SYMMETRY ANALYSIS AND SOLITARY WAVE SOLUTIONS OF NONLINEAR ION-ACOUSTIC WAVES EQUATION

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Abstract. The problem of nonlinear ion-acoustic waves equation in a magnetized plasma, known as Zakharov-Kuznetsov equation, is investigated by using symmetry analysis. The carryover of the symmetry analysis has led to certain similarity reductions of this equation. Furthermore, exact solutions of similarity reductions are obtained by modified Exp-Function method with computational symbolic. Some figures are obtained to show the properties of the solutions.

1. Introduction

There are many well-known methods to obtain exact solutions [1–5]. In order to unite and widen various specialized solution method for partial differential equations Lie introduced the notion of continuous groups now know as Lie groups. Continuing his investigations he shown that partial differential equation can be reduced to many ordinary differential equations which is led to varied solutions. In the last century, the application of the Lie groups has been developed by a number of researchers. Ovsiannikov [6], Olver [7], Ibragimov [8], and Bluman et al. [9] are some of the mathematicians who have huge number of studies in that field.
Consider the nonlinear ion-acoustic waves equation which is called (1+3)-dimensional Zakharov-Kuznetsov (Zk) equation \[10, 11\] in the following form:

$$ u_t + p_1 u u_x + p_2 u_{x,x,x} + p_3 u_{x,y,y} + p_4 u_{x,z,z} = 0 \tag{1} $$

where $p_1$, $p_2$, $p_3$ and $p_4$ are nonzero constants. ZK \[10\] is described the diffusion of nonlinear ion-acoustic waves in magnetized plasma \[10\]. This equation was devoted to study many properties including presence and stability of solitary wave solutions for the ZK model \[10, 13−15\].

2. Determination of the symmetries

Firstly, we shall conclude the similarity reductions using Lie group method \[16−22\]. In order to apply Lie group method we can write the one parameter Lie group of infinitesimal transformations as follow:

$$ t^* = t + \varepsilon A(t, x, y, z, u) + o(\varepsilon^2), \quad x^* = x + \varepsilon B(t, x, y, z, u) + o(\varepsilon^2), $$

$$ y^* = y + \varepsilon C(t, x, y, z, u) + o(\varepsilon^2), \quad z^* = z + \varepsilon D(t, x, y, z, u) + o(\varepsilon^2), $$

$$ u^* = u + \varepsilon E(t, x, y, z, u) + o(\varepsilon^2). \tag{2} $$

If we set

$$ \Delta = u_t + p_1 u u_x + p_2 u_{x,x,x} + p_3 u_{x,y,y} + p_4 u_{x,z,z} = 0 \tag{3} $$

where subscripts $t, x, y$ and $z$ to the function $u$ denote differentiation with respect to these variables. The infinitesimal generator $V$ associated with the above mentioned group of transformations can be presented as following expression

$$ V = A \frac{\partial}{\partial t} + B \frac{\partial}{\partial x} + C \frac{\partial}{\partial y} + D \frac{\partial}{\partial z} + E \frac{\partial}{\partial u}, \tag{4} $$

when the following invariance condition is satisfied:

$$ \Gamma^{(3)}(\Delta) = 0, \tag{5} $$

where $\Gamma^{(3)}$ is the third order prolongation of the operator $V$

$$ \Gamma^{(3)} = V + E_x \frac{\partial}{\partial u_x} + E_t \frac{\partial}{\partial u_t} + E_{xxx} \frac{\partial}{\partial u_{xxx}} + E_{xyy} \frac{\partial}{\partial u_{xyy}} + E_{xz} \frac{\partial}{\partial u_{xz}}, \tag{6} $$

where the components $E_x, E_{xx}, E_{xy}, E_{xz}, E_{tx}....$ can be determined from the following expressions:

$$ E_x = D_x E - u_x D_x A - u_x D_x B, $$

$$ E_{xt} = D_t E_x - u_{tx} D_t A - u_{tx} D_t B. \tag{7} $$
Substituting (3) into invariance condition (5), yields an identity components \( A_x, A_{xx}, B_t, B_x, \ldots \) hence we collect the coefficients of \( u_x, u_{x,x}, \ldots \) and equate it to zero, which led to obtain a system of linear differential equations of the infinitesimals \( A, B, C \) and \( E \)

