ON THE BEHAVIOR NEAR THE ORIGIN OF A SINE SERIES WITH COEFFICIENTS OF MONOTONE TYPE

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Abstract. In this paper we have obtained some asymptotic equalities of the sum function of a trigonometric sine series expressed in terms of its special type of coefficients.

1. Introduction

Let us consider the sine series

$$
\sum_{m=1}^{\infty} a_m \sin mx
$$

with coefficients tending to zero and such that the sequence $\{a_m\}$ satisfies condition $\Delta a_m = a_m - a_{m+1} \geq 0$ or $\Delta^2 a_m = \Delta a_m - \Delta a_{m+1} \geq 0$ for all $m$. It is a well-known fact that under such conditions the series (1.1) converges for all $x$ (see [12], page 95). We denote by $g(x)$ its sum.

As usually we write $g(u) \sim h(u)$, $u \to 0$ if there exist absolute positive constants $A$ and $B$ such that $Ah(u) \leq g(u) \leq Bh(u)$ is in a neighborhood of the point $u = 0$, and write $g(u) \approx h(u)$ if $\lim_{u \to 0} g(u)/h(u) = 1$. Likewise, throughout this paper the constants in the $O$-expression denote positive absolute constants and they may be different in different relations.

Several authors have investigated the behavior of the sum $g(x)$ near the origin expressed in terms of the coefficients $a_m$. Seemingly, the first was Young [11] who consider this problem, and he was concerned solely about estimates of $|g(x)|$ from above. Then Salem ([3], [4], Theorem 1) proved that if the sequence $\{ma_m\}$ is monotone decreasing, then the following order equality holds

$$
g(x) \sim \sum_{m=1}^{\ell} ma_m x,
$$

where $x \in I_{\ell} := \left(\frac{\pi}{2\ell+1}, \frac{\pi}{2}\right]$, $\ell = 1, 2, \ldots$, $x \to 0$.

Later on, Aljančić, Bojanić and Tomić ([5], Theorem 2) give asymptotic expression for $g(x)$ as $x \to 0$, when the coefficients $a_m$ are convex ($\Delta^2 a_m \geq 0$) and can be represent as the values $A(m)$ of a slowly varying (in Karamata’s sense) function

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A(z), i.e. for each \( t > 0 \)

\[
\lim_{z \to \infty} \frac{A(tz)}{A(z)} = 1. \tag{1.2}
\]

Their result is equivalent to the following statement which can be deduced from one result given by Telyakovskiĭ ([6], Theorem 2) and it is formulated as a corollary in this form:

Corollary 1.1. Suppose that the coefficients \( a_m \) of the series (1.1) are convex and that \( a_m = A(m) \), for a slowly varying function \( A(z) \). Then the following asymptotic equality holds true:

\[ g(x) \approx a_\ell \frac{1}{x}, \quad x \in I_\ell, \quad x \to 0. \]

Telyakovskiĭ deduced this result after the proof, in the same paper, of the following two theorems:

Theorem 1.1. Assume that \( a_m \downarrow 0 \). Then for \( x \in I_\ell \) the following estimate is valid

\[ g(x) = \sum_{m=1}^{\ell} ma_m x + O\left(\frac{1}{\ell^2} \sum_{m=1}^{\ell} m^3 a_m\right). \]

Theorem 1.2. Let \( a_m \to 0 \) and let the sequence \( \{a_m\} \) be convex. If \( x \in I_\ell \), where \( \ell \geq 11 \), then the following estimate holds true

\[ \frac{a_\ell}{2} \cot \frac{x}{2} + \frac{1}{2\ell} \sum_{m=1}^{\ell-1} m^2 \Delta a_m \leq g(x) \leq \frac{a_\ell}{2} \cot \frac{x}{2} + 6 \sum_{m=1}^{\ell-1} m^2 \Delta a_m. \]

Note also that the above theorems as well as some of [1] are generalized and extended in [7]-[10].

