ON FUZZY ORDERED LA-SEMIHYPERGROUPS

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Abstract. We introduce the notion of fuzzy ordered LA-semihypergroups and provide different examples. We also discuss some results related with fuzzy left and right hyperideals.

1. Introduction

The theory of algebraic hyperstructure was introduced by Marty in 1934, when Marty [1] defined hypergroups. Since then many hyperstructures were being studied by several authors, for instance, Bonansinga and Corsini [2], Davvaz [3], Freni [4], Hila et al. [5], Leoreanu [6], Salvo et al. [7] and many others. The concept of ordered semihypergroup was studied by Heidari and Davvaz in [8], where they used a binary relation ” ≤ ” on semihypergroup \((H, ◦)\) such that the binary relation is a partial order and the structure \((H, ◦, ≤)\) is known as ordered semihypergroup. There are several authors who study the ordering of hyperstructures, for instance, Bakhshi and Borzooei [9], Chvalina [10], Hoskova [11], Kondo and Lekkoksung [12] and Novak [13].

Another non-associative algebraic hyperstructure known as LA-semihypergroup which is a useful generalization of semigroup, semihypergroups and LA-semigroups was introduced by Hilla and Dine [14] in 2011 based on left invertive law given by Kazim and Naseerudin [15] in 1972. Yaqoob et al. [16] extended the work...
of Hila and Dine and characterized intra-regular left almost semihypergroups by their hyperideals using pure
left identity. The ordering in LA-semihypergroups was introduced by Yaqoob and Gulistan [17].

The concept of fuzzy set was introduced by Zadeh in 1965 [18]. Rosenfeld [19] introduced fuzzy sets in
the context of group theory and formulated the concept of fuzzy subgroup of a group in 1971. Later many
researcher are engaged in extending the concept of abstract algebra to the frame work of fuzzy setting.
Fuzzy hyperstructures have been already considered by many researchers, for instance, Corsini et al. [20,21],
Davvaz [22, 23], Hila and Abdullah [24], Khan et al. [25], Pibaljommee et al. [26, 27], Tang et al. [28–31],
Tipachot and Pibaljommee [32] and Zhan et al. [33,34].

As a further study of ordered LA-semihypergroups, we attempt in the present paper to study the fuzzy
ordered LA-semihypergroups in detail.

2. Preliminaries

Let $H$ be a non-empty set. Then the map $\circ : H \times H \rightarrow P^+(H)$ is called hyperoperation or join operation
on the set $H$, where $P^+(H) = P(H)\setminus\{\emptyset\}$ denotes the set of all non-empty subsets of $H$. A hypergroupoid is
a set $H$ together with a (binary) hyperoperation. For any non-empty subsets $A,B$ of $H$, we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b$$

Instead of $\{a\} \circ A$ and $B \circ \{a\}$, we write $a \circ A$ and $B \circ a$, respectively.

Recently, in [14,16] authors introduced the notion of LA-semihypergroups as a generalization of semi-
groups, semihypergroups, and LA-semigroups. A hypergroupoid $(H, \circ)$ is called an LA-semihypergroup if
for every $x,y,z \in H$, we have $(x \circ y) \circ z = (z \circ y) \circ x$. The law $(x \circ y) \circ z = (z \circ y) \circ x$ is called a left
invertive law. An element $e \in H$ is called a left identity (resp., pure left identity) if for all $x \in H$, $x \in e \circ x$
(resp., $x = e \circ x$). In an LA-semihypergroup, the medial law $(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w)$ holds for all
$x,y,z,w \in H$. An LA-semihypergroup may or may not contains a left identity and pure left identity. In an
LA-semihypergroup $H$ with pure left identity, the paramedial law $(x \circ y) \circ (z \circ w) = (w \circ z) \circ (y \circ x)$ holds
for all $x,y,z,w \in H$. If an LA-semihypergroup contains a pure left identity, then by using medial law, we
get $x \circ (y \circ z) = y \circ (x \circ z)$ for all $x,y,z \in H$.

Definition 2.1. [17] Let $H$ be non-empty set and $\leq$ be an ordered relation on $H$. The triplet $(H, \circ, \leq)$ is
called an ordered LA-semihypergroup if the following conditions are satisfied.