\[
A_x = A_y = A_z = A_u = A_{t,t} = 0,
\]

\[
B_y = B_z = B_u = B_{t,t} = 0, B_x = \frac{1}{3} A_t,
\]

\[
C_t = C_x = B_u = 0, C_y = \frac{1}{3} A_t, C_z = \frac{-p_3}{p_4} D_y,
\]

\[
D_t = D_x = D_u = D_{y,y} = 0, D_z = \frac{1}{3} A_t,
\]

\[
-p_1^2 u^2 B_x + p_1 u A_x - p_1 u B_x - p_1 E + A_t = 0,
\]

\[
3p_2 C_{x,x} + p_3 C_{y,y} + p_4 C_{z,z} = 0,
\]

(8)

Solving resulting of partial differential equations system, we got:

\[
A = c_1 t + c_2, \quad B = \frac{1}{3} c_1 x + p_1 c_7 t + c_4
\]

\[
C = \frac{1}{3} c_1 y - c_3 \frac{p_3}{p_4} z + c_5, \quad D = \frac{1}{3} c_1 z + c_3 y + c_6,
\]

\[
E = \frac{-2}{3} c_1 u + c_7.
\]

(9)

We can be easily write the vector field operator \( V \) from (9) as

\[
V = V_1(c_1) + V_2(c_2) + V_3(c_3) + V_4(c_4) + V_5(c_5) + V_6(c_6) + V_7(c_7),
\]

(10)

where

\[
V_1 = \frac{\partial}{\partial t} + \frac{1}{3} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - 2 u \frac{\partial}{\partial u} \right),
\]

\[
V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{-p_3}{p_4} z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z},
\]

\[
V_4 = \frac{\partial}{\partial x}, \quad V_5 = \frac{\partial}{\partial y}, \quad V_6 = \frac{\partial}{\partial z},
\]

\[
V_7 = p_1 t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}.
\]

(11)

The commutator relations are given in Table 1.
Table 1: The commutator table

<table>
<thead>
<tr>
<th></th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$V_3$</th>
<th>$V_4$</th>
<th>$V_5$</th>
<th>$V_6$</th>
<th>$V_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>0</td>
<td>$-V_2$</td>
<td>0</td>
<td>$\frac{-1}{3}V_4$</td>
<td>$\frac{-1}{3}V_5$</td>
<td>$\frac{-1}{3}V_6$</td>
<td>$\frac{2}{3}V_7$</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$V_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$V_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-V_6$</td>
<td>$\frac{p_3}{p_4}V_5$</td>
<td>0</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$\frac{1}{3}V_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$\frac{1}{3}V_5$</td>
<td>0</td>
<td>0</td>
<td>$V_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$\frac{1}{3}V_6$</td>
<td>0</td>
<td>$-\frac{p_3}{p_4}V_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$V_7$</td>
<td>$\frac{-2}{3}V_7$</td>
<td>$-p_1V_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

From the commutator relations in table 1, we utilized the following six non-equivalent possibilities of Lie algebra:

(I)$V_1 + m_1V_2 + m_2V_4 + m_3V_5 + m_4V_6 + m_5V_7$,

(II)$V_2 + m_1V_4 + m_2V_5 + m_3V_6$

(III)$V_4 + m_1V_5 + m_2V_7$

(IV)$V_2 + m_1V_5 + m_2V_7$

(V)$V_2 + m_1V_4 + m_2V_7$

(VI)$V_2 + m_1V_4 + m_2V_5$

3. Reductions and exact solutions

In order to obtain the invariant transformation, we can write the characteristic equation as follow

$$\frac{dt}{A(t,x,y,z,u)} = \frac{dx}{B(t,x,y,z,u)} = \frac{dy}{C(t,x,y,z,u)} = \frac{dz}{D(t,x,y,z,u)} = \frac{du}{E(t,x,y,z,u)}.$$  (12)

This equation is solved for the above six cases the invariant variables, then the corresponding reductions to partial differential equations are obtained and by using the similarity transformations the govern partial differential equations reduced to ordinary differential equations.
Table 2: The invariant variables and their corresponding partial differential equations

