COEFFICIENT ESTIMATES OF MEROMORPHIC BI-STARLIKE FUNCTIONS OF COMPLEX ORDER

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Abstract. In the present investigation, we define a new subclass of meromorphic bi-univalent functions class $\Sigma'$ of complex order $\gamma \in \mathbb{C} \setminus \{0\}$, and obtain the estimates for the coefficients $|b_0|$ and $|b_1|$. Further we pointed out several new or known consequences of our result.

1. Introduction and Definitions

Denote by $\mathcal{A}$ the class of analytic functions of the form

$$ f(z) = z + \sum_{n=2}^{\infty} a_n z^n $$

which are univalent in the open unit disc $\Delta = \{z : |z| < 1\}$. Also denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions $f(0) = 0 = f'(0) - 1$ in $\Delta$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ includes the class $\mathcal{S}^*(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order $\alpha$ in $\Delta$ and the class $\mathcal{K}(\alpha)(0 \leq \alpha < 1)$ of convex functions of order $\alpha$

$$ \Re \left( \frac{z f'(z)}{f(z)} \right) > \alpha \quad \text{or} \quad \Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) > \alpha, (z \in \Delta) $$

respectively. Further a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\gamma)$ of univalent function of complex order $\gamma (\gamma \in \mathbb{C} \setminus \{0\})$ if and only if

$$ \frac{f(z)}{z} \neq 0 \quad \text{and} \quad \Re \left( 1 + \frac{1}{\gamma} \left[ \frac{zf''(z)}{f'(z)} - 1 \right] \right) > 0, z \in \Delta. $$

By taking $\gamma = (1 - \alpha)\cos \beta \ e^{-i\beta}$, $|\beta| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, the class $\mathcal{S}((1 - \alpha)\cos \beta \ e^{-i\beta}) \equiv \mathcal{S}(\alpha, \beta)$ called the generalized class of $\beta$-spiral-like functions of order $\alpha (0 \leq \alpha < 1)$.

An analytic function $\varphi$ is subordinate to an analytic function $\psi$, written by

$$ \varphi(z) \prec \psi(z), $$

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provided there is an analytic function $\omega$ defined on $\Delta$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1$$

satisfying

$$\varphi(z) = \psi(\omega(z)).$$

Ma and Minda [9] unified various subclasses of starlike and convex functions for which either of the quantity

$$zf'(z) f(z) \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function $\phi$ with positive real part in the unit disk $\Delta$, $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi$ maps $\Delta$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

The class of Ma-Minda starlike functions consists of functions $f \in A$ satisfying the subordination

$$zf'(z) f(z) \prec \phi(z).$$

Similarly, the class of Ma-Minda convex functions consists of functions $f \in A$ satisfying the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z).$$

It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z, \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \quad (1.2)$$

A function $f \in A$ given by (1.1), is said to be bi-univalent in $\Delta$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\Delta$, these classes are denoted by $\Sigma$. Earlier, Brannan and Taha [2] introduced certain subclasses of bi-univalent function class $\Sigma$, namely bi-starlike functions $S^*_\Sigma(\alpha)$ and bi-convex function $K_{\Sigma}(\alpha)$ of order $\alpha$ corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$ respectively. For each of the function classes $S^*_\Sigma(\alpha)$ and $K_{\Sigma}(\alpha)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found [2, 17]. But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} := \{1, 2, 3, \cdots\})$$

is still an open problem (see [1, 2, 8, 10, 17]). Recently several interesting subclasses of the bi-univalent function class $\Sigma$ have been introduced and studied in the literature (see [15, 18, 19]).

