SOME PROPERTIES OF GEODESIC STRONGLY E-B-VEX FUNCTIONS

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Abstract. Geodesic E-b-vex sets and geodesic E-b-vex functions on a Riemannian manifold are extended to the so called geodesic strongly E-b-vex sets and geodesic strongly E-b-vex functions. Some basic properties of geodesic strongly E-b-vex sets are also studied.

1. Introduction

Convexity and its generalizations play an important role in optimization theory, convex analysis and Minkowski space [3, 4, 6, 9, 10].

Youness [17] defined E-convex sets and E-convex functions by relaxing the definitions of convex sets and convex functions, which have some important applications in various branches of mathematical sciences [1, 12, 13]. Also, Youness [18] extended the definitions of E-convex sets and E-convex functions to strongly E-convex sets and strongly E-convex functions. The B-vex functions which shares many properties with convex functions was introduced by Bector and Singh [2]. Some researchers studied some new generalizations of convex functions by relaxing definitions of E-convex functions and B-vex functions such as E-B-vex functions [15] and strongly E-B-vex functions [19]. Also, generalization of convexity on Riemannian manifolds were presented in [5, 8, 14, 16]).
In this paper, a new class of sets on Riemannian manifolds, called geodesic strongly E-b-vex sets, and a new class of functions defined on them, called geodesic strongly E-convex functions, have been proposed. Also, some of their properties have been discussed. This paper divides into three sections. In section 2, some of definitions and properties which will be used throughout this work are presented that can be found in many books on differential geometry such as [16]. In section 3, a geodedic strongly E-b-vex set and geodesic strongly E-b-vex function are studied with some of their properties.

2. Preliminaries

Now, let $\mathbb{N}$ is a $C^\infty$ n-dimensional Riemannian manifold, also $\mu_1, \mu_2 \in \mathbb{N}$ and $\delta: [0, 1] \rightarrow \mathbb{N}$ be a geodesic joining the points $\mu_1$ and $\mu_2$, which means that $\delta_{\mu_1, \mu_2}(0) = \mu_2$ and $\delta_{\mu_1, \mu_2}(1) = \mu_1$.

Strongly E-convex sets (SEC) and strongly E-convex (SEC) functions were introduced in [18] such as:

**Definition 2.1.**  
1. A subset $\Omega \subseteq \mathbb{R}^n$ is strongly E-convex (SEC) set if there is a map $\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that
\[ \delta(\alpha \mu_1 + \varepsilon(\mu_1)) + (1 - \delta)(\alpha \mu_2 + \varepsilon(\mu_2)) \in B \]
for each $\mu_1, \mu_2 \in \Omega$, $\alpha \in [0, 1]$ and $\delta \in [0, 1]$.

2. A function $g: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ where $\Omega$ is a SEC set and
\[ g(\delta(\alpha \mu_1 + \varepsilon(\mu_1)) + (1 - \delta)(\alpha \mu_2 + \varepsilon(\mu_2))) \leq g(\alpha \mu_1) + (1 - \delta)g(\alpha \mu_2), \]
$\forall \mu_1, \mu_2 \in \Omega$, $\alpha \in [0, 1]$ and $\delta \in [0, 1]$.

**Definition 2.2.** [5]

1. Considering $\varepsilon: \mathbb{N} \rightarrow \mathbb{N}$ is a map. A subset $\Omega \subset \mathbb{N}$ is geodesic E-convex iff there exists a unique geodesic $\eta_{\varepsilon(\mu_1), \varepsilon(\mu_2)}(\delta)$ of length $d(\mu_1, \mu_2)$, which belongs to $\Omega$, $\forall \mu_1, \mu_2 \in \Omega$ and $\delta \in [0, 1]$.

2. A function $g: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ where $\Omega$ is a GEC set in $\mathbb{N}$ is geodesic E-convex if
\[ g(\eta_{\varepsilon(\mu_1), \varepsilon(\mu_2)}(\delta)) \leq g(\varepsilon(\mu_1)) + (1 - \delta)g(\varepsilon(\mu_2)), \]
$\forall \mu_1, \mu_2 \in \Omega$ and $\delta \in [0, 1]$.