<table>
<thead>
<tr>
<th>Case</th>
<th>The invariant variables</th>
<th>Corresponding partial differential equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>I(i)</td>
<td>$\frac{x-m_5 p_1 t+n}{(t+m_1)^2}$, $\frac{y+n_3}{(t+m_1)^2}$, $\frac{z+n_4}{(t+m_1)^2}$, $\frac{F}{(t+m_1)^2} + \frac{m_5}{2}$</td>
<td>$2F + \zeta_1 F_{\zeta_1} + \zeta_2 F_{\zeta_2} + \zeta_3 F_{\zeta_3} - 3p_1 F F_{\zeta_1}$ $- 3p_2 F_{\zeta_1} F_{\zeta_2} F_{\zeta_2} - 3p_4 F_{\zeta_1} F_{\zeta_3} F_{\zeta_3} = 0$.</td>
</tr>
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<tr>
<th>Case</th>
<th>The invariant variables</th>
<th>Corresponding partial differential equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>I(ii)</td>
<td>$\frac{x-m_5 p_1 t+n}{(t+m_1)^2}$, $\frac{y+n_3}{(t+m_1)^2}$, $\frac{z+n_4}{(t+m_1)^2}$, $\frac{F}{(t+m_1)^2} + \frac{m_5}{2}$</td>
<td>If we put $p_1 F F_{\zeta_1} + (p_2 F_{\zeta_1} F_{\zeta_2} + p_3 F_{\zeta_2} F_{\zeta_2})<em>{\zeta_1} = 0$, we conclude that $[2F + \zeta_1 F</em>{\zeta_1} + \zeta_2 F_{\zeta_2} + \zeta_3 F_{\zeta_3}] = 0$.</td>
</tr>
</tbody>
</table>

II  $x - m_1 t$, $y-m_2 t$, $z-m_3 t$, $F$  | $m_1 F_{\zeta_1} + m_2 F_{\zeta_2} + m_3 F_{\zeta_3} - p_1 F F_{\zeta_1}$ $- p_2 F_{\zeta_1} F_{\zeta_2} F_{\zeta_2} - p_4 F_{\zeta_1} F_{\zeta_3} F_{\zeta_3} = 0$. |

III  $t$, $y-m_1 x$, $z-m_2 x$, $F$  | $F_{\zeta_1} - p_1 F (m_1 F_{\zeta_1} + m_2 F_{\zeta_2}) - m_1 (p_2 m_1^2$ $+ p_3 F_{\zeta_1} F_{\zeta_2} F_{\zeta_2} - m_2 (3p_2 m_1^2 + p_3 m_1)$ $F_{\zeta_2} F_{\zeta_2} F_{\zeta_2} - m_1 (3p_2 m_2^2 + p_4 m_1) F_{\zeta_2} F_{\zeta_3} F_{\zeta_3}$ $- m_2 (3p_2 m_2^2 + p_4) F_{\zeta_3} F_{\zeta_3} F_{\zeta_3} = 0$. |

IV  $x$, $y-m_1 t$, $z-m_2 t$, $F$  | $m_1 F_{\zeta_1} + m_2 F_{\zeta_2} - p_1 F F_{\zeta_1} - p_2 F_{\zeta_1} F_{\zeta_1} F_{\zeta_1}$ $- p_3 F_{\zeta_1} F_{\zeta_2} F_{\zeta_2} - p_4 F_{\zeta_1} F_{\zeta_3} F_{\zeta_3} = 0$. |

V  $x-m_1 t$, $y$, $z-m_2 t$, $F$  | $m_1 F_{\zeta_1} + m_2 F_{\zeta_2} - p_1 F F_{\zeta_1} - p_2 F_{\zeta_1} F_{\zeta_1} F_{\zeta_1}$ $- p_3 F_{\zeta_1} F_{\zeta_2} F_{\zeta_2} - p_4 F_{\zeta_1} F_{\zeta_3} F_{\zeta_3} = 0$. |

VI  $x-m_1 t$, $y-m_2 t$, $z$, $F$  | $m_1 F_{\zeta_1} + m_2 F_{\zeta_2} - p_1 F F_{\zeta_1} - p_2 F_{\zeta_1} F_{\zeta_1} F_{\zeta_1}$ $- p_3 F_{\zeta_1} F_{\zeta_2} F_{\zeta_2} - p_4 F_{\zeta_1} F_{\zeta_3} F_{\zeta_3} = 0$. |

where $n_3 = 3m_3$, $n_4 = 3m_4$, $n = 3m_2 - \frac{n}{2} p_1 m_2 m_5$
Case I(i):

