IDEAL CONVERGENT SEQUENCE SPACES WITH RESPECT TO INVARIANT MEAN AND A MUSIELAK-ORLICZ FUNCTION OVER $n$-NORMED SPACES

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Abstract. In the present paper we defined $I$-convergent sequence spaces with respect to invariant mean and a Musielak-Orlicz function $M = (M_k)$ over $n$-normed spaces. We also make an effort to study some topological properties and prove some inclusion relation between these spaces.

1. Introduction and preliminaries

Let $\sigma$ be an injective mapping from the set of the positive integers to itself such that $\sigma^p(n) \neq n$ for all positive integers $n$ and $p$, where $\sigma^p(n) = \sigma(\sigma^{p-1}(n))$. An invariant mean or a $\sigma$-mean is a continuous linear functional defined on the space $\ell_\infty$ such that for all $x = (x_n) \in \ell_\infty$:

1. If $x_n \geq 0$ for all $n$, then $\phi(x) \geq 0$,
2. $\phi(e) = 1$,
3. $\phi(Sx) = \phi(x)$, where $Sx = (x_{\sigma(n)})$.

$V_\sigma$ denotes the set of bounded sequences all of whose invariant means are equal which is also called as the space of $\sigma$-convergent sequences. In [26], it is defined by

$$V_\sigma = \{ x \in \ell_\infty : \lim_k t_{kn}(x) = \ell, \text{ uniformly in } n, \ell = \sigma - \lim x \},$$
where \( t_{kn}(x) = \frac{x_n + x_{n+1} + \cdots + x_{n+k}}{k+1} \).

\( \sigma \)-mean is called a Banach limit if \( \sigma \) is the translation mapping \( n \to n+1 \). In this case, \( V_\sigma \) becomes the set of almost convergent sequences which is denoted by \( \hat{c} \) and defined in [11] as

\[
\hat{c} = \left\{ x \in \ell_\infty : \lim_{k} d_{kn}(x) \text{ exists uniformly in } n \right\},
\]

where \( d_{kn}(x) = \frac{x_n + x_{n+1} + \cdots + x_{n+k}}{k+1} \).

The space of strongly almost convergent sequences was introduced by Maddox [12] as follow:

\[
\hat{c} = \left\{ x \in \ell_\infty : \lim_{k} d_{kn}(|x-\ell e|) \text{ exists uniformly in } n \text{ for some } \ell \right\}.
\]

The notion of ideal convergence was first introduced by P. Kostyrko [8] as a generalization of statistical convergence which was further studied in topological spaces by Das, Kostyrko, Wilczynski and Malik see [1].

More applications of ideals can be seen in ([1], [2]). Mursaleen and Sharma [19] continue in this direction and introduced \( I \)-convergence of generalized sequences with respect to Musielak-Orlicz function.

A family \( I \subset 2^X \) of subsets of a non empty set \( X \) is said to be an ideal in \( X \) if

1. \( \phi \in I \)
2. \( A, B \in I \) imply \( A \cup B \in I \)
3. \( A \in I, B \subset A \) imply \( B \in I \),

while an admissible ideal \( I \) of \( X \) further satisfies \( \{x\} \in I \) for each \( x \in X \) see [8].

A sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) is said to be \( I \)-convergent to \( x \in X \), if for each \( \epsilon > 0 \) the set \( A(\epsilon) = \left\{ n \in \mathbb{N} : ||x_n - x|| \geq \epsilon \right\} \) belongs to \( I \).

A sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) is said to be \( I \)-bounded to \( x \in X \) if there exists an \( K > 0 \) such that \( \{ n \in \mathbb{N} : |x_n| > K \} \in I \). For more details about ideal convergence sequence spaces (see [7], [9], [15], [16], [17], [18], [21], [25], [26], [27]) and references therein.