1. $(H, \circ)$ is an LA-semihypergroup,

2. $(H, \leq)$ is a partially ordered set,

3. for every $a,b,c \in H$, $a \leq b$ implies $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$, where $a \circ c \leq b \circ c$ means that for
   $x \in a \circ c$ there exist $y \in b \circ c$ such that $x \leq y$. 
Definition 2.2. [17] If \((H, \circ, \leq)\) is an ordered LA-semihypergroup and \(A \subseteq H\), then \((A)\) is the subset of \(H\) defined as follows:

\[
(A) = \{ t \in H : t \leq a, \text{ for some } a \in A \}.
\]

Definition 2.3. [17] A non-empty subset \(A\) of an ordered LA-semihypergroup \((H, \circ, \leq)\) is called an LA-subsemihypergroup of \(H\) if \((A \circ A) \subseteq (A)\).

Definition 2.4. [17] A non-empty subset \(A\) of an ordered LA-semihypergroup \((H, \circ, \leq)\) is called a right (resp., left) hyperideal of \(H\) if

1. \(A \circ H \subseteq A\) (resp., \(H \circ A \subseteq A\)),
2. for every \(a \in H\), \(b \in A\) and \(a \leq b\) implies \(a \in A\).

If \(A\) is both right hyperideal and left hyperideal of \(H\), then \(A\) is called a hyperideal (or two sided hyperideal) of \(H\).

3. Fuzzy ordered LA-semihypergroups

Let \(x \in H\), then \(A_x = \{(y, z) \in H \circ H : x \leq y \circ z\}\). Let \(f\) and \(g\) be two fuzzy subsets of an ordered LA-semihypergroup \(H\), then \(f \ast g\) is defined as

\[
(f \ast g)(x) = \begin{cases} 
\bigvee_{(y, z) \in A_x} \{ f(y) \land g(z) \} & \text{if } x \leq y \circ z, \text{ for some } y, z \in H \\
0 & \text{otherwise.}
\end{cases}
\]

Let \(F(H)\) denote the set of all fuzzy subsets of an ordered LA-semihypergroup.

Theorem 3.1. Let \(H\) be an ordered LA-semihypergroup. Then the set \((F(H), \ast, \subseteq)\) is an ordered LA-semihypergroup.

Proof. Clearly \(F(H)\) is closed. Let \(f, g\) and \(h\) be in \(F(H)\) and let \(x\) be any element of \(H\) such that it is not expressible as product of two elements in \(H\). Then we have,

\[
((f \ast g) \ast h)(x) = 0 = ((h \ast g) \ast f)(x).
\]
Let \( A_x \neq \emptyset \). Then there exist \( y \) and \( z \) in \( H \) such that \( (y, z) \in A_x \). Therefore by using left invertive law, we have

\[
((f \ast g) \ast h)(x) = \bigvee_{(y, z) \in A_x} \{(f \ast g)(y) \land h(z)\}
\]

\[
= \bigvee_{(y, z) \in A_x} \left\{ \bigvee_{(p, q) \in A_y} \{f(p) \land g(q)\} \land h(z) \right\}
\]

\[
= \bigvee_{x \leq (pq) \circ z} \{f(p) \land g(q) \land h(z)\}
\]

\[
= \bigvee_{x \leq (zq) \circ p} \{h(z) \land g(q) \land f(p)\}
\]

\[
= \bigvee_{(w, p) \in A_x} \left\{ \bigvee_{(z, q) \in A_w} (h(z) \land g(q) \land f(p)) \right\}
\]

\[
= \bigvee_{(w, p) \in A_x} \{(h \ast g)(w) \land f(p)\} = ((h \ast g) \ast f)(x).
\]

Hence \((F(H), \circ)\) is an LA-semihypergroup. Assume that \( f \subseteq g \) and let \( A_x = \emptyset \) for any \( x \in H \), then

\[
(f \ast h)(x) = 0 = (g \ast h)(x) \implies f \ast h \subseteq g \ast h.
\]

Similarly we can show that \( f \ast h \supseteq g \ast h \). Let \( A_x \neq \emptyset \). Then there exist \( y \) and \( z \) in \( H \) such that \( (y, z) \in A_x \), therefore

\[
(f \ast h)(x) = \bigvee_{(y, z) \in A_x} \{f(y) \land h(z)\} \leq \bigvee_{(y, z) \in A_x} \{g(y) \land h(z)\} = (g \ast h)(x),
\]

Similarly we can show that \( f \ast h \supseteq g \ast h \). It is easy to see that \( F(H) \) is a poset. Thus \((F(H), \ast, \subseteq)\) is an ordered LA-semihypergroup. \(\square\)

**Theorem 3.2.** Let \( H \) be an ordered LA-semihypergroup. Then the property

\[
(f \ast g) \ast (h \ast k) = (f \ast h) \ast (g \ast k)
\]

holds in \( F(H) \), for all \( f, g, h \) and \( k \) in \( F(H) \).