A function $f$ is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both $f$ and $f^{-1}$ are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $S^*_\Sigma(\phi)$ and $K_{\Sigma}(\phi)$. In the sequel, it is assumed that $\phi$ is an analytic function with positive real part in the unit disk $\Delta$, satisfying
\(\phi(0) = 1, \phi'(0) > 0\) and \(\phi(\Delta)\) is symmetric with respect to the real axis. Such a function has a series expansion of the form
\[
\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0).
\]

Let \(\Sigma'\) denote the class of meromorphic univalent functions \(g\) of the form
\[
g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}
\]
defined on the domain \(\Delta^* = \{z : 1 < |z| < \infty\}\). Since \(g \in \Sigma'\) is univalent, it has an inverse \(g^{-1} = h\) that satisfies
\[
g^{-1}(g(z)) = z, \quad (z \in \Delta^*)
\]
and
\[
g(g^{-1}(w)) = w, \quad (M < |w| < \infty, M > 0)
\]
where
\[
g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{C_n}{wn^1}, \quad (M < |w| < \infty).
\]

Analogous to the bi-univalent analytic functions, a function \(g \in \Sigma'\) is said to be meromorphic bi-univalent if \(g^{-1} \in \Sigma'\). We denote the class of all meromorphic bi-univalent functions by \(M_{\Sigma'}\). Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature, for example, Schiffer [13] obtained the estimate \(|b_2| \leq \frac{2}{3}\) for meromorphic univalent functions \(g \in \Sigma'\) with \(b_0 = 0\) and Duren [3] gave an elementary proof of the inequality \(|b_n| \leq \frac{2}{(n+1)^2}\) on the coefficient of meromorphic univalent functions \(g \in \Sigma'\) with \(b_k = 0\) for \(1 \leq k < \frac{n}{2}\).

For the coefficient of the inverse of meromorphic univalent functions \(h \in M_{\Sigma'}\), Springer [14] proved that \(|C_3| \leq 1\) and \(|C_3 + \frac{1}{2} C_1^2| \leq \frac{1}{2}\) and conjectured that \(|C_{2n-1}| \leq \frac{(2n-1)^2}{n(n-1)^2}\), \((n = 1, 2, \ldots)\).

In 1977, Kubota [7] has proved that the Springer conjecture is true for \(n = 3, 4, 5\) and subsequently Schober [12] obtained a sharp bounds for the coefficients \(C_{2n-1}, 1 \leq n \leq 7\) of the inverse of meromorphic univalent functions in \(\Delta^*\). Recently, Kapoor and Mishra [6] (see [16]) found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order \(\alpha\) in \(\Delta^*\).

Motivated by the earlier work of [4, 5, 6, 20], in the present investigation, a new subclass of meromorphic bi-univalent functions class \(\Sigma'\) of complex order \(\gamma \in \mathbb{C}\setminus\{0\}\), is introduced and estimates for the coefficients \(|b_0|\) and \(|b_1|\) of functions in the newly introduced subclass are obtained. Several new consequences of the results are also pointed out.

**Definition 1.1.** For \(0 \leq \lambda \leq 1, \mu \geq 0, \mu > \lambda\) a function \(g(z) \in \Sigma'\) given by (1.4) is said to be in the class \(M_{\Sigma'}(\lambda, \mu, \phi)\) if the following conditions are satisfied:
\[
\frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right)^\mu + \lambda g'(z) \left( \frac{g(z)}{z} \right)^{\mu-1} - 1 \right] \prec \phi(z)
\]
and
\[
\frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right)^\mu + \lambda h'(w) \left( \frac{h(w)}{w} \right)^{\mu-1} - 1 \right] \prec \phi(w)
\]
where \(z, w \in \Delta^*, \gamma \in \mathbb{C}\setminus\{0\}\) and the function \(h\) is given by (1.5).
By suitably specializing the parameters $\lambda$ and $\mu$, we state the new subclasses of the class meromorphic bi-univalent functions of complex order $M^\gamma_{\Sigma'}(\lambda, \mu, \phi)$ as illustrated in the following Examples.