Let \( A = A_{ij} \) be an infinite matrix of complex numbers \( a_{ij} \), where \( i, j, \in \mathbb{N} \). We write \( Ax = (A_i(x)) \) if \( A_i(x) = \sum_{j=1}^{\infty} a_{ij}x_j \) converges for each \( i \in \mathbb{N} \). Throughout the paper, by \( t_{kn}(Ax) \), we mean

\[
t_{kn}(Ax) = \frac{A_n(x) + A_{\sigma^1(n)}(x) + \cdots + A_{\sigma^k(n)}(x)}{k+1}, \text{ for all } k, n \in \mathbb{N}.
\]

A sequence space \( X \) is called as solid (or normal) if \( (\alpha_k x_k) \in X \) whenever \( (x_k) \in X \) and \( (\alpha_k) \) is a sequence of scalars such that \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \).

Let \( X \) be a sequence space and \( K = \{k_1 < k_2 < \cdots \} \subseteq \mathbb{N} \). The sequence space \( Z_K^X = \{(x_{kn}) \in w : (x_n) \in X \} \) is called \( K \)-step space of \( X \).
A canonical preimage of a sequence \((x_{kn}) \in Z^X_K\) is a sequence \((y_n) \in w\) defined by
\[
y_n = \begin{cases} x_n, & \text{if } n \in \mathbb{N}; \\ 0, & \text{otherwise}. \end{cases}
\]
A sequence space \(X\) is monotone if it contains the canonical preimages of all its step spaces.

An Orlicz function \(M\) is a function, which is continuous, non-decreasing and convex with \(M(0) = 0, M(x) > 0\) for \(x > 0\) and \(M(x) \to \infty\) as \(x \to \infty\).

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let \(w\) be the space of all real or complex sequences \(x = (x_k)\), then
\[
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \right\}
\]
which is called as an Orlicz sequence space. The space \(\ell_M\) is a Banach space with the norm
\[
||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.
\]

It is shown in [10] that every Orlicz sequence space \(\ell_M\) contains a subspace isomorphic to \(\ell_p(p \geq 1)\). The \(\Delta_2\)–condition is equivalent to \(M(Lx) \leq kLM(x)\) for all values of \(x \geq 0\), and for \(L > 1\).

A sequence \(M = (M_k)\) of Orlicz function is called a Musielak-Orlicz function see ([13],[20]). A sequence \(N = (N_k)\) defined by
\[
N_k(v) = \sup \{|v|u - (M_k) : u \geq 0\}, \quad k = 1, 2, \ldots
\]
is called the complementary function of a Musielak-Orlicz function \(M\). For a given Musielak-Orlicz function \(M\), the Musielak-Orlicz sequence space \(t_M\) and its subspace \(h_M\) are defined as follows
\[
t_M = \left\{ x \in w : I_M(cx) < \infty \text{ for some } c > 0 \right\},
\]
\[
h_M = \left\{ x \in w : I_M(cx) < \infty \text{ for all } c > 0 \right\},
\]
where \(I_M\) is a convex modular defined by
\[
I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_M.
\]

We consider \(t_M\) equipped with the Luxemburg norm
\[
||x|| = \inf \left\{ k > 0 : I_M \left( \frac{x}{k} \right) \leq 1 \right\}
\]
or equipped with the Orlicz norm
\[
||x||^0 = \inf \left\{ \frac{1}{k} \left( 1 + I_M(kx) \right) : k > 0 \right\}.
\]

For more details about sequence spaces defined by Orlicz function see ([22], [23], [24]) and reference therein.
\( \text{n-normed spaces one can see in Misiak [14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([4],[5]) and Gunawan and Mashadi [6]. Let } n \in \mathbb{N} \text{ and } X \text{ be a linear space over the field } K, \text{ where } K \text{ is field of real or complex numbers of dimension } d, \text{ where } d \geq n \geq 2. \text{ A real valued function } ||\cdot||^{\alpha} \text{ on } X^n \text{ satisfying the following four conditions:}

\begin{enumerate}
\item \( ||x_1, x_2, \cdots, x_n|| = 0 \) if and only if \( x_1, x_2, \cdots, x_n \) are linearly dependent in \( X \);
\item \( ||x_1, x_2, \cdots, x_n|| \) is invariant under permutation;
\item \( || \alpha x_1, x_2, \cdots, x_n || = ||x_1, x_2, \cdots, x_n|| \) for any \( \alpha \in K \), and
\item \( ||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n|| \)
\end{enumerate}

is called a \( n \)-norm on \( X \), and the pair \((X, ||\cdot||^{\alpha})\) is called a \( n \)-normed space over the field \( K \).