For an integer \( k \geq 0 \) and a real sequence \( \{a_m\}_{m=0}^{\infty} \) denote

\[
\Delta_k a_m = \sum_{i=0}^{k} (-1)^i C_k^i a_{m+i} \quad (\Delta_0 a_m = a_m),
\]

\[
\{\Delta\}_k a_m = \sum_{i=0}^{k} C_k^i a_{m+i} \quad (\{\Delta\}_0 a_m = a_m).
\]

Definition 1.1 ([2]). A sequence \( \{a_m\}_{m=0}^{\infty} \) is said to be \((k,s)\)-monotone if \( a_m \to 0 \) as \( m \to \infty \) and \( \Delta_k (\{\Delta\}_s a_m) \geq 0 \), for some \( k \geq 0, s \geq 0 \) and all \( m \).

It is easy to see that that if a sequence \( \{a_m\} \) \( (a_m \to 0 \text{ as } m \to \infty) \) is non-increasing, then it is \((1,s)\)-monotone for all \( s = 0,1,2,\ldots \). The converse statement is not always true. For example, if we consider the sequence \( \{a_m\} \) such that \( a_m \to 0 \) as \( m \to \infty \) and \( a_2m = 0, a_{2m+1} \geq a_{2m+3} \) for \( m = 0,1,2,\ldots \), then this sequence is not non-increasing but it is \((1,1)\)-monotone.

Chronologically this definition arises the following question: What is the behavior near the origin of the series (1.1) with \((k,s)\)-monotone coefficients? The answer to this question is the main goal of this paper. Precisely, we shall answer to this question only considering the cases when the series (1.1) has: \((1,1)\)-monotone, or \((1,2)\)-monotone, or \((2,1)\)-monotone, or \((2,2)\)-monotone coefficients.

For the proof of our results we need the following two lemmas proved in [2].
Lemma 1.1. Let \( \{a_m\}_{m=0}^\infty \) be a sequence such that \( a_m \to 0 \) as \( m \to \infty \) and \( \Delta^k a_m \geq 0 \) for some \( k \geq 1 \) and all \( m \). Then for all \( r = 0, 1, \ldots, k-1 \) and all \( m \) the following inequality holds.

Lemma 1.2. Let \( \{a_m\}_{m=0}^\infty \) be a \((k,s)\)-monotone sequence. If \( k = 1, s = 1 \) or \( s = 2 \), then

\[
g(x) = \frac{a_0}{2} \left( 1 - \tan \frac{x}{2} \right) + \frac{1}{(2 \cos \frac{x}{2})^2} \sum_{m=1}^\infty \{\Delta\} a_{m-1} \sin (m s - 2 + s) \frac{x}{2},
\]

almost everywhere.

Lemma 1.3. Let \( B_m(x) = \sum_{i=0}^m \sin (i-1) \frac{x}{2} \). Then the following estimates hold:

\[
\|B_m(x)\| \leq \frac{2\pi}{x}, \quad 0 < x \leq \pi.
\]

Proof. After some elementary calculation we have

\[
\|B_m(x)\| = \left| \frac{1}{2 \sin \frac{x}{2}} \sum_{i=0}^m \left( \cos (i - 2) \frac{x}{2} - \cos \frac{ix}{2} \right) \right|
\]

\[
= \left| \frac{\cos \frac{x}{2} + \cos x - \cos (m - 1) \frac{x}{2} - \cos \frac{mx}{2}}{2 \sin \frac{x}{2}} \right|
\]

\[
\leq \frac{2}{\sin \frac{x}{2}} \leq \frac{2\pi}{x}, \quad 0 < x \leq \pi.
\]

2. Main Results

The following theorem considers sine series with \((1,1)\)-monotone sequence.

Theorem 2.1. Assume that \( \{a_m\}_{m=1}^\infty \) is a \((1,1)\)-monotone sequence. Then for \( x \in I_\ell \) the following estimate is valid

\[
g(x) = \frac{1}{2 \cos \frac{x}{2}} \left( \frac{1}{2} \sum_{m=1}^\ell m \{\Delta\} a_{m-1} x + O \left( \frac{1}{\ell^3} \sum_{m=1}^\ell m^3 \{\Delta\} a_m \right) \right).
\]

Proof. By the Lemma 1.2 \((a_0 = 0)\) we have

\[
g(x) = \frac{1}{2 \cos \frac{x}{2}} \sum_{m=1}^\infty \{\Delta\} a_{m-1} \sin (m-1) \frac{x}{2}.
\]