**Example 1.1.** For $0 \leq \lambda < 1, \mu = 1$ a function $g \in \Sigma'$ given by (1.4) is said to be in the class $M^\gamma_{\Sigma'}(\lambda, 1, \phi) \equiv F^\gamma_{\Sigma'}(\lambda, \phi)$ if it satisfies the following conditions respectively:

$$1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right) + \lambda g'(z) - 1 \right] < \phi(z)$$

and

$$1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right) + \lambda h'(w) - 1 \right] < \phi(w)$$

where $z, w \in \Delta^*$, $\gamma \in \mathbb{C}\{0\}$ and the function $h$ is given by (1.5).

**Example 1.2.** For $\lambda = 1, 0 \leq \mu < 1$ a function $g \in \Sigma'$ given by (1.4) is said to be in the class $M^\gamma_{\Sigma'}(1, \mu, \phi) \equiv B^\gamma_{\Sigma'}(\mu, \phi)$ if it satisfies the following conditions respectively:

$$1 + \frac{1}{\gamma} \left[ g'(z) \left( \frac{g(z)}{z} \right)^{\mu-1} - 1 \right] < \phi(z)$$

and

$$1 + \frac{1}{\gamma} \left[ h'(w) \left( \frac{h(w)}{w} \right)^{\mu-1} - 1 \right] < \phi(w)$$

where $z, w \in \Delta^*$, $\gamma \in \mathbb{C}\{0\}$ and the function $h$ is given by (1.5).

**Example 1.3.** For $\lambda = 1, \mu = 0$, a function $g \in \Sigma'$ given by (1.4) is said to be in the class $M^\gamma_{\Sigma'}(1, 0, \phi) \equiv S^\gamma_{\Sigma'}(\phi)$ if it satisfies the following conditions respectively:

$$1 + \frac{1}{\gamma} \left( \frac{z g'(z)}{g(z)} - 1 \right) < \phi(z)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{w h'(w)}{h(w)} - 1 \right) < \phi(w)$$

where $z, w \in \Delta^*$, $\gamma \in \mathbb{C}\{0\}$ and the function $h$ is given by (1.5).

2. **Coefficient estimates for the function class $M^\gamma_{\Sigma'}(\lambda, \mu, \phi)$**

In this section we obtain the coefficients $|b_0|$ and $|b_1|$ for $g \in M^\gamma_{\Sigma'}(\lambda, \mu, \phi)$ associating the given functions with the functions having positive real part. In order to prove our result we recall the following lemma.

**Lemma 2.1.** [11] If $\Phi \in \mathcal{P}$, the class of all functions with $\Re(\Phi(z)) > 0, (z \in \Delta)$ then

$$|c_k| \leq 2, \text{ for each } k,$$

where

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \text{ for } z \in \Delta.$$
Define the functions $p$ and $q$ in $\mathcal{P}$ given by

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots$$

and

$$q(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + \frac{q_1}{z} + \frac{q_2}{z^2} + \cdots.$$  

It follows that

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[ \frac{p_1}{z} + \left( \frac{p_2}{2} - \frac{p_1^2}{2} \right) \frac{1}{z^2} + \cdots \right]$$

and

$$v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ \frac{q_1}{z} + \left( \frac{q_2}{2} - \frac{q_1^2}{2} \right) \frac{1}{z^2} + \cdots \right].$$

Note that for the functions $p(z), q(z) \in \mathcal{P}$, we have $|p_i| \leq 2$ and $|q_i| \leq 2$ for each $i$.

**Theorem 2.1.** Let $g$ be given by (1.4) be in the class $\mathcal{M}_{\gamma}^{\mathcal{L}}(\lambda, \mu, \phi)$. Then

(2.1)  
$$|b_0| \leq \left| \frac{\gamma B_1}{\mu - \lambda} \right|$$

and

(2.2)  
$$|b_1| \leq \left| \gamma \sqrt{\frac{(\mu - 1)\gamma B_1^2}{2(\mu - \lambda)^2} + \left( \frac{B_2}{\mu - 2\lambda} \right)^2} \right|$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda \leq 1, \mu \geq 0, \mu > \lambda$ and $z, w \in \Delta^*$. 