**Definition 2.3.** [7]

1. Considering $\varepsilon: \mathbb{N} \rightarrow \mathbb{N}$ is a map. A subset $\Omega \subset \mathbb{N}$ is geodesic strongly E-convex (GSEC) iff there exists a unique geodesic $\eta_{\alpha \mu_1 + \varepsilon(\mu_1), \alpha \mu_2 + \varepsilon(\mu_2)}(\delta)$ of length $d(\mu_1, \mu_2)$, which belongs to $\Omega$, $\forall \mu_1, \mu_2 \in \Omega$, $\alpha \in [0, 1]$ and $\delta \in [0, 1]$ and.
(2) A function \( g: \Omega \subseteq \mathbb{R} \rightarrow \mathbb{R} \), where \( \Omega \) is a GSEC set in \( \mathbb{R} \), is geodesic strongly E-convex (GSEC) if
\[
g(\eta_{\alpha \mu_1 + \varepsilon(\mu_1), \alpha \mu_2 + \varepsilon(\mu_2)}(\delta)) \leq \delta g(\varepsilon(\mu_1)) + (1 - \delta)g(\varepsilon(\mu_2)),
\]
\( \forall \mu_1, \mu_2 \in \Omega \) and \( \delta \in [0, 1] \).

3. Geodesic Strongly E-b-vex Sets and Geodesic Strongly E-b-vex Functions

In this part of work, a geodesic strongly E-b-vex (GSE-b-vex) set and a geodesic strongly E-b-convex (GSE-b-convex) function in a Riemannian manifold \( \mathbb{R} \) are given and some of their properties are discussed.

Definition 3.1. A subset \( \Omega \) of \( \mathbb{R} \) is called a geodesic strongly E-b-vex (GSE-b-vex) set iff there exists a unique geodesic \( \eta_{\alpha \mu_1 + \varepsilon(\mu_1), \alpha \mu_2 + \varepsilon(\mu_2)}(\delta b) \) of length \( d(\mu_1, \mu_2) \), which belongs to \( \Omega \), \( \forall \mu_1, \mu_2 \in \Omega, \alpha \in [0, 1] \) and \( \delta \in [0, 1] \).

Remark 3.1. (1) Every GSE-b-vex set is a GSEC set when \( b(\mu_1, \mu_2, \delta) = 1 \).

(2) Every GSE-b-vex set is a GE-b-vex set when \( \alpha = 0 \).

(3) When
\[
\eta_{\alpha \mu_1 + \varepsilon(\mu_1), \alpha \mu_2 + \varepsilon(\mu_2)}(\delta b) = \delta b(\alpha \mu_1 + \varepsilon(\mu_1)) + (1 - \delta b)(\alpha \mu_2 + \varepsilon(\mu_2)),
\]
then we have strongly E-B-vex set.

Now, some properties of GSE-b-vex sets are proposed.

Proposition 3.1. Every convex set \( \Omega \subseteq \mathbb{R} \) is a GSE-b-vex set.

The proof of the above proposition is direct that by taking \( \varepsilon: \mathbb{R} \rightarrow \mathbb{R} \) as the identity map, \( b(\mu_1, \mu_2, \delta) = 1 \) and \( \alpha = 0 \).

Proposition 3.2. Let \( \Omega \subseteq \mathbb{R} \) be a GSE-b-vex set, then \( \varepsilon(\Omega) \subseteq \Omega \).

Proof. Since \( \Omega \) is a GSE-b-vex set, then
\[
\eta_{\alpha \mu_1 + \varepsilon(\mu_1), \alpha \mu_2 + \varepsilon(\mu_2)}(\delta b) \in \Omega,
\]
\( \mu_1, \mu_2 \in \Omega, \alpha \in [0, 1] \) and \( \delta \in [0, 1] \). Let \( \delta b = 1 \) and \( \alpha = 0 \), then \( \eta_{\varepsilon(\mu_1), \varepsilon(\mu_2)} = \varepsilon(\mu_2) \in \Omega \), then \( \varepsilon(\Omega) \subseteq \Omega \).

Theorem 3.1. Suppose that a set \( \{\Omega_j\}_{j=1,2,\ldots,n} \) is an arbitrary collection of GSE-v-vex subsets of \( \mathbb{R} \), then \( \bigcap_{j=1,2,\ldots,n} \Omega_j \) is a GSE-b-vex set.

Proof. Considering \( \{\Omega_j\}_{j=1,2,\ldots,n} \) is a collection of GSE-b-vex subsets of \( \Omega \). If \( \bigcap_{j=1,2,\ldots,n} \Omega_j \) is an empty set, then the result is obvious. Assume that \( \mu_1, \mu_2 \in \bigcap_{j=1,2,\ldots,n} \Omega_j \), then \( \mu_1, \mu_2 \in \Omega_j \). Hence, \( \eta_{\alpha \mu_1 + \varepsilon(\mu_1), \alpha \mu_2 + \varepsilon(\mu_2)}(\delta b) \in \Omega_j, \forall \alpha \in [0, 1] \) and \( \delta \in [0, 1] \). Hence, \( \eta_{\alpha \mu_1 + \varepsilon(\mu_1), \alpha \mu_2 + \varepsilon(\mu_2)}(\delta b) \in \bigcap_{j=1,2,\ldots,n} \Omega_j, \forall \alpha \in [0, 1] \) and \( \delta \in [0, 1] \).
Remark 3.2. However, the above theorem is not true for the union of GSE-b-vex sets.