**Proof.** Straightforward. \(\square\)

**Theorem 3.3.** If an ordered LA-semihypergroup \( H \) has a pure left identity, then the following properties hold in \( F(H) \).

(i) \( (f \ast g) \ast (h \ast k) = (k \ast h) \ast (g \ast f) \),

(ii) \( f \ast (g \ast h) = g \ast (f \ast h) \),

for all \( f, g, h \) and \( k \) in \( F(H) \).

**Proof.** Straightforward. \(\square\)
Proposition 3.1. An ordered LA-semihypergroup \((F(H),*,\subseteq)\) with \(F(H) = (F(H))^2\) is a commutative ordered semihypergroup if and only if \((f * g) * h = f * (h * g)\) holds for all fuzzy subsets \(f,g,h \in F(H)\).

Proof. Let an ordered LA-semihypergroup \(F(H)\) be a commutative ordered semihypergroup. For any fuzzy subsets \(f,g,h \in F(H)\), if \(A_x = \emptyset\) for any \(x \in H\), then \((f * g) * h(x) = 0 = (f * (h * g))(x)\). Let \(A_x \neq \emptyset\), then there exist \(s\) and \(t\) in \(H\) such that \((s,t) \in A_x\), therefore by use of left invertive law and commutative law, we get

\[
(f * g)(x) = \bigvee_{x \subseteq \text{mon}} \{(f * g)(s) \wedge h(t)\}
\]

Thus \((f * g) * h = f * (h * g)\). Conversely, let \((f * g) * h = f * (h * g)\) holds for all fuzzy subsets \(f,g,h \in F(H)\). We have to show that \(H\) is a commutative ordered semihypergroup. Let \(f\) and \(g\) be any arbitrary fuzzy subsets of \(H\), if \(A_x = \emptyset\) for any \(x \in H\), then \((f * g)(x) = 0 = (g * f)(x)\). Let \(A_x \neq \emptyset\), then there exist \(s\) and \(t\) in \(H\) such that \((s,t) \in A_x\). Since \(F(H) = (F(H))^2\), so \(f = h * k\), where \(h\) and \(k\) are any fuzzy subsets of \(H\). Now by left invertive law, we have

\[
(f * g)(x) = ((h * k) * g)(x) = \bigvee_{x \subseteq \text{mon}} \{(h * k)(s) \wedge g(t)\}
\]

Thus \((f * g) * h = f * (h * g)\). Conversely, let \((f * g) * h = f * (h * g)\) holds for all fuzzy subsets \(f,g,h \in F(H)\). We have to show that \(H\) is a commutative ordered semihypergroup. Let \(f\) and \(g\) be any arbitrary fuzzy subsets of \(H\), if \(A_x = \emptyset\) for any \(x \in H\), then \((f * g)(x) = 0 = (g * f)(x)\). Let \(A_x \neq \emptyset\), then there exist \(s\) and \(t\) in \(H\) such that \((s,t) \in A_x\). Since \(F(H) = (F(H))^2\), so \(f = h * k\), where \(h\) and \(k\) are any fuzzy subsets of \(H\). Now by left invertive law, we have

\[
(f * g)(x) = ((h * k) * g)(x) = \bigvee_{x \subseteq \text{mon}} \{(h * k)(s) \wedge g(t)\}
\]

Thus \((f * g) * h = f * (h * g)\). Conversely, let \((f * g) * h = f * (h * g)\) holds for all fuzzy subsets \(f,g,h \in F(H)\). We have to show that \(H\) is a commutative ordered semihypergroup. Let \(f\) and \(g\) be any arbitrary fuzzy subsets of \(H\), if \(A_x = \emptyset\) for any \(x \in H\), then \((f * g)(x) = 0 = (g * f)(x)\). Let \(A_x \neq \emptyset\), then there exist \(s\) and \(t\) in \(H\) such that \((s,t) \in A_x\). Since \(F(H) = (F(H))^2\), so \(f = h * k\), where \(h\) and \(k\) are any fuzzy subsets of \(H\). Now by left invertive law, we have

\[
(f * g)(x) = ((h * k) * g)(x) = \bigvee_{x \subseteq \text{mon}} \{(h * k)(s) \wedge g(t)\}
\]