Then the use of Abel’s transformation gives

\[
H(x) = \lim_{p \to \infty} \left( \sum_{m=1}^{p-1} \Delta (\{\Delta\} a_{m-1}) B_m(x) + \{\Delta\} a_{p-1} B_p(x) + \{\Delta\} a_0 \sin \frac{x}{2} \right)
\]

\[
= \sum_{m=1}^\infty \Delta (\{\Delta\} a_{m-1}) B_m(x) + \{\Delta\} a_0 \sin \frac{x}{2} := H_{(1)}^{(1)}(x) + H_{(2)}^{(2)}(x),
\]

where

\[
H_{(1)}^{(1)}(x) = \sum_{m=1}^{\ell+1} \Delta (\{\Delta\} a_{m-1}) B_m(x) + \{\Delta\} a_0 \sin \frac{x}{2},
\]
and
\[ H^{(2)}_\ell(x) = \sum_{m=\ell+2}^{\infty} \triangle \{ \triangle \}_{1} a_{m-1} \mathcal{B}_m(x). \]

Let us estimate first \( H^{(1)}_\ell(x) \). Based on Lemma 1.3, our assumption \( \triangle \{ \triangle \}_{1} a_m \geq 0 \) for all \( m \), the well-known relation \( \sin t = t + \mathcal{O}(t^3) \), as \( t \to 0 \), and \( x \in I_\ell \) we have
\[ H^{(1)}_\ell(x) = \sum_{m=0}^{\ell+1} \{ \triangle \}_{1} a_m \mathcal{B}_m(x) + \{ \triangle \}_{1} a_0 \sin \frac{x}{2} \]
\[ = \sum_{m=0}^{\ell} \{ \triangle \}_{1} a_m \mathcal{B}_{m+1}(x) - \{ \triangle \}_{1} a_{\ell+1} \mathcal{B}_{\ell+1}(x) \]
\[ = \frac{1}{2} \sum_{m=1}^{\ell} m \{ \triangle \}_{1} a_m x + \mathcal{O} \left( \frac{1}{\ell^3} \sum_{m=1}^{\ell} m^3 \{ \triangle \}_{1} a_m \right) + \mathcal{O} (\ell \{ \triangle \}_{1} a_\ell). \]

By virtue of monotonicity of \( \{ \triangle \}_{1} a_m \) we obtain
\[ \ell \{ \triangle \}_{1} a_\ell \leq \frac{4}{\ell^3} \left( \frac{\ell (\ell + 1)}{2} \right)^2 \{ \triangle \}_{1} a_\ell \leq \frac{4}{\ell^3} \sum_{m=1}^{\ell} m^3 \{ \triangle \}_{1} a_m. \]

Thus,
\[ H^{(1)}_\ell(x) = \frac{1}{2} \sum_{m=1}^{\ell} m \{ \triangle \}_{1} a_m x + \mathcal{O} \left( \frac{1}{\ell^3} \sum_{m=1}^{\ell} m^3 \{ \triangle \}_{1} a_m \right). \]

Furthermore, since \( x \in I_\ell \) and \( |\mathcal{B}_m(x)| = \mathcal{O} \left( \frac{x}{2} \right) \) by the Lemma 1.2, we notice that
\[ H^{(2)}_\ell(x) = \mathcal{O} \left( \frac{1}{\ell^3} \sum_{m=\ell+2}^{\infty} (\{ \triangle \}_{1} a_{m-1} - \{ \triangle \}_{1} a_m) \right) \]
\[ = \mathcal{O} (\ell + 1) \{ \triangle \}_{1} a_{\ell+1} = \mathcal{O} (\ell \{ \triangle \}_{1} a_\ell) \]
\[ = \mathcal{O} \left( \frac{1}{\ell^3} \sum_{m=1}^{\ell} m^3 \{ \triangle \}_{1} a_m \right). \]

Finally, relations (2.2)-(2.5) prove completely estimation (2.1). \( \square \)

**Corollary 2.1.** Let \( \{ a_m \}_{m=1}^{\infty} \) be a \((1,1)\)-monotone sequence and the series
\[ \sum_{m=1}^{\infty} m (a_m + a_{m+1}) \]
converges. Then the following asymptotic equality
\[ \lim_{x \to 0} \frac{g(x)}{x} = \frac{1}{4} \sum_{m=1}^{\infty} m (a_m + a_{m+1}) \]
holds true.
Proof. In accordance with Theorem 2.1 it is enough to prove that

$$\frac{1}{\ell^2} \sum_{m=1}^{\ell} m^3 \{\triangle\}_1 a_m \to 0, \; \text{as} \; \ell \to \infty.$$ 