**Proof.** It follows from (1.6) and (1.7) that

(2.3)  
$$1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right)^\mu + \lambda g'(z) \left( \frac{g(z)}{z} \right)^{\mu - 1} - 1 \right] = \phi(u(z))$$

and

(2.4)  
$$1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right)^\mu + \lambda h'(w) \left( \frac{h(w)}{w} \right)^{\mu - 1} - 1 \right] = \phi(v(w)).$$

In light of (1.4), (1.5), (1.6) and (1.7), we have

(2.5)  
$$1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right)^\mu + \lambda g'(z) \left( \frac{g(z)}{z} \right)^{\mu - 1} - 1 \right]
= 1 + \frac{1}{\gamma} \left[ (\mu - \lambda) \frac{b_0}{z} + (\mu - 2\lambda) \frac{(\mu - 1)b_0}{2} + b_1 \frac{1}{z^2} + \cdots \right]
= 1 + B_1 p_1 \frac{1}{2z} + \left[ \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2 p_1^2 \right] \frac{1}{z^2} + \cdots$$
and

\[(2.6) \quad 1 + \frac{1}{\gamma} \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right)^{\mu} + \lambda h'(w) \left( \frac{h(w)}{w} \right)^{\mu-1} \right] = 1 + \frac{1}{\gamma} \left[ -(\mu - \lambda) \frac{b_0}{z} + (\mu - 2\lambda) \left( \frac{b_0}{2} - b_1 \frac{1}{2z^2} + \ldots \right) \right] = 1 + B_1 q_1 \frac{1}{2w} + \left[ \frac{1}{2} B_1 (q_2 - \frac{q_1^2}{2}) + \frac{1}{4} B_2 q_1^2 \right] \frac{1}{w^2} + \ldots \]

Now, equating the coefficients in (2.5) and (2.6), we get

\[(2.7) \quad \frac{1}{\gamma} (\mu - \lambda) b_0 = \frac{1}{2} B_1 p_1, \]

\[(2.8) \quad \frac{1}{\gamma} (\mu - 2\lambda) \left( \mu - 1 \right) \frac{b_0^2}{2} + b_1 = \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2 p_1^2, \]

\[(2.9) \quad -\frac{1}{\gamma} (\mu - \lambda) b_0 = \frac{1}{2} B_1 q_1, \]

and

\[(2.10) \quad \frac{1}{\gamma} (\mu - 2\lambda) \left( \mu - 1 \right) \frac{b_0^2}{2} - b_1 = \frac{1}{2} B_1 (q_2 - \frac{q_1^2}{2}) + \frac{1}{4} B_2 q_1^2. \]

From (2.7) and (2.9), we get

\[(2.11) \quad p_1 = -q_1 \]

and

\[8(\mu - \lambda)^2 b_0^2 = \gamma^2 B_1^2 (p_1^2 + q_1^2). \]

Hence,

\[(2.12) \quad b_0^2 = \frac{\gamma^2 B_1^2 (p_1^2 + q_1^2)}{8(\mu - \lambda)^2}. \]

Applying Lemma (2.1) for the coefficients $p_1$ and $q_1$, we have

\[|b_0| \leq \left| \frac{\gamma B_1}{\mu - \lambda} \right|. \]

Next, in order to find the bound on $|b_1|$ from (2.8), (2.10) and (2.11), we obtain

\[(2.13) \quad (\mu - 2\lambda)^2 b_1^2 = (\mu - 2\lambda)^2 (\mu - 1)^2 \frac{b_0^4}{4} - \gamma^2 \left( \frac{B_1^2}{4} p_2 q_2 + (B_2 - B_1)(p_2 + q_2) \frac{p_1^2}{8} + (B_1 - B_2)^2 \frac{p_1^4}{16} \right). \]