Thus \((f * g) * h = f * (h * g)\). Conversely, let \((f * g) * h = f * (h * g)\) holds for all fuzzy subsets \(f,g,h \in F(H)\). We have to show that \(H\) is a commutative ordered semihypergroup. Let \(f\) and \(g\) be any arbitrary fuzzy subsets of \(H\), if \(A_x = \emptyset\) for any \(x \in H\), then \((f * g)(x) = 0 = (g * f)(x)\). Let \(A_x \neq \emptyset\), then there exist \(s\) and \(t\) in \(H\) such that \((s,t) \in A_x\). Since \(F(H) = (F(H))^2\), so \(f = h * k\), where \(h\) and \(k\) are any fuzzy subsets of \(H\). Now by left invertive law, we have

\[
(f * g)(x) = ((h * k) * g)(x) = \bigvee_{x \subseteq \text{mon}} \{(h * k)(s) \wedge g(t)\}
\]
This shows that \( f * g = g * (h * k) = g * f \). Therefore commutative law holds in \( F(H) \).

Now if \( A_x = \emptyset \) for any \( x \in H \), then \((f * g) * k)(x) = 0 = (f * (g * k))(x)\). Let \( A_x \neq \emptyset \), then there exist \( s \) and \( t \) in \( H \) such that \((s, t) \in A_x\), therefore by using left invertive law and commutative law, we get

\[
((f * g) * k)(x) = \bigvee_{(s, t) \in A_x} \{ (f * g)(s) \land k(t) \}
\]

\[
= \bigvee_{(s, t) \in A_x} \left\{ \bigvee_{(m, n) \in A_x} f(m) \land g(n) \land k(t) \right\}
\]

\[
= \bigvee_{x \leq \{ \text{not} \}} \{ f(m) \land g(n) \land k(t) \}
\]

\[
= \bigvee_{x \leq \{ \text{not} \}} \{ f(m) \land g(n) \land k(t) \}
\]

\[
= \bigvee_{x \leq \{ \text{not} \}} \left\{ \bigvee_{(m, n) \in A_x} f(m) \land g(n) \land k(t) \right\}
\]

\[
= \bigvee_{(m, p) \in A_x} \left\{ \bigvee_{(n, t) \in A_x} f(m) \land g(n) \land k(t) \right\}
\]

\[
= \bigvee_{(m, p) \in A_x} \{ f(m) \land (g * k)(p) \}
\]

\[
= (f * (g * k))(x).
\]

Therefore associative law holds in \( F(H) \). Thus \( F(H) \) is commutative ordered semihypergroup. \( \square \)

**Theorem 3.4.** \( \mathcal{C}_M = \{ f \mid f \in F(H), f * \alpha = f, \text{ where } \alpha = \alpha * \alpha \} \) is a commutative monoid in \( H \).

**Proof.** The fuzzy subset \( \mathcal{C}_M \) of \( H \) is non-empty since \( \alpha * \alpha = \alpha \), which implies that \( \alpha \) is in \( \mathcal{C}_M \). Let \( f \) and \( \gamma \) be fuzzy subsets of \( H \) in \( \mathcal{C}_M \), then \( f * \alpha = f \) and \( \gamma * \alpha = \gamma \). If \( A_x = \emptyset \) for \( x \in H \), then \((f * \gamma)(x) = 0 = ((f * \gamma) * \alpha)(x)\). Let \( A_x \neq \emptyset \), then there exist \( y \) and \( z \) in \( H \) such that \((y, z) \in A_x\). Therefore
by using medial law, we have

\[(f * \gamma)(x) = \bigvee_{(y,z) \in A_x} \{(f * \alpha)(y) \land (\gamma * \alpha)(z)\}\]

\[= \bigvee_{(y,z) \in A_x} \left\{ \bigvee_{(p,q) \in A_y} \{f(p) \land \alpha(q)\} \land \bigvee_{(u,v) \in A_z} \{\gamma(u) \land \alpha(v)\} \right\}\]

\[= \bigvee_{x \leq (p \circ q) \circ (u \circ v)} \left\{ f(p) \land \gamma(q) \land \alpha(v) \right\}\]

\[= \bigvee_{x \leq (p \circ q) \circ (u \circ v)} \left\{ f(p) \land \gamma(u) \land \alpha(q) \land \alpha(v) \right\}\]

\[= \bigvee_{(m,n) \in A_x} \left\{ \bigvee_{(p,u) \in A_m} \{f(p) \land \gamma(u)\} \land \bigvee_{(q,v) \in A_n} \{\alpha(q) \land \alpha(v)\} \right\}\]

\[= \bigvee_{(m,n) \in A_x} \{(f * \gamma)(m) \land (\alpha * \alpha)(n)\} = ((f * \gamma) * (\alpha * \alpha))(x).\]

Thus \(f * \gamma = (f * \gamma) * (\alpha * \alpha) = (f * \gamma) * \alpha\), which implies that \(C_M\) is closed.