Indeed, for an arbitrary natural number \(M\) we can write

$$\frac{1}{\ell^2} \sum_{m=1}^{\ell} m^3 \{\triangle\}_1 a_m \leq \frac{1}{\ell^2} \sum_{m=1}^{M} m^3 \{\triangle\}_1 a_m + \sum_{m=M+1}^{\infty} m \{\triangle\}_1 a_m.$$ 

If a number \(\varepsilon > 0\) be chosen, then by hypothesis a number \(M = M(\varepsilon)\) exists, such that

$$\sum_{m=M+1}^{\infty} m \{\triangle\}_1 a_m < \frac{\varepsilon}{2}.$$ 

Likewise, for all sufficiently large \(\ell\)

$$\frac{1}{\ell^2} \sum_{m=1}^{M} m^3 \{\triangle\}_1 a_m < \frac{\varepsilon}{2}.$$ 

Then obviously, for such \(\ell\) we have

$$\frac{1}{\ell^2} \sum_{m=1}^{\ell} m^3 \{\triangle\}_1 a_m < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

\[ \Box \]

The following statements can be proved similarly therefore we will skip their proofs.

**Theorem 2.2.** Assume that \(\{a_m\}_{m=1}^{\infty}\) is a \((1,2)\)-monotone sequence. Then for \(x \in \mathcal{I}_\ell\) the following estimate is valid

$$g(x) = \frac{1}{(2 \cos \frac{\pi}{4})^2} \left\{ \sum_{m=0}^{\ell} (m+1) \{\triangle\}_2 a_m x + O \left( \frac{1}{\ell^2} \sum_{m=0}^{\ell} (m+1)^3 \{\triangle\}_2 a_m \right) \right\}.$$ 

**Corollary 2.2.** Suppose that \(\{a_m\}_{m=1}^{\infty}\) is a \((1,2)\)-monotone sequence and the series

$$\sum_{m=0}^{\infty} (m+1) (a_m + 2a_{m+1} + a_{m+2})$$

converges. Then the following asymptotic equality

$$\lim_{x \to 0} \frac{g(x)}{x} = \frac{1}{4} \sum_{m=0}^{\infty} (m+1) (a_m + 2a_{m+1} + a_{m+2})$$

holds true.

The proof of the next statement is more complicated and that is why we will sketch it in more details.
Theorem 2.3. Assume that \( \{a_m\}_{m=1}^{\infty} \) is a \((2,2)\)-monotone sequence. Then for \( x \in I_{\ell}, \ell \geq 11 \) the following estimate is valid

\[
\frac{\{\triangle\} a_{m-1}}{2} \cot \frac{x}{2} + \frac{x}{2} \sum_{m=1}^{\ell-1} m^2 \triangle (\{\triangle\} a_{m-1}) \leq g(x) \left( 2 \cos \frac{x}{2} \right)^2 \leq \frac{\{\triangle\} a_{m-1}}{2} \cot \frac{x}{2} + \frac{6}{\ell} \sum_{m=1}^{\ell-1} m^2 \triangle (\{\triangle\} a_{m-1}).
\]

Proof. By the Lemma 1.1 the condition \( \triangle_2 (\{\triangle\} a_m) \geq 0 \) implies \( \triangle (\{\triangle\} a_m) \geq 0 \). Therefore by the Lemma 1.2 we have

\[
g(x) = \frac{1}{(2 \cos \frac{x}{2})^2} \sum_{m=1}^{\infty} \{\triangle\} a_{m-1} \sin mx.
\]

Applying Abel’s transformation we obtain

\[
(2.6) \quad g(x) = \frac{1}{(2 \cos \frac{x}{2})^2} \sum_{m=1}^{\infty} \triangle (\{\triangle\} a_{m-1}) \tilde{D}_m(x),
\]

where \( \tilde{D}_m(x) = \sum_{i=1}^{m} \sin ix \) is the conjugate Dirichlet kernel.