For example, we may take \( X = \mathbb{R}^n \) being equipped with the Euclidean \( n \)-norm \( ||x_1, x_2, \cdots, x_n||_E = \text{the volume of the } n\text{-dimensional parallelopipied spanned by the vectors } x_1, x_2, \cdots, x_n \text{ which may be given explicitly by the formula} \)

\[
||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,
\]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots, n \). Let \((X, ||\cdot||^{\alpha})\) be a \( n \)-normed space of dimension \( d \geq n \geq 2 \) and \( \{a_1, a_2, \cdots, a_n\} \) be linearly independent set in \( X \). Then the following function \( ||\cdot|| \) on \( X^{n-1} \) defined by

\[
||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}
\]
defines an \((n-1)\)-norm on \( X \) with respect to \( \{a_1, a_2, \cdots, a_n\} \).

A sequence \((x_k)\) in a \( n \)-normed space \((X, ||\cdot||)\) is said to converge to some \( L \in X \) if

\[
\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.
\]

A sequence \((x_k)\) in a \( n \)-normed space \((X, ||\cdot||)\) is said to be Cauchy if

\[
\lim_{k, p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.
\]

If every cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be \( n \)-Banach space.

In the present paper, we define some new sequence spaces by using the concept of ideal convergence, invariant mean, Musielak-Orlicz function, \( n \)-normed and \( A \) transform as follows:

\[
\mathcal{I} - c_0^n(A, M, p, ||\cdot||) = \left\{ x \in w : \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{\|l_k(A(x))\|}{\rho}, \rho, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \geq \epsilon \right\} \in \mathcal{I}, \text{ for all } n \in \mathbb{N} \right\},
\]
The main goal of this paper is to introduce the sequence spaces $\ell^p(A, \mathcal{M}, p, ||\cdot||)$ and $\ell^\infty(A, \mathcal{M}, p, ||\cdot||)$ defined by a Musielak-Orlicz function $\mathcal{M} = (\mu_k)$ over $n$-normed spaces. We also make an effort to study some topological properties and prove some inclusion relation between these spaces.

2. Main Results

Theorem 2.1 Let $\mathcal{M} = (\mu_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then the spaces $\ell^p(A, \mathcal{M}, p, ||\cdot||)$, $\ell^\infty(A, \mathcal{M}, p, ||\cdot||)$ and $\ell^\infty(A, \mathcal{M}, p, ||\cdot||)$ are linear.

Proof. Let $x, y \in \mathcal{I} - c^p(A, \mathcal{M}, p, ||\cdot||)$ and let $\alpha, \beta$ be scalars. Then there exist positive numbers $\rho_1$ and $\rho_2$ such that for every $\epsilon > 0$

$$D_1 = \left\{ k \in \mathbb{N} : \left[ k_{\nu k} \left( \frac{A(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \geq \frac{\epsilon}{2D} \right\} \in \mathcal{I}, \quad (2.1)$$

$$D_1 = \left\{ k \in \mathbb{N} : \left[ k_{\nu k} \left( \frac{A(y)}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \geq \frac{\epsilon}{2D} \right\} \in \mathcal{I}, \quad (2.2)$$
Let $\rho_3 = \max \{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since $\mathcal{M} = (M_k)$ is non-decreasing, convex function and so by using inequality (1.1), we have