Now if \(A_x = \emptyset\) for \(x \in H\), then \((f * \gamma)(x) = 0 = (\gamma * f)(x)\). Let \(A_x \neq \emptyset\), then there exist \(y\) and \(z\) in \(H\) such that \((y,z) \in A_x\). Therefore by using left invertive law, we have

\[(f * \gamma)(x) = \bigvee_{(y,z) \in A_x} \{(f * \alpha)(y) \land \gamma(z)\}\]

\[= \bigvee_{(y,z) \in A_x} \left\{ \bigvee_{(p,q) \in A_y} f(p) \land \alpha(q) \land \gamma(z) \right\}\]

\[= \bigvee_{x \leq (p \circ q) \circ z} \left\{ \gamma(z) \land \alpha(q) \land f(p) \right\}\]

\[= \bigvee_{x \leq (z \circ q) \circ p} \left\{ \gamma(z) \land \alpha(q) \land f(p) \right\}\]

\[= \bigvee_{(t,p) \in A_x} \left\{ \bigvee_{(z,q) \in A_t} \gamma(z) \land \alpha(q) \land f(p) \right\}\]

\[= \bigvee_{(t,p) \in A_x} \{(\gamma * \alpha)(t) \land f(p)\} = ((\gamma * \alpha) * f)(x).\]

Thus \(f * \gamma = (\gamma * \alpha) * f = \gamma * f\), which implies that commutative law holds in \(C_M\) and associative law holds in \(C_M\) due to commutativity. Since for any fuzzy subset \(f\) in \(C_M\), we have \(f * \alpha = f\) (where \(\alpha\) is fixed) implies that \(\alpha\) is a right identity in \(H\) and hence an identity.

For an ordered LA-semihypergroup \(H\), the fuzzy subset \(H\) of \(H\) is defined as follows:

\[H : H \rightarrow [0,1]|x \rightarrow H(x) := 1.\]

**Lemma 3.1.** In an ordered LA-semihypergroup with a left identity \(H * H = H\).
Proof. Let \( x \in H \). Then \( x \leq e \circ x \), that is \((e, x) \in A_x\), where \( e \) is the left identity of \( H \). Therefore

\[
(H \ast H)(x) = \bigvee_{(e, x) \in A_x} (H(e) \wedge H(x)) = 1 = H(x).
\]

So \( H \ast H = H \).


4. Fuzzy hyperideals in ordered LA-semihypergroups

In this section, we define the concept of a fuzzy right (resp., left) hyperideal and give relationships between them.

**Definition 4.1.** Let \((H, \circ, \leq)\) be an ordered LA-semihypergroup. A fuzzy subset \( f : H \to [0, 1] \) is called fuzzy LA-subsemihypergroup of \( H \) if the following assertion are satisfied:

\[
(i) \quad \bigwedge_{z \leq a \circ b} f(z) \geq \min\{f(a), f(b)\},
\]

\[
(ii) \quad \text{if } a \leq b \text{ implies } f(a) \geq f(b),
\]

for every \( a, b \in H \).

**Example 4.1.** We consider a set \( H = \{x, y, z\} \) with the following hyperoperation \( \circ \) and the order \( \leq \):

\[
\begin{array}{ccc}
\circ & x & y & z \\
x & \{x, y\} & \{x, y\} & z \\
y & \{x, z\} & \{x, z\} & z \\
z & z & z & z \\
\end{array}
\]

\( \leq := \{(x, x), (y, y), (z, x), (z, y), (z, z)\} \).

We give the covering relation \( \prec \) and the figure of \( H \) as follows:

\[
\prec = \{(z, x), (z, y)\}.
\]

Then \((H, \circ, \leq)\) is an ordered LA-semihypergroup. Now let \( f \) be a fuzzy subset of \( H \) such that

\[
f(x) = 0.5, \ f(y) = 0.3, \ f(z) = 0.9.
\]

Clearly \( f \) is a fuzzy LA-subsemihypergroup of \( H \).
Theorem 4.1. A fuzzy subset $f$ of an ordered LA-semihypergroup $H$ is a fuzzy LA-subsemihypergroup of $H$ if and only if

(i) $f * f \subseteq f$,

(ii) if $a \leq b$ implies $f(a) \geq f(b)$, for every $a, b \in H$.