Now, we introduce the definition of a geodesic E-b-vex (GSE-b-vex) function on $\mathbb{N}$.

**Definition 3.2.** Assume that $\Omega \subset \mathbb{N}$ is a GSE-b-vex set. A function $g: \Omega \to \mathbb{R}$ is called a geodesic strongly E-b-vex (GSE-b-vex) if

$$g(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \leq (1-\gamma)g(\varepsilon(\mu_1)) + \gamma g(\varepsilon(\mu_2)), \quad (3.1)$$

\[\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1] \text{ and } \delta \in [0,1].\]

If the inequality (3.1) is strict, then $g$ is called a strictly GSE-b-vex function.

**Example 3.1.** Assume that $g: \mathbb{R} \to \mathbb{R}$ such that $g(\mu) = -|\mu|$. Also, assume that $\varepsilon: \mathbb{R} \to \mathbb{R}$ is defined as $\varepsilon(\mu) = \alpha\mu$ where $0 < \alpha \leq 1$, $\forall \mu \in \mathbb{R}$ and the geodesic $\eta$ is given as

$$\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b) = \begin{cases} \frac{1}{2\alpha}[\alpha\mu_2 + \varepsilon(\mu_2) + \delta b(\alpha\mu_1+\varepsilon(\mu_1) - \alpha\mu_2 - \varepsilon(\mu_2))] & : \mu_1\mu_2 \geq 0, \\ \frac{1}{2\alpha}[\alpha\mu_2 + \varepsilon(\mu_2) + \delta b(\alpha\mu_1+\varepsilon(\mu_1) - \alpha\mu_1 - \varepsilon(\mu_1))] & : \mu_1\mu_2 < 0 \end{cases}$$

then $g$ is GSE-b-vex function.

**Proposition 3.3.** Let $g: \Omega \to \mathbb{R}$ be a GSE-b-vex function on a GSE-b-vex set $\Omega \times \mathbb{N}$, then $g(\alpha\mu + \varepsilon(\mu)) \leq g(\varepsilon(\mu)), \mu \in \Omega$ and $\alpha \in [0,1]$.

**Proof.** Since $g$ is GSE-b-vex function on GSE-b-vex set $\Omega$, then

$$g(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \leq \delta bg(\varepsilon(\mu_1)) + (1-\delta b)g(\varepsilon(\mu_2)),$$

then for $\delta b = 1$, we have

$$g(\alpha\mu_1 + \varepsilon(\mu_1)) \leq g(\varepsilon(\mu_1)).$$

\[\square\]

**Theorem 3.2.** If $g_1: \Omega \to \mathbb{R}$ is a GSE-b-vex function on a GSE-b-vex set $\Omega \subset \mathbb{N}$ and $g_2: U \to \mathbb{R}$ is a non-decreasing convex function such that $\text{rang}(g_1) \subset U$, then the composite function $g_2og_1$ is GSE-b-vex function on $\Omega$. 
Proof. By using the hypothesis, we can write all \( x_1, x_2 \in B, \alpha \in [0,1] \) and \( \gamma \in [0,1] \),
\[
\begin{align*}
g_1(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b)) & \leq \delta b g_1(\varepsilon(\mu_1)) + (1 - \delta b) g_1(\varepsilon(\mu_2)), \\
\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1] \text{ and } \delta \in [0,1] \text{ and since } g_2 \text{ is a non-decreasing convex function, then we get} \\
g_2 \circ g_1(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b)) &= g_2 \left( g_2(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b)) \right) \\
& \leq g_2 \left( \delta b g_1(\varepsilon(\mu_1)) + (1 - \delta b) g_1(\varepsilon(\mu_2)) \right) \\
& \leq \delta b g_2(g_1(\varepsilon(\mu_1))) + (1 - \delta b) g_2(g_1(\varepsilon(\mu_2))) \\
& = \delta b (g_2 \circ g_1)(\varepsilon(\mu_1)) + (1 - \delta b) (g_2 \circ g_1)(\varepsilon(\mu_2))
\end{align*}
\]
hence, \( g_2 \circ g_1 \) is GSE-b-vex on \( \Omega \). Moreover, \( g_2 \circ g_1 \) is a strictly GSE-b-vex function if \( g_2 \) is a strictly non-decreasing convex function.