$$M_k\left(\|\frac{t_{kn}(A(x) + \beta y)}{\rho_3}, z_1, \cdots, z_{n-1}\|\right)^{p_k} \leq M_k\left(\|\frac{t_{kn}(\alpha x)}{\rho_3}, z_1, \cdots, z_{n-1}\|\right)^{p_k} + M_k\left(\|\frac{t_{kn}(\beta y)}{\rho_3}, z_1, \cdots, z_{n-1}\|\right)^{p_k} \leq M_k\left(\|\frac{t_{kn}(A(x))}{\rho_1}, z_1, \cdots, z_{n-1}\|\right)^{p_k} + M_k\left(\|\frac{t_{kn}(A(y))}{\rho_2}, z_1, \cdots, z_{n-1}\|\right)^{p_k}.$$ 

Now by (2.1) and (2.2), we have

$$\left\{k \in \mathbb{N} : M_k\left(\|\frac{t_{kn}(A(x) + \beta y)}{\rho_3}, z_1, \cdots, z_{n-1}\|\right)^{p_k} > \epsilon \right\} \subset D_1 \cup D_2.$$

Therefore $\alpha x + \beta y \in \mathcal{I} - c_0^\sigma(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||)$. Hence $\mathcal{I} - c_0^\sigma(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||)$ is a linear space. Similarly we can prove that $\mathcal{I} - c^\sigma(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||)$ and $\mathcal{I} - \ell^\sigma_{\infty}(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||)$ are linear spaces.

**Theorem 2.2** Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then

$$\mathcal{I} - c_0^\sigma(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \subset \mathcal{I} - c^\sigma(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \subset \mathcal{I} - \ell^\sigma_{\infty}(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||).$$

**Proof.** The first inclusion is obvious. For second inclusion, let $x \in \mathcal{I} - c^\sigma(A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||)$. Then there exists $\rho_1 > 0$ such that for every $\epsilon > 0$

$$A_1 = \left\{k \in \mathbb{N} : M_k\left(\|\frac{t_{kn}(A(x) - L)}{\rho_1}, z_1, \cdots, z_{n-1}\|\right)^{p_k} \geq \epsilon \right\} \in \mathcal{I}.$$

Let us define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, we have

$$M_k\left(\|\frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1}\|\right) \leq M_k\left(\|\frac{t_{kn}(A(x) - L)}{\rho_1}, z_1, \cdots, z_{n-1}\|\right) + M_k\left(\|\frac{t_{kn}(L)}{\rho_1}, z_1, \cdots, z_{n-1}\|\right).$$

Suppose that $k \notin A_1$. Hence by above inequality and (1.1), we have

$$M_k\left(\|\frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1}\|\right)^{p_k} \leq \left\{M_k\left(\|\frac{t_{kn}(A(x) - L)}{\rho_1}, z_1, \cdots, z_{n-1}\|\right)^{p_k} + M_k\left(\|\frac{t_{kn}(L)}{\rho_1}, z_1, \cdots, z_{n-1}\|\right)^{p_k} \right\}$$

$$< D\left\{\epsilon + M_k\left(\|\frac{t_{kn}(L)}{\rho}, z_1, \cdots, z_{n-1}\|\right)^{p_k}\right\}.$$ 

Because of the fact that $M_k\left(\|\frac{t_{kn}(L)}{\rho_1}, z_1, \cdots, z_{n-1}\|\right)^{p_k} \leq \max\{1, M_k\left(\|\frac{t_{kn}(L)}{\rho_1}, z_1, \cdots, z_{n-1}\|\right)^{H}\}$, we have

$$M_k\left(\|\frac{t_{kn}(L)}{\rho}, z_1, \cdots, z_{n-1}\|\right)^{p_k} < \infty.$$
Put $K = D\left\{ \epsilon + \left[ M_k\left( \frac{t_{kn}(L)}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \right\}$. It follows that

$$\left\{ k \in \mathbb{N} : \left[ M_k\left( \frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} > K \right\} \in \mathcal{I}$$

which means $x \in \mathcal{I} - \ell_\infty^\sigma(A, M, \rho, \|\cdot\|, \|\cdot\|)$. This completes the proof of the theorem.

