a^{m} b^{n-m}. \quad (4.5)$$

Proof. Let

$$G(t) = \left( \frac{(2Li_k(1 - e^{-at})(2Li_k(1 - e^{-bt}))}{(e^{at} + 1)(e^{bt} + 1)^2} \right) \frac{e^{abt} + 1}{e^{abt} + 1} \exp(ab(y + u)t)C_0(abxt)C_0(abwt)$$

$$G(t) = \left( \frac{2Li_k(1 - e^{-at})}{e^{at} + 1} \right) \exp(abyt)C_0(abxt) \left( \frac{e^{abt} + 1}{e^{abt} + 1} \right) \left( \frac{2Li_k(1 - e^{-bt})}{e^{bt} + 1} \right) \exp(abwt)C_0(abwt) \left( \frac{e^{abt} + 1}{e^{abt} + 1} \right)$$

$$\times \exp(abyt)C_0(abxt) \left( \frac{e^{abt} + 1}{e^{abt} + 1} \right).$$

\begin{align*}
&= \left( \frac{2\text{Li}_k(1 - e^{-at})}{e^{at} + 1} \right) \exp(abyt)C_0(abxt) \sum_{i=0}^{a-1} (-1)^i e^{bt} \left( \frac{2\text{Li}_k(1 - e^{-bt})}{e^{bt} + 1} \right) \\
& \quad \times \exp(abut)C_0(abwt) \sum_{j=0}^{b-1} (-1)^j e^{atj}.
\end{align*}

\begin{align*}
&= \left( \frac{2\text{Li}_k(1 - e^{-t})}{e^{at} + 1} \right) C_0(abxt) \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} e^{(by + \frac{a}{b}i + j)at} \sum_{m=0}^{\infty} L_{m}^{(k)}(au, aw) \frac{bt^m}{m!} \\
& = \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \sum_{m=0}^{\infty} \binom{n}{m} L_{m}^{(k)}(by + \frac{b}{a}i + j, bx) \frac{bt^m}{m!} \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} L_{m}^{(k)}(au, aw) b^m a^{n-m} \frac{t^n}{n!}, 
\end{align*}

\begin{align*}
G(t) &= \sum_{n=0}^{\infty} \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} L_{m}^{(k)}(ay + \frac{a}{b}i + j, ax) \frac{bt^m}{m!} \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} L_{m}^{(k)}(bu, bw) a^m b^{n-m} \frac{t^n}{n!},
\end{align*}

On the other hand

\begin{align*}
G(t) &= \sum_{n=0}^{\infty} \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} L_{m}^{(k)}(by + \frac{b}{a}i + j, bx) \frac{bt^m}{m!} \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} L_{m}^{(k)}(au, aw) b^m a^{n-m} \frac{t^n}{n!}.
\end{align*}

On comparing the coefficients of \( \frac{t^n}{n!} \) in (4.6) and (4.7), we arrive at the desired result (4.5).

**Acknowledgement.** All authors would like to thank Integral University, Lucknow, India, for providing the manuscript number IU/R&D/2017-MCN000240 for the present research work.

**References**