Proof. Consider that $f$ is a fuzzy LA-subsemihypergroup of $H$. Let $A_x = \emptyset$ for any $x \in H$. Then $(f * f)(x) = 0 = f(x)$. Let $A_x \neq \emptyset$. Then for $x, y, a \in H$, we have

$$(f * f)(a) = \bigvee_{a \leq x \circ y} \{f(x) \wedge f(y)\} \leq \bigvee_{a \leq x \circ y} f(a) = f(a).$$

Thus $f * f \subseteq f$.

Conversely, assume that $f * f \subseteq f$. Let $x, y, a \in H$ with $a \in x \circ y$. We have

$$f(a) \geq (f * f)(a) = \bigvee_{a \leq x \circ y} \{f(x) \wedge f(y)\} \geq f(x) \wedge f(y).$$

Thus $\bigwedge_{a \leq x \circ y} f(a) \geq \min \{f(x), f(y)\}$. Thus $f$ is a fuzzy LA-subsemihypergroup of $H$. \qed

Definition 4.2. Let $(H, \circ, \leq)$ be an ordered LA-semihypergroup. A fuzzy subset $f : H \to [0, 1]$ is called a fuzzy right (resp., left) hyperideal of $H$ if

1. $\bigwedge_{z \leq a \circ b} f(z) \geq f(a)$ (resp., $\bigwedge_{z \leq a \circ b} f(z) \geq f(b)$),

2. $a \leq b$ implies $f(a) \geq f(b)$,

for every $a, b \in H$.

If $f$ is both fuzzy right hyperideal and fuzzy left hyperideal of $H$, then $f$ is called a fuzzy hyperideal of $H$.

Definition 4.3. We consider a set $H = \{x, y, z\}$ with the following hyperoperation "\circ" and the order "\leq":

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$\leq := \{(x, x), (x, y), (x, z), (y, y), (z, z), (w, x), (w, y), (w, z), (w, w)\}$.

We give the covering relation "\prec" and the figure of $H$ as follows:

$\prec := \{(x, y), (x, z), (w, x)\}$
Then \((H, \circ, \leq)\) is an ordered LA-semihypergroup. Now let \(f\) be a fuzzy subset of \(H\) such that

\[
f(a) = \begin{cases} 
0.6 & \text{if } a = x \\
0.4 & \text{if } a = y \\
0.2 & \text{if } a = z \\
0.9 & \text{if } a = w
\end{cases}
\]

Then \(f\) is a fuzzy two sided hyperideal of \(H\).

**Theorem 4.2.** A fuzzy subset \(f\) of an ordered LA-semihypergroup \(H\) is a fuzzy left (resp., right) hyperideal of \(H\) if and only if

(i) \(H \ast f \subseteq f\) (resp., \(f \ast H \subseteq f\))

(ii) if \(a \leq b\) implies \(f(a) \geq f(b)\), for every \(a, b \in H\).

**Proof.** The proof is similar to the proof of the Theorem 4.1. \(\square\)

**Definition 4.4.** Let \(H\) be an ordered LA-semihypergroup and \(f\) be a fuzzy subset of \(H\). Then for every \(t \in [0, 1]\) the set

\[
f_t = \{x : x \in H, f(x) \geq t\}
\]

is called the level set of \(H\).

**Definition 4.5.** Let \(H\) be an ordered LA-semihypergroup and \(\emptyset \neq A \subseteq H\). Then the characteristic function \(\chi_A\) of \(A\) is defined as:

\[
\chi_A : H \to [0, 1] : x \mapsto \chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A.
\end{cases}
\]

**Theorem 4.3.** Let \(H\) be an ordered LA-semihypergroup and \(f\) be a fuzzy subset of \(H\). Then \(f\) is a fuzzy LA-subsemihypergroup (resp., right hyperideal, left hyperideal) of \(H\) if and only if for every \(t \in [0, 1]\), the non-empty level subset \(f_t\) is a fuzzy LA-subsemihypergroup (resp., right hyperideal, left hyperideal) of \(H\).

**Proof.** Assume that \(f\) is a fuzzy right hyperideal of \(H\). Let \(t \in [0, 1]\) with \(f_t \neq \phi\). Let \(a \in f_t \circ H\). We have \(a \in b \circ h\) for some \(b \in f_t\) and \(h \in H\). By assumption, \(t \leq f(b) \leq \bigwedge_{a \in b \circ h} f(a)\), we have \(f(a) \geq t\). This implies
Let \( f_t \circ H \subseteq f_t \). Let \( x \in f_t \) and \( y \in H \) with \( y \leq x \). Since \( t \leq f(x) \leq f(y) \), we obtain \( y \in f_t \). Therefore, \( f_t \) is right hyperideal of \( H \).