**Theorem 2.3** Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $\mathcal{I} - \ell_\infty^\sigma(A, M, \rho, \|\cdot\|, \|\cdot\|)$ is a paranormed space with paranorm defined by

$$g(x) = \inf \left\{ \rho > 0 : \left[ M_k\left( \frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \leq 1 \right\}.$$

**Proof.** It is clear that $g(x) = g(-x)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Let us take $x, y \in \mathcal{I} - c_{00}(A, M, \rho, \|\cdot\|, \|\cdot\|)$. Let

$$B(x) = \left\{ \rho > 0 : \left[ M_k\left( \frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \leq 1 \right\},$$

$$B(y) = \left\{ \rho > 0 : \left[ M_k\left( \frac{t_{kn}(A(y))}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \leq 1 \right\}.$$

Let $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$. If $\rho = \rho_1 + \rho_2$, then we have

$$M_k\left( \frac{t_{kn}(A(x+y))}{\rho}, z_1, \cdots, z_{n-1} \right) \leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) M_k\left( \frac{t_{kn}(A(x))}{\rho_1}, z_1, \cdots, z_{n-1} \right) + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) M_k\left( \frac{t_{kn}(A(y))}{\rho_2}, z_1, \cdots, z_{n-1} \right).$$

Thus

$$\left[ M_k\left( \frac{t_{kn}(A(x+y))}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \leq 1$$

and

$$g(x + y) \leq \inf \left\{ (\rho_1 + \rho_2) > 0 : \rho_1 \in B(x), \rho_2 \in B(y) \right\} \leq \inf \left\{ \rho_1 > 0 : \rho_1 \in B(x) \right\} + \inf \left\{ \rho_2 > 0 : \rho_2 \in B(y) \right\} = g(x) + g(y).$$

Let $\eta^s \to \eta$ where $\eta, \eta^s \in \mathbb{C}$ and let $g(x^s - x) \to 0$ as $s \to \infty$. We have to show that $g(\eta^s x^s - \eta x) \to 0$ as $s \to \infty$. Let

$$B(x^s) = \left\{ \rho_s > 0 : \left[ M_k\left( \frac{t_{kn}(A(x^s))}{\rho_s}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \leq 1 \right\},$$

$$B(x^s - x) = \left\{ \rho'_s > 0 : \left[ M_k\left( \frac{t_{kn}(A(x^s) - x)}{\rho'_s}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \leq 1 \right\}.$$
If \( \rho_s \in B(x^s) \) and \( \rho'_s \in B(x^s - x) \) then we observe that

\[
M_k\left(\|\frac{t_{kn}(A(x^s - \eta x))}{\rho_s|\eta^s - \eta| + \rho'_s|\eta|}, z_1, \cdots, z_{n-1}\|\right)
\leq M_k\left(\|\frac{t_{kn}(A(x^s))}{\rho_s}, z_1, \cdots, z_{n-1}\|\right) + \frac{|\eta^s - \eta|}{\rho_s|\eta^s - \eta| + \rho'_s|\eta|} M_k\left(\|\frac{t_{kn}(A(x^s - x))}{\rho_s}, z_1, \cdots, z_{n-1}\|\right)
\leq \frac{|\eta^s - \eta|}{\rho_s|\eta^s - \eta| + \rho'_s|\eta|} M_k\left(\|\frac{t_{kn}(A(x^s))}{\rho_s}, z_1, \cdots, z_{n-1}\|\right)
+ \frac{|\eta|}{\rho_s|\eta^s - \eta| + \rho'_s|\eta|} M_k\left(\|\frac{t_{kn}(A(x^s - x))}{\rho_s}, z_1, \cdots, z_{n-1}\|\right).
\]