In this case, we put \( \theta = k_1 \zeta_1 + k_2 \zeta_2 + k_3 \zeta_3 \), then, the equation can be written in the form:

\[
2F + \theta F' - 3p_1 k_1 FF' - 3k_1 (p_2 k_1^2 + p_3 k_2^2 + p_4 k_3^2) F''' = 0. \tag{13}
\]

To obtain the solution for the ODE corresponding to this case, we assume that this solution takes the following form

\[
F = a_0 + a_1 \theta + a_2 \theta^2 + \frac{b_1}{\theta} + \frac{b_2}{\theta^2}. \tag{14}
\]

Substituting Eq. (14) into Eq. (13), equating to zero the coefficients of all powers of \( \theta \) yields a set of algebraic equations for \( a_0, a_1, a_2, b_1, b_2 \), solving the system of algebraic equations with the aid of Maple, we obtain the following results: \( a_0 = a_1 = b_1 = b_2 = 0, \ a_1 = \frac{1}{k_1 p_1} \), then, the final solution of Eq. (1) can be written in the form:

\[
u(t, x, y, z) = \frac{1}{k_1 p_1 (t + m_1)} [k_1 (x - \frac{m_5}{2} p_1 t + n_1) + k_2 (y + n_3) + k_3 (z + n_4)] + \frac{m_5}{2}. \tag{15}
\]

Case I(ii): In this case we have to solve the following two PDEs

\[
p_1 FF_{\zeta_1} + p_2 F_{\zeta_1, \zeta_1} + p_3 F_{\zeta_1, \zeta_2, \zeta_2} + p_4 F_{\zeta_1, \zeta_3, \zeta_3} = 0, \tag{16}
\]

\[
2F + \zeta_1 F_{\zeta_1} + \zeta_2 F_{\zeta_2} + \zeta_3 F_{\zeta_3} = 0. \tag{17}
\]

We now introduce the simplified form of Lie-group transformations namely, the scaling group of transformation

\[
F = e^{\epsilon t} \bar{F}, \quad \zeta_1 = e^{\epsilon \zeta_1} \bar{\zeta_1}, \quad \zeta_2 = e^{\epsilon \zeta_2} \bar{\zeta_2}, \quad \zeta_3 = e^{\epsilon \zeta_3} \bar{\zeta_3}. \tag{18}
\]

Substituting from (18) into (17) we have \( \epsilon_1 = \epsilon_2 = \epsilon_3 = -\epsilon \).

This mean that (16) is invariant under the transformation (18) and the characteristic equation can be written as

\[
\frac{d\zeta_1}{\zeta_1} = \frac{d\zeta_2}{\zeta_2} = \frac{d\zeta_3}{\zeta_3} = -\frac{dF}{F}. \tag{19}
\]

We get the similarity variables

\[
\eta_1 = \frac{\zeta_1}{\zeta_2}, \quad \eta_2 = \frac{\zeta_3}{\zeta_2}, \quad F = \frac{f}{\zeta_2^2}. \tag{20}
\]

Substituting from (20) into (17) we find that it is satisfied. Also substituting from (20) to Eq.(16) we obtain

\[
p_1 f f_{\eta_1} + p_2 f_{\eta_1, \eta_1, \eta_1} + 12p_3 f_{\eta_1} + 8p_3 \eta_1 f_{\eta_1} + 8p_3 \eta_2 f_{\eta_2} \\
p_3 \eta_1^2 f_{\eta_1, \eta_1, \eta_1} + p_3 \eta_2^2 f_{\eta_1, \eta_1, \eta_2} + 2p_3 \eta_1 \eta_2 f_{\eta_1, \eta_2, \eta_2} + p_4 f_{\eta_1, \eta_2, \eta_2} = 0. \tag{21}
\]

By using \( \theta = \eta_1 + \eta_2 \) (21) can be written in form

\[
p_1 f f' + p_2 f''' + 4p_3 (3 + 2\theta) f' + (p_3 \theta^2 + p_4) f'''' + 4\theta + \theta^2 + 2 - \theta = 0. \tag{22}
\]
Using the same method in the previous case, hence, we have obtained the following exact solution to ODE corresponding this case in the form \( u(t, x, z) = \frac{m_5}{2} - \frac{12(p_2 + p_4)}{p_1(n + n_4 - \frac{m_5}{2}t + x + z)^2} \). (23)