Using (2.12) and applying Lemma (2.1) once again for the coefficients $p_1, p_2$ and $q_2$, we get

\[|b_1| \leq \frac{\gamma \left( \frac{((\mu - 1) \gamma B_1^2)^2}{2(\mu - \lambda)^2} + \frac{B_2}{\mu - 2\lambda} \right)^2}{2}. \]
Corollary 2.1. Let \( g(z) \) is given by (1.4) be in the class \( \mathcal{F}_\gamma^\prime(\lambda, \phi) \). Then
\[ (2.14) \quad |b_0| \leq \left| \frac{\gamma B_1}{1 - \lambda} \right| \]
and
\[ (2.15) \quad |b_1| \leq \left| \frac{\gamma B_2}{2\lambda - 1} \right| \]
where \( \gamma \in \mathbb{C}\setminus\{0\} \), \( 0 < \lambda < 1 \) and \( z, w \in \Delta^* \).

Corollary 2.2. Let \( g(z) \) is given by (1.4) be in the class \( \mathcal{B}_\gamma^\prime(\mu, \phi) \). Then
\[ (2.16) \quad |b_0| \leq \left| \frac{\gamma B_1}{\mu - 1} \right| \]
and
\[ (2.17) \quad |b_1| \leq \left| \gamma \sqrt{\left( \frac{\gamma B_1^2}{2(\mu - 1)} \right)^2 + \left( \frac{B_2}{\mu - 2} \right)^2} \right| \]
where \( \gamma \in \mathbb{C}\setminus\{0\} \), \( 0 < \mu < 1 \) and \( z, w \in \Delta^* \).

Corollary 2.3. Let \( g(z) \) is given by (1.4) be in the class \( \mathcal{S}_\gamma^\prime(\phi) \). Then
\[ (2.18) \quad |b_0| \leq |\gamma B_1| \]
and
\[ (2.19) \quad |b_1| \leq \left| \frac{\gamma}{2} \sqrt{\gamma^2 B_1^4 + B_2^2} \right| \]
where \( \gamma \in \mathbb{C}\setminus\{0\} \) and \( z, w \in \Delta^* \).

3. Corollaries and concluding Remarks

Analogous to (1.3), by setting \( \phi(z) \) as given below:
\[ (3.1) \quad \phi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha = 1 + 2az + 2a^2z^2 + \cdots \quad (0 < \alpha \leq 1), \]
we have
\[ B_1 = 2\alpha, \quad B_2 = 2\alpha^2. \]

For \( \gamma = 1 \) and \( \phi(z) \) is given by (3.1) we state the following corollaries:

Corollary 3.1. Let \( g \) is given by (1.4) be in the class \( \mathcal{M}_\Sigma^\prime(\lambda, \mu, \left( \frac{1+z}{1-z} \right)^\alpha) \equiv \mathcal{M}_\Sigma(\lambda, \alpha) \). Then
\[ (2.18) \quad |b_0| \leq \left| \frac{2\alpha}{\mu - \lambda} \right| \]
and
\[ (2.19) \quad |b_1| \leq \left| 2\alpha^2 \sqrt{\frac{(\mu - 1)^2}{(\mu - 2l)^2} + \frac{1}{(\mu - 2\lambda)^2}} \right| \]
where \( 0 < \lambda \leq 1, \mu \geq 0, \mu > \lambda \) and \( z, w \in \Delta^* \).
Corollary 3.2. Let \( g(z) \) be given by (1.4) be in the class \( F_{\Sigma}^1(\lambda, \left( \frac{1+z}{1-z} \right)^\alpha) \equiv F_{\Sigma}(\lambda, \alpha) \), then
\[
|b_0| \leq \frac{2\alpha}{|1-\lambda|}
\]
and
\[
|b_1| \leq \frac{2\alpha^2}{|1-2\lambda|}
\]
where \( 0 \leq \lambda < 1 \) and \( z, w \in \Delta^* \).