From the above inequality, it follows that

\[
\left[M_k\left(\|\frac{t_{kn}(A(x^s - \eta x))}{\rho_s|\eta^s - \eta| + \rho'_s|\eta|}, z_1, \cdots, z_{n-1}\|\right)\right]^{p_k} \leq 1
\]

and consequently,

\[
g(\eta^s x^s - \eta x) \leq \inf \left\{ \left( \rho_s|\eta^s - \eta| + \rho'_s|\eta| \right) > 0 : \rho_s \in B(x^s), \rho'_s \in B(x^s - x) \right\}
\leq (|\eta^s - \eta|) \inf \left\{ \rho > 0 : \rho \in B(x^s) \right\}
+ (|\eta|) \inf \left\{ (\rho'_s)^{\frac{p}{p-1}} : \rho'_s \in B(x^s - x) \right\}
\rightarrow 0 \text{ as } s \rightarrow \infty.
\]

This completes the proof of the theorem.

\[\square\]

**Theorem 2.4** Let \( \mathcal{M}' = (M'_k) \) and \( \mathcal{M}'' = (M''_k) \) are Musielak-Orlicz functions that satisfies the \( \Delta_2 \)-condition. Then

(i) \( \mathcal{I} - c_0^p (A, \mathcal{M}', p, ||\cdot||) \subseteq \mathcal{I} - c_0^p (A, \mathcal{M}' \circ \mathcal{M}'', p, ||\cdot||) \)

(ii) \( \mathcal{I} - c^p (A, \mathcal{M}', p, ||\cdot||) \subseteq \mathcal{I} - c^p (A, \mathcal{M}' \circ \mathcal{M}'', p, ||\cdot||) \)

(iii) \( \mathcal{I} - l_\infty^p (A, \mathcal{M}', p, ||\cdot||) \subseteq \mathcal{I} - l_\infty^p (A, \mathcal{M}' \circ \mathcal{M}'', p, ||\cdot||) \).

**Proof.** (i) We prove the theorem in two parts. Firstly, let \( M'_k\left(\|\frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1}\|\right) > \delta \). Since \( \mathcal{M}' \) is nondecreasing, convex and satisfies \( \Delta_2 \)-condition, we have

\[
\left[M'_k\left(\|\frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1}\|\right)\right]^{p_k}
\leq (K\delta^{-1} M''_2(2^{p_k}))^{p_k} M'_k\left(\|\frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1}\|\right)\]
\[
\leq \max\{1, (K\delta^{-1} M''_2(2^{H}))^{H} M'_k\left(\|\frac{t_{kn}(A(x))}{\rho}, z_1, \cdots, z_{n-1}\|\right)^{p_k},
\]
where $K \geq 1$ and $\delta < 1$. From the last inequality, the inclusion
\[
\{ k \in \mathbb{N} : \left[ M''_k \left( \left[ M'_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right] \right) \right]^{p_k} \geq \epsilon \} \\
\subseteq \{ k \in \mathbb{N} : \left[ M''_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right]^{p_k} \geq \frac{\epsilon}{\max\{1, (K \delta^{-1} M''_k(2^H))\}} \}
\]
is obtained. If $x \in \mathcal{I} - c_0^0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot)$, then the set in the right side of the above inclusion belongs to the ideal and so
\[
\{ k \in \mathbb{N} : \left[ M''_k \left( \left[ M'_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right) \right] \right]^{p_k} \geq \epsilon \} \subseteq \mathcal{I}.
\]
Secondly, suppose that $M''_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \leq \delta$. Since $M''_k$ is continuous, we have
\[
M''_k \left( \left[ M'_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right] \right) < \epsilon \text{ for all } \epsilon > 0
\]
which implies
\[
\mathcal{I} \lim_k \left[ M''_k \left( \left[ M'_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right] \right) \right]^{p_k} = 0 \text{ as } \epsilon \rightarrow 0.
\]
This completes the proof of (i) part. Similarly, we can prove other parts. \hfill \square

**Theorem 2.5** Let $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ are Musielak-Orlicz functions that satisfies the $\Delta_2$-condition. Then

(i) $\mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot) \cap \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot) \subseteq \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot)$

(ii) $\mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot) \cap \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot) \subseteq \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot)$

(iii) $\mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot) \cap \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot) \subseteq \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot)$.