\textbf{Theorem 3.3.} Considering \( g_i : \Omega \rightarrow \mathbb{R}, i = 1,2,...,n \) are GSE-b-vex functions. Then, the function
\[
g = \sum_{i=1}^{n} \xi_i g_i
\]
is also GSE-b-vex geodesic on \( \Omega, \forall \xi_i \in \mathbb{R}, \xi_i \geq 0 \).

Proof. Since \( g_i, i = 1,2,...,n \) are GSE-b-vex functions, then
\[
\begin{align*}
g_i(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b)) & \leq \delta b g_i(\varepsilon(\mu_1)) + (1 - \delta b) g_i(\varepsilon(\mu_2)), \\
\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1] \text{ and } \delta \in [0,1] \text{, Hence,} \\
\xi_i g_i(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b)) & \leq \delta b \xi_i g_i(\varepsilon(\mu_1)) + (1 - \delta b) \xi_i g_i(\varepsilon(\mu_2)).
\end{align*}
\]
This implies to,
\[
\begin{align*}
g(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b)) &= \sum_{i=1}^{n} \xi_i g_i(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b)) \\
& \leq \delta b \sum_{i=1}^{n} \xi_i g_i(\varepsilon(\mu_1)) + (1 - \delta b) \sum_{i=1}^{n} \xi_i g_i(\varepsilon(\mu_2)) \\
& = \delta b g(\varepsilon(\mu_1)) + (1 - \delta b) g(\varepsilon(\mu_2)).
\end{align*}
\]
Then \( g \) is GSE-b-vex function.

Next, we show that a function is GSE-b-vex iff its epigraph is a GSE-b-vex set.

\textbf{Definition 3.3.} Assume that \( \Omega \subset \mathbb{R} \times \mathbb{R}, E : \mathbb{R} \rightarrow \mathbb{R}, b : \Omega \times \Omega \times [0,1] \rightarrow \mathbb{R}_+ \) and \( F : \mathbb{R} \rightarrow \mathbb{R} \). A set \( \Omega \) is called a geodesic strongly \( E \times F \)-convex (GSE \( \times F \)-b-vex) if
\[
(\eta_{\alpha_1+\varepsilon(\mu_1), \alpha_2+\varepsilon(\mu_2)}(\delta b), \delta b F(\xi_1) + (1 - \delta b) F(\xi_2)) \in \Omega,
\]
\( \forall (\mu_1, \xi_1), (\mu_2, \xi_2) \in \Omega, \alpha \in [0,1] \) and \( \gamma \in [0,1] \).
Remark 3.3. From Definition 3.3, we have found \( \Omega \subseteq \mathbb{R} \) is a GSE-b-vex set iff \( \Omega \times \mathbb{R} \) is a GSE \( \times F \)-b-vex set.

Now, the epigraph of a function \( g: \Omega \subset \mathbb{R} \to \mathbb{R} \) is given as

\[
E(g) = \{ (\mu, a) : \mu \in \Omega, a \in \mathbb{R}, g(\mu) \leq a \}.
\]  

(3.2)

Theorem 3.4. Suppose that \( \Omega \subseteq \mathbb{R} \) is a GSE-b-vex set, \( g: \Omega \to \mathbb{R} \) is a function and \( F: \mathbb{R} \to \mathbb{R} \) is a map such that \( F(g(\mu)) + a = g(\varepsilon(\mu)) + a, \forall a \in \mathbb{R}, a \geq 0 \). Then, \( g \) is a GSE-b-vex on \( \Omega \) iff \( E(g) \) is a GSE \( \times F \)-b-vex on \( \Omega \times \mathbb{R} \).

Proof. Let \( (\mu_1, a_1), (\mu_2, a_2) \in E(g) \). Since \( \Omega \) is GSE-b-vex, then

\[
\eta_{\alpha\mu_1 + \varepsilon(\mu_1), \alpha\mu_2 + \varepsilon(\mu_2)}(\delta b) \in \Omega,
\]

\( \forall \alpha \in [0,1] \) and \( \delta \in [0,1] \). When \( \alpha = 0 \) and \( \delta b = 1 \), we have \( \varepsilon(\mu_1) \in \Omega \) also, when \( \alpha = 0 \) and \( \delta b = 0 \) we get \( \varepsilon(\mu_2) \in \Omega \). Assume that \( F(a_1) \) and \( F(a_2) \) where \( g(\varepsilon(\mu_1)) \leq F(a_1) \) and \( g(\varepsilon(\mu_2)) \leq F(a_2) \). Then

\[
(\varepsilon(\mu_1), F(a_1)), (\varepsilon(\mu_2), F(a_2)) \in E(g).
\]