Conversely, we assume that for every \( t \in [0,1] \), \( f_t \) is a right hyperideal of \( H \). We show that \( f_a \leq \bigwedge_{c \in a \circ b} f(c) \) for \( a, b \in H \). We put \( t_\circ = f(a) \). By assumption \( f_t \circ \) is a right hyperideal of \( H \). Since \( a \in f_{t_\circ} \), \( a \circ b \subseteq f_{t_\circ} \). Then, for every \( c \in a \circ b \), we obtain \( t_\circ \leq f(c) \) and hence, \( f(a) = t_\circ \leq \bigwedge_{c \in a \circ b} f(c) \). Let \( a, b \in H \) with \( a \leq b \). Since \( a \leq b, b \in f_{f(b)} \) and \( f_{f(b)} \) is a right hyperideal of \( H \). We have \( a \in f_{f_1} \). So, \( f(b) \leq f(a) \). Therefore \( f \) is a fuzzy right hyperideal of \( H \).

\[ \square \]

**Corollary 4.1.** Let \( H \) be an ordered LA-semihypergroup and \( \chi_I \) be the characteristic function of \( I \). Then, then \( I \) is an LA-subsemihypergroup (resp., right hyperideal, left hyperideal) of \( H \) if and only if \( \chi_I \) is a fuzzy LA-subsemihypergroup (resp., right hyperideal, left hyperideal) of \( H \).

**Theorem 4.4.** If \( \{ f_i \}_{i \in J} \) is a family of fuzzy left hyperideals (resp., right hyperideals) of an ordered LA-semihypergroup \( H \), then \( \bigwedge_{i \in J} f_i \) is a fuzzy left hyperideal (resp., right hyperideal) of \( H \), where

\[
\bigwedge_{i \in J} f_i = \bigwedge_{i \in J} f_i \text{ and } \bigwedge_{i \in J} f_i (x) = \inf \{ f_i (x) : i \in J, x \in H \} .
\]

**Proof.** Straightforward. \( \square \)

**Proposition 4.1.** The fuzzy product of two fuzzy right (resp., left) hyperideals of an ordered LA-semihypergroup \( H \) is again a fuzzy right (resp., left) hyperideal of \( H \).

**Proof.** Let \( f_1 \) and \( f_2 \) be two fuzzy right hyperideals of an ordered LA-semihypergroup \( H \). Let \( x, y \in H \) such that \( x \leq y \). Let \( (a,b) \in A_y \) then \( y \leq a \circ b \). Since \( x \leq y \), so \( x \leq a \circ b \) implies \( (a,b) \in A_x \). Hence \( A_y \subseteq A_x \). Now

\[
(f_1 \ast f_2)(y) = \bigvee_{(a,b) \in A_y} \{ f_1(a) \land f_2(b) \} = \bigvee_{(a,b) \in A_y \subseteq A_x} \{ f_1(a) \land f_2(b) \} \leq \bigvee_{(a,b) \in A_x} \{ f_1(a) \land f_2(b) \} = (f_1 \ast f_2)(x)
\]

Thus \( (f_1 \ast f_2)(x) \geq (f_1 \ast f_2)(y) \). If \( A_x = \emptyset \), for any \( x \in H \). Then

\[
\bigwedge_{x \leq a \circ b} (f_1 \ast f_2)(x) = 0 = (f_1 \ast f_2)(a).
\]
If $A_x \neq \emptyset$. Since $f_1$ and $f_2$ are a fuzzy right hyperideals of $H$, then
\[
(f_1 \ast f_2)(x) = \bigvee_{(a,b) \in A_x} \{ f_1(a) \wedge f_2(b) \}
\]
\[
\leq \bigvee_{(a \circ y, b \circ y) \in A_{x \circ y}} \left\{ \left( \bigwedge_{m \leq a \circ y} f_1(m) \right) \wedge \left( \bigwedge_{n \leq b \circ y} f_2(n) \right) \right\}
\]
\[
= \bigwedge_{q \leq x \circ y} (f_1 \ast f_2)(q).
\]
Thus $f_1 \ast f_2$ is a fuzzy right hyperideal of $H$. \hfill \Box

**Theorem 4.5.** Let $f_1$ be a fuzzy right hyperideal and $f_2$ a fuzzy left hyperideal of $H$. Then $f_1 \ast f_2 \subseteq f_1 \cap f_2$.