Corollary 3.3. Let \( g(z) \) be given by (1.4) be in the class \( B_{\Sigma}^1(\lambda, \left( \frac{1+z}{1-z} \right)^\alpha) \equiv B_{\Sigma}(\mu, \alpha) \), then
\[
|b_0| \leq \frac{2\alpha}{|\mu-1|}
\]
and
\[
|b_1| \leq \left| \frac{2\alpha^2}{(\mu-1)^2} + \frac{1}{(\mu-2)^2} \right|
\]
where \( 0 \leq \mu < 1 \) and \( z, w \in \Delta^* \).

Corollary 3.4. Let \( g(z) \) be given by (1.4) be in the class \( S_{\Sigma}^1(\left( \frac{1+z}{1-z} \right)^\alpha) \equiv S_{\Sigma}^1(\alpha) \), then
\[
|b_0| \leq 2\alpha
\]
and
\[
|b_1| \leq \alpha^2 \sqrt{5}
\]
where \( z, w \in \Delta^* \).

On the other hand if we take
\[
(3.2) \quad \phi(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1),
\]
then
\[
B_1 = B_2 = 2(1-\beta).
\]

For \( \gamma = 1 \) and \( \phi(z) \) is given by (3.2) we state the following corollarys:

Corollary 3.5. Let \( g \) be given by (1.4) be in the class \( M_{\Sigma}^1(\lambda, \mu, \left( \frac{1+(1-2\beta)z}{1-z} \right)^\alpha) \equiv M_{\Sigma}(\lambda, \mu, \beta) \). Then
\[
|b_0| \leq \frac{2(1-\beta)}{|\mu-\lambda|}
\]
and
\[
|b_1| \leq \left| 2(1-\beta) \sqrt{\frac{(\mu-1)^2(1-\beta)^2}{(\mu-\lambda)^2} + \frac{1}{(\mu-2\lambda)^2}} \right|
\]
where \( 0 \leq \lambda \leq 1, \mu \geq 0, \mu > \lambda \) and \( z, w \in \Delta^* \).

Remark 3.1. We obtain the estimates \( |b_0| \) and \( |b_1| \) as obtained in the Corollaries 3.2 to 3.4 for function \( g \) given by (1.4) are in the subclasses defined in Examples 1.1 to 1.3.
Concluding Remarks: Let a function $g \in \Sigma'$ given by (1.4). By taking $\gamma = (1 - \alpha) \cos \beta e^{-i\beta}$, $|\beta| < \frac{\pi}{2}$, $0 \leq \alpha < 1$ the class $M_{\gamma}^{\alpha}(\lambda, \mu, \phi) \equiv M_{\beta}^{\alpha}(\alpha, \lambda, \mu, \phi)$ called the generalized class of $\beta$-bi spiral-like functions of order $\alpha(0 \leq \alpha < 1)$ satisfying the following conditions.

\[
\begin{align*}
\exp^{i\beta} & \left[ (1 - \lambda) \left( \frac{g(z)}{z} \right)^{\mu} + \lambda g'(z) \left( \frac{g(z)}{z} \right)^{\mu - 1} \right] \prec \left[ \phi(z)(1 - \alpha) + \alpha \cos \beta + i \sin \beta \right] \\
\exp^{i\beta} & \left[ (1 - \lambda) \left( \frac{h(w)}{w} \right)^{\mu} + \lambda h'(w) \left( \frac{h(w)}{w} \right)^{\mu - 1} \right] \prec \left[ \phi(w)(1 - \alpha) + \alpha \cos \beta + i \sin \beta \right]
\end{align*}
\]

where $0 \leq \lambda \leq 1$, $\mu \geq 0$ and $z, w \in \Delta^*$ and the function $h$ is given by (1.5).

For function $g \in M_{\gamma}^{\alpha}(\alpha, \lambda, \mu, \phi)$ given by (1.4), by choosing $\phi(z) = \left( \frac{1 + z}{1 - z} \right)$, (or $\phi(z) = \frac{1 + A_1 + B z}{1 - A_2 + B z}$, $-1 \leq B < A \leq 1$), we obtain the estimates $|b_0|$ and $|b_1|$ by routine procedure (as in Theorem 2.1) and so we omit the details.

References


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