For \( x \in (0, \pi] \) and \( m = 0, 1, 2, \ldots \), introduce the functions

\[
\varphi_m(x) := \frac{-\cos (m+1/2)x}{2 \sin x/2}
\]

and

\[
\psi_m(x) := \sum_{i=0}^{m} \varphi_i(x) = \frac{-\sin (m+1)x}{4 \sin^2(x/2)}.
\]

Denoting \( H(x) := \sum_{m=1}^{\infty} \triangle (\{\triangle\} a_{m-1}) \tilde{D}_m(x) \) one can write

\[
H(x) = \sum_{m=1}^{\ell-1} \triangle (\{\triangle\} a_{m-1}) \tilde{D}_m(x) + \sum_{m=\ell}^{\infty} \triangle (\{\triangle\} a_{m-1}) \left( \frac{1}{2} \cot \frac{x}{2} + \varphi_m(x) \right)
\]

\[
= \frac{\{\triangle\} a_{\ell-1}}{2} \cot \frac{x}{2} + \sum_{m=1}^{\ell-1} \triangle (\{\triangle\} a_{m-1}) \tilde{D}_m(x) + \sum_{m=\ell}^{\infty} \triangle (\{\triangle\} a_{m-1}) \varphi_m(x)
\]

\[
(2.7) \quad = \frac{a_{\ell-1} + 2a_\ell + a_{\ell+1}}{2} \cot \frac{x}{2} + E_\ell(x) + F_\ell(x).
\]

We shall make use of the representation (2.7) for \( x \in I_{\ell} \), and from now and till the end of the proof of our theorem we suppose that \( x \in I_{\ell} \) but we shall not remind of it.

The following estimate is true in view of the monotonous decay of \( \triangle (\{\triangle\} a_{m-1}) \) and the positivity of \( \tilde{D}_m(x) \) for \( m \leq \ell \):

\[
E_\ell(x) \geq \triangle (\{\triangle\} a_{\ell-1}) \sum_{m=1}^{\ell-1} \left( \frac{1}{2} \cot \frac{x}{2} + \varphi_m(x) \right)
\]

\[
(2.8) \quad = \triangle (\{\triangle\} a_{\ell-1}) \left( \frac{\ell}{2} \cot \frac{x}{2} + \psi_{\ell-1}(x) \right) = \frac{\triangle (\{\triangle\} a_{\ell-1})}{4 \sin^2(x/2)} (\ell \sin x - \sin \ell x).
\]
Let us estimate $F_\ell(x)$ from above. Applying Abel’s transformation we have

$$|F_\ell(x)| = \lim_{n \to \infty} \left| \sum_{m=\ell}^{n-1} \Delta_2 \left( \{\triangle\} a_{m-1} \right) \psi_m(x) + \Delta \left( \{\triangle\} a_{n-1} \right) \psi_n(x) - \Delta \left( \{\triangle\} a_{\ell-1} \right) \psi_{\ell-1}(x) \right|$$

$$\leq \sum_{m=\ell}^{\infty} \Delta_2 \left( \{\triangle\} a_{m-1} \right) |\psi_m(x) - \psi_{\ell-1}(x)|$$

(2.9)

From (2.8) and (2.9), in a similar way as Telyakovski˘ı did [6], for $\ell \geq 11$ we can show that

$$\frac{1}{2} \Delta \left( \{\triangle\} a_{\ell-1} \right) (1 + \sin \ell x).$$

Further, if $m < \ell$, then

$$\tilde{D}_m(x) \geq \sum_{i=1}^{m} \frac{2}{\pi} i x \geq \frac{m(m+1)}{\ell+1} > \frac{m^2}{\ell}.$$

Therefore,

(2.10) \hspace{1cm} \frac{1}{2} E_\ell(x) \geq \frac{1}{2\ell} \sum_{m=1}^{\ell-1} m^2 \Delta \left( \{\triangle\} a_{m-1} \right).$$

From (2.10), (2.7), and (2.6) we obtain the estimate of $g(x)$ from below

$$g(x) \geq \frac{1}{(2 \cos \frac{x}{2})^2} \left( a_{\ell-1} + 2a_{\ell} + a_{\ell+1} \cot \frac{x}{2} + \frac{1}{2\ell} \sum_{m=1}^{\ell-1} m^2 \Delta \left( \{\triangle\} a_{m-1} \right) \right).$$