**Proof.** (i) Let $x \in \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot) \cap \mathcal{I} - c_0(\mathcal{M}', A, p, ||\cdot||, \cdots, \cdot)$. Then there exists $K_1 > 0$ and $K_2 > 0$ such that
\[
A_1 = \{ k \in \mathbb{N} : \left[ M_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right]^{p_k} \geq K_1 \} \subseteq \mathcal{I}
\]
and
\[
A_2 = \{ k \in \mathbb{N} : \left[ M_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right]^{p_k} \geq K_2 \} \subseteq \mathcal{I}
\]
for some $\rho > 0$. Let $k \notin A_1 \cup A_2$. Then we have
\[
\left[ (M_k + M'_k) \left( \left[ \frac{t_k(x)}{\rho} \right), z_1, \cdots, z_{n-1} \right) \right]^{p_k}
\]
\[
\leq D \left\{ \left( M_k \left( \left[ \frac{t_k(x)}{\rho_1} \right), z_1, \cdots, z_{n-1} \right) \right)^{p_k} + \left( M'_k \left( \left[ \frac{t_k(x)}{\rho} \right), z_1, \cdots, z_{n-1} \right) \right)^{p_k} \right\}
\]
\[
< \{ K_1 + K_2 \}.
\]
\[ k \notin B = \left\{ k \in \mathbb{N} : \left( (M_k' + M_k) \left( \frac{t_kn(A(x))}{\rho}, z_1, \ldots, z_{n-1} \right) \right)^p > K \right\}. \] We have \( A_1 \cup A_2 \in \mathcal{I} \) and so \( B \subset A_1 \cup A_2 \) which implies \( B \in \mathcal{I} \). This means that \( x \in \mathcal{I} - c_0^p(A, \mathcal{M}' \cup \mathcal{M}, p, ||\cdot||) \). This completes the proof of (i) part of the theorem. Similarly, we can prove (ii) and (iii) part. 

**Theorem 2.6** If \( \sup_k [M_k(t)]^p < \infty \) for all \( t > 0 \), then we have 
\[ \mathcal{I} - c^p(A, \mathcal{M}, p, ||\cdot||) \subseteq \mathcal{I} - \ell_\infty^p(A, \mathcal{M}, p, ||\cdot||). \]

*Proof.* Let \( x \in \mathcal{I} - c^p(A, \mathcal{M}, p, ||\cdot||) \). By using inequality (1.1), we have 
\[
\left[ M_k \left( \frac{t_kn(A(x))}{\rho} \right) \right]^p \leq D \{ \left[ M_k \left( \frac{t_kn(A(x) - L)}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^p + \left[ M_k \left( \frac{t_kn(L)}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^p \},
\]
where \( \rho = 2\rho_1 \). Hence, we have 
\[
\left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{t_kn(A(x))}{\rho} \right) \right]^p \geq K \right\} \subseteq \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{t_kn(A(x) - L)}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^p \geq \epsilon \right\}
\]
for all \( n \) and some \( K > 0 \). Since the set in the right side of the above inclusion belongs to the ideal, all of its subsets are in the ideal. Hence 
\[
\left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{t_kn(A(x))}{\rho} \right) \right]^p \geq K \right\} \in \mathcal{I}
\]
which completes the proof. 