**Proof.** Let $A_x = \emptyset$ for any $x \in H$. Then $(f_1 \ast f_2)(x) = 0 = (f_1 \cap f_2)(x)$. Given that $f_1$ is a fuzzy right hyperideal of $H$, i.e. $\bigwedge_{a \leq x \circ y} f_1(a) \geq f_1(x)$ also given that $f_2$ is fuzzy left hyperideal of $H$, i.e. $\bigwedge_{a \leq x \circ y} f_2(a) \geq f_2(y)$.

Let $A_x \neq \emptyset$. Then for $x, y, a \in H$, we have
\[
(f_1 \ast f_2)(a) = \bigvee_{a \leq x \circ y} \{ f_1(x) \wedge f_2(y) \} \leq \bigvee_{a \leq x \circ y} \left\{ \bigwedge_{a \leq x \circ y} f_1(a) \wedge \bigwedge_{a \leq x \circ y} f_2(a) \right\}
\]
\[
= (f_1 \cap f_2)(a).
\]
Thus $f_1 \ast f_2 \subseteq f_1 \cap f_2$. \hfill \Box

**Lemma 4.1.** Let $H$ be an ordered LA-semihypergroup with left identity. Then every fuzzy right hyperideal of $H$ is fuzzy left hyperideal of $H$.

**Proof.** Let $H$ be an LA-semihypergroup with pure left identity $e$, and $f$ be a fuzzy right hyperideal of $H$. Since $f$ is a fuzzy right hyperideal of $H$, so $f \ast H \subseteq f$. Thus by Lemma 3.1, and left invertive law, we have
\[
H \ast f = (H \ast H) \ast f = (f \ast H) \ast H \subseteq f \ast H \subseteq f.
\]
Thus $H \ast f \subseteq f$. Hence $f$ is a fuzzy left hyperideal of $H$. \hfill \Box

**Theorem 4.6.** If $f$ is a fuzzy left hyperideal of $H$ with left identity, then $f \cup (f \ast H)$ is a fuzzy hyperideal of $H$.

**Proof.** Let $f$ be a fuzzy left hyperideal of $H$. We have to show that $f \cup (f \ast H)$ is fuzzy hyperideal. Let
\[
(f \cup (f \ast H)) \ast H = (f \ast H) \cup (f \ast H) \ast H
\]
\[
= (f \ast H) \cup (H \ast H) \ast f
\]
\[
= (f \ast H) \cup (H \ast f)
\]
\[
\subseteq (f \ast H) \cup f
\]
\[
= f \cup (f \ast H).
\]
Hence \( f \cup (f \ast H) \) is fuzzy right hyperideal of \( H \). Since every fuzzy right hyperideal of an ordered LA-semihypergroup with left identity is a fuzzy left hyperideal of \( H \), so \( f \cup (f \ast H) \) is a fuzzy hyperideal of \( H \).

\[ \square \]

**Definition 4.6.** A fuzzy hyperideal \( f \) of an ordered LA-semihypergroup \( H \) is called idempotent if

\[ f \ast f = f. \]

**Proposition 4.2.** Every idempotent fuzzy left hyperideal of an ordered LA-semihypergroup \( H \) is a fuzzy hyperideal of \( H \).

**Proof.** Let \( f \) be a fuzzy left hyperideal of \( H \) which is idempotent. Then

\[ f \ast H = (f \ast f) \ast H = (H \ast f) \ast f \subseteq f \ast f = f. \]

Hence \( f \) is a fuzzy right hyperideal of \( H \) and so \( f \) is a fuzzy hyperideal of \( H \).

\[ \square \]

**Proposition 4.3.** If \( f \) is an idempotent fuzzy set in an ordered LA-semihypergroup \( H \) with left identity. Then \( H \ast f \) and \( f \ast H \) are idempotents.

**Proof.** Let \( f \) be an idempotent element in an ordered LA-semihypergroup \( H \) with left identity. Then by using medial law, we have

\[ (H \ast f) \ast (H \ast f) = (H \ast H) \ast (f \ast f) = H \ast f. \]

Thus \( (H \ast f) \ast (H \ast f) = H \ast f \). The case for \( f \ast H \) can be seen in a similar way.

\[ \square \]

5. Conclusion

Fuzzy set theory is a mathematical tools for dealing with uncertainties. This paper is devoted to the discussion of the combinations of fuzzy set in ordered LA-semihypergroup. We combined these concepts to introduce fuzzy left (resp., right) hyperideals and discussed some interesting results.

**References**