Since

$$\tilde{D}_m(x) \leq m^2 x \leq \frac{\pi m^2}{\ell},$$

then

(2.11) \hspace{1cm} E_\ell(x) \leq \frac{\pi}{\ell} \sum_{m=1}^{\ell-1} m^2 \Delta \left( \{\triangle\} a_{m-1} \right).$$

For the estimate (2.9) we can write

$$|F_\ell(x)| \leq \frac{\Delta \left( \{\triangle\} a_{\ell-1} \right)}{2 \sin^2(x/2)} \leq \Delta \left( \{\triangle\} a_{\ell-1} \right) \frac{\pi^2}{2x^2} \leq \frac{(\ell+1)^2}{2} \Delta \left( \{\triangle\} a_{\ell-1} \right),$$

and for $\ell \geq 11$

$$\frac{(\ell+1)^2}{2} < \frac{2.4}{\ell} \sum_{m=1}^{\ell-1} m^2,$$

hence, by reason of the monotonicity of $\Delta \left( \{\triangle\} a_{\ell-1} \right)$ we get

(2.12) \hspace{1cm} |F_\ell(x)| \leq \frac{2.4}{\ell} \sum_{m=1}^{\ell-1} m^2 \Delta \left( \{\triangle\} a_{m-1} \right).
Estimates (2.12), (2.13), and (2.7) give the estimate of $g(x)$ from above

$$g(x) \leq \frac{1}{\left(2 \cos \frac{x}{2}\right)^2} \left(\frac{a_{\ell-1} + 2a_\ell + a_{\ell+1}}{2} \cot \frac{x}{2} + \frac{6}{\ell} \sum_{m=1}^{\ell-1} m^2 \Delta \left(\{\Delta\} a_{m-1}\right)\right).$$

The proof is completed. \(\square\)

It follows from Theorem 2.3 that for $x \in I_{\ell}$ in a sufficiently small neighbourhood of the origin we have

$$(2.13) g(x) = \frac{1}{2} \left(1 + \cos \frac{x}{2}\right) \left(\frac{1}{2} \sum_{m=1}^{\ell-1} m^2 \Delta \left(\{\Delta\} a_{m-1}\right)\right).$$

**Corollary 2.3.** Assume that $\{a_m\}_{m=1}^{\infty}$ is a $(2, 2)$-monotone sequence. Then the following order equality is true

$$g(x) \sim (\ell - 1)\{\Delta\} a_{\ell-1} + \frac{1}{\ell} \sum_{m=1}^{\ell-1} m\{\Delta\} a_{m-1}. $$

**Proof.** Since $\lim_{x \to 0} x \cot x = 1$, then it is enough to prove that

$$\frac{1}{\ell} \sum_{m=1}^{\ell-1} (2m - 1)\{\Delta\} a_{m-1} - (\ell - 1)\{\Delta\} a_{\ell-1} \leq \frac{1}{\ell} \sum_{m=1}^{\ell-1} m^2 \Delta \left(\{\Delta\} a_{m-1}\right)$$

and

$$\frac{1}{\ell} \sum_{m=1}^{\ell-1} m^2 \Delta \left(\{\Delta\} a_{m-1}\right) \leq \frac{1}{\ell} \sum_{m=1}^{\ell-1} (2m - 1)\{\Delta\} a_{m-1}. $$

Indeed, putting $\{\Delta\} a_{m-1} := b_{m-1}$, we can write

$$\frac{1}{\ell} \sum_{m=1}^{\ell-1} m^2 \Delta b_{m-1} = \frac{1}{\ell} \left[b_0 + 3b_1 + 5b_2 + \cdots + (2\ell - 3)b_{\ell-2} - (\ell - 1)^2 b_{\ell-1}\right]$$

(2.14)

$$\leq \frac{1}{\ell} \sum_{m=1}^{\ell-1} (2m - 1)b_{m-1} \leq \frac{1}{\ell} \sum_{m=1}^{\ell-1} (2m - 1)\{\Delta\} a_{m-1},$$

because by the Lemma 1.1, $b_{m-1} \geq 0$ holds true.

On the other hand we get

$$(\ell - 1)^2 b_{\ell-1} \leq \ell(\ell - 1)b_{\ell-1},$$

therefore the proof of the corollary is completed. \(\square\)

**Remark 2.1.** Similar statement with Theorem 2.3 holds true for the series (1.1) with $(2, 1)$-monotone coefficients.

**References**


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