**Theorem 2.7** Let \( 0 < p_k \leq q_k < \infty \) for each \( k \in \mathbb{N} \) and \( \left( \frac{p_k}{q_k} \right) \) be bounded. Then following inclusions hold
\[
(i) \mathcal{I} - c_0^p(A, \mathcal{M}, q, ||\cdot||) \subseteq \mathcal{I} - c_0^q(A, \mathcal{M}, p, ||\cdot||)
\]
\[
(ii) \mathcal{I} - c^p(A, \mathcal{M}, q, ||\cdot||) \subseteq \mathcal{I} - c^q(A, \mathcal{M}, p, ||\cdot||).
\]

*Proof.* (i) Let \( x \in \mathcal{I} - c_0^p(A, \mathcal{M}, q, ||\cdot||) \). Write \( \alpha_k = \frac{p_k}{q_k} \). By hypothesis, we have \( 0 < \alpha \leq \alpha_k \leq 1 \). If 
\[
\left[ M_k \left( \frac{t_kn(A(x))}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^{q_k} \geq 1,
\]
the inequality 
\[
\left[ M_k \left( \frac{t_kn(A(x))}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \leq \left[ M_k \left( \frac{t_kn(A(x))}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^{q_k}
\]
holds. This implies the inclusion 
\[
\left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{t_kn(A(x))}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \geq \epsilon \right\}
\]
\[
\subseteq \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{t_kn(A(x))}{\rho}, z_1, \ldots, z_{n-1} \right) \right]^{q_k} \geq \epsilon \right\}
\]
and so the result is obvious. Conversely, if \( M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \) for each \( k \in \mathbb{N} \), we obtain the following inclusion
\[
\left\{ k \in \mathbb{N} : M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \right\}^{p_k} \geq \epsilon \}
\subseteq \left\{ k \in \mathbb{N} : M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \right\}^{q_k} \geq \epsilon^\frac{1}{q_k}
\]
since then the inequality
\[
\left[ M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} \leq \left( \left[ M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{q_k} \right)^{\alpha}
\]
holds. Hence we conclude that \( x \in \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \). This completes the proof of (i) part. Similarly, we can prove (ii) part.

\[ \square \]

**Theorem 2.8** If \( 0 < \inf p_k \leq p_k \leq 1 \) for each \( k \in \mathbb{N} \). Then the following inclusions hold:

(i) \( \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \subseteq \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, ||\cdot||, \cdots, ||\cdot||) \)

(ii) \( \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \subseteq \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, ||\cdot||, \cdots, ||\cdot||) \).

**Proof.** Let \( x \in \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \). Suppose that \( k \notin \left\{ k \in \mathbb{N} : M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \right\}^{p_k} \geq \epsilon \} \) for \( 0 < \epsilon < 1 \). By hypothesis, the inequality
\[
M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \geq \left[ M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k}
\]
holds. Then we have \( k \notin \left\{ k \in \mathbb{N} : M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \geq \epsilon \} \) which implies
\[
\left\{ k \in \mathbb{N} : M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \geq \epsilon \right\} \subseteq \left\{ k \in \mathbb{N} : M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \right\}^{p_k} \geq \epsilon \right\}.
\]
Hence \( x \in \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, ||\cdot||, \cdots, ||\cdot||) \) since the set
\[
\left\{ k \in \mathbb{N} : M_k \left( \frac{\|f_k(A(x))\|}{\rho}, z_1, \cdots, z_{n-1} \right) \geq \epsilon \right\} \subseteq \mathcal{I}.
\]
This completes the proof of (i) part. Similarly, we can prove (ii) part.

\[ \square \]

**Corollary 2.9** If \( 0 < \inf p_k \leq p_k \leq 1 \) for each \( k \in \mathbb{N} \). Then the following inclusions hold:

(i) \( \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, ||\cdot||, \cdots, ||\cdot||) \subseteq \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \)

(ii) \( \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, ||\cdot||, \cdots, ||\cdot||) \subseteq \mathcal{I} - \mathcal{C}_0^\sigma (A, \mathcal{M}, p, ||\cdot||, \cdots, ||\cdot||) \).

**Proof.** The proof is obvious by Theorem 2.8.

\[ \square \]
References