**Case II:** We take \( \theta = k_1\zeta_1 + k_2\zeta_2 + k_3\zeta_3 \), then equation of case (II) can be written in the form

\[
(k_1m_1 + k_2m_2 + k_3m_3)F'' - p_1k_1FF' - k_1(p_2k_1^2 + p_3k_2^2 + p_4k_3^2)F''' = 0. \tag{24}
\]

To utilize the solution for the ODE corresponding to this case, we used modified Exp-Function method \([13,23]\), which is expressed in the form:

\[
F(\theta) = \sum_{n=-c}^{p} a_n[\phi(\theta)]^n - \sum_{m=-d}^{q} b_m[\phi(\theta)]^m = \\frac{a_{-c}[\phi(\theta)]^{-c} + \ldots + a_p[\phi(\theta)]^p}{b_{-d}[\phi(\theta)]^{-d} + \ldots + b_q[\phi(\theta)]^q} \tag{25}
\]

where \( \phi(\theta) \) satisfies the following Riccati equation

\[
\phi'(\theta) = A + B \phi(\theta) + C \phi^2(\theta). \tag{26}
\]

see \([24-25]\).

We can freely choose the values of \( n \) and \( m \) in (25), that the solution does not depend on the balancing of the highest order linear and nonlinear terms \([24]\).

\[
F''' = \frac{a_1[\phi(\theta)]^{-c-8d-3} + \ldots + a_2[\phi(\theta)]^{p+8d+3}}{b_1[\phi(\theta)]^{-9d} + \ldots + b_2[\phi(\theta)]^{9q}}, \tag{27}
\]

\[
FF' = \frac{a_3[\phi(\theta)]^{-2c-7d-3} + \ldots + a_4[\phi(\theta)]^{2p+7d+3}}{b_3[\phi(\theta)]^{-9d} + \ldots + b_4[\phi(\theta)]^{9q}}, \tag{28}
\]

where \( a_i \) and \( b_i \) are determined coefficients only for simplicity. From balancing the lowest order and highest order of \( \phi \) (27-28) we obtain \(-c-8d-3 = -2c-7d-3\), which leads to the limit \( c = d \) and \( p+8d+3 = 2p+7d+3 \), which leads to the limit \( p = q \). For simplicity, we set \( p=q=1 \), we have

\[
F = \frac{a_{-1}[\phi(\theta)]^{-1} + a_0 + a_1[\phi(\theta)]}{b_{-1}[\phi(\theta)]^{-1} + b_0 + b_1[\phi(\theta)]}, \tag{29}
\]

\[
= \frac{a_{-1} + a_0[\phi(\theta)] + a_1[\phi(\theta)]^2}{b_{-1} + b_0[\phi(\theta)] + b_1[\phi(\theta)]^2}
\]
Substituting (29) into (24), equating to zero the coefficients of all powers of $\phi(\theta)$ yields a set of algebraic equations for $a_i$ and $b_i$. By aid Maple we solve this algebraic equations, we get:

$$a_{-1} = \left[ \frac{1}{k_1p_1} (k_1m_1 + k_2m_2 + k_3m_3 - k_1(B^2 + 8AC)(p_2k_1^2 + p_3k_2^2 + p_4k_3^2)) \right] b_{-1},$$

$$a_0 = \left[ \frac{-12BC}{p_1} (p_2k_1^2 + p_3k_2^2 + p_4k_3^2) \right] b_{-1},$$

$$a_1 = \left[ \frac{-12C^2}{p_1} (p_2k_1^2 + p_3k_2^2 + p_4k_3^2) \right] b_{-1}, \quad b_0 = b_1 = 0. \quad (30)$$

The corresponding traveling wave solutions to (1) are:

Case 1: $A \neq 0$, $B \neq 0$, $C \neq 0$.

$$u(t, x, y, z) = \frac{1}{k_1p_1} (k_1m_1 + k_2m_2 + k_3m_3 - k_1(B^2 + 8AC)(p_2k_1^2 + p_3k_2^2 + p_4k_3^2))$$

$$- \frac{12BC}{p_1} (p_2k_1^2 + p_3k_2^2 + p_4k_3^2) \left[ \frac{-B}{2C} + \frac{\sqrt{4AC - B^