Considering \( g \) is a GSE-b-vex on \( \Omega \), then

\[
g(\eta_{\alpha\mu_1 + \varepsilon(\mu_1), \alpha\mu_2 + \varepsilon(\mu_2)}(\delta b)) \leq \delta b g(\varepsilon(\mu_1)) + (1 - \delta b) g(\varepsilon(\mu_2))
\]

\[
\leq \delta b F(a_1) + (1 - \delta b) F(a_2).
\]

This is leading to, \( (\eta_{\alpha\mu_1 + \varepsilon(\mu_1), \alpha\mu_2 + \varepsilon(\mu_2)}(\delta b), \delta b F(a_1) + (1 - \delta b) F(a_2)) \in E(g) \), which means that \( E(g) \) is GSE \( \times \hat{E} \)-b-vex on \( \Omega \times \mathbb{R} \).

Conversely, let us take \( E(g) \) is GSE \( \times \hat{E} \)-b-vex on \( \Omega \times \mathbb{R} \). Assume that \( \mu_1, \mu_2 \in \Omega, \alpha \in [0,1] \) and \( \delta \in [0,1] \), then \( (\mu_1, g(\mu_1)) \in E(g) \) and \( (\mu_2, g(\mu_2)) \in E(g) \).

In addition, \( (\eta_{\alpha\mu_1 + \varepsilon(\mu_1), \alpha\mu_2 + \varepsilon(\mu_2)}(\delta b), \delta b F(g(\mu_1)) + (1 - \delta b) F(g(\mu_2))) \in E(g) \) \( \implies \)

\[
g(\eta_{\alpha\mu_1 + \varepsilon(\mu_1), \alpha\mu_2 + \varepsilon(\mu_2)}(\delta b)) \leq \delta b F(g(\mu_1)) + (1 - \delta b) F(g(\mu_2))
\]

\[
= \delta b g(\varepsilon(\mu_1)) + (1 - \delta b) g(\varepsilon(\mu_2)).
\]

Hence, the result. \( \square \)

Theorem 3.5. Let \( \{ \Omega_j \}_{j=1,\ldots,n} \) be a family of GSE \( \times F \)-b-vex sets. Then \( \cap_{j=1,\ldots,n} \Omega_j \) is also GSE \( \times F \)-b-vex set.

Proof. Let \( (\mu_1, a_1), (\mu_2, a_2) \in \cap_{j=1,\ldots,n} \Omega_j \), then \( (\mu_1, a_1), (\mu_2, a_2) \in \Omega_j, \forall j \). \( \implies \)

\[
(\eta_{\alpha\mu_1 + \varepsilon(\mu_1), \alpha\mu_2 + \varepsilon(\mu_2)}(\delta b), \delta b F(a_1) + (1 - \delta b) F(a_2)) \in \Omega_j,
\]
∀α ∈ [0, 1] and δ ∈ [0, 1]. Hence,

\( (η_{αμ_1+ε(μ_1),αμ_2+ε(μ_2)}(δb), δbF(a_1) + (1 − δb)F(a_2)) ∈ \cap_{j=1,...,n}Ω_j, \)

∀α ∈ [0, 1] and δ ∈ [0, 1]. This shows that \( \cap_{j=1,...,n}Ω_j \) is GSE × F-b-vex set.

**Theorem 3.6.** Suppose that \( G : \mathbb{R} \rightarrow \mathbb{R} \) such that \( G(g(x) + μ) = g(ε(x)) + μ, \forall μ \in \mathbb{R}, μ ≥ 0 \). Let \( \{g_i\}_{i \in I} \) be a family of real valued functions that is defined on a GSE-b-vex set \( Ω \) and bounded from above. Then, \( g(x) = \sup_{i \in I} g_i(x), x ∈ Ω \) is GSE-b-vex on \( Ω \).

**Proof.** Let \( g_i, i \in I \) be a GSE-b-vex function on \( Ω \), then

\[ E(g_i) = \{(x, μ) : x ∈ Ω, μ ∈ \mathbb{R}, g_i(x) ≤ μ\} \]

are GSE × F-b-vex on \( Ω × \mathbb{R} \). Hence,

\[ \cap_{i ∈ I} E(g_i) = \{(x, μ) : x ∈ Ω, μ ∈ \mathbb{R}, g_i(x) ≤ μ, i ∈ I\} \]

\[ = \{(x, μ) : x ∈ Ω, μ ∈ \mathbb{R}, g(x) ≤ μ\} \]

is GSE × F-b-vex set. Then, by Theorem 3.4 \( g \) is a GSE-b-vex function.

**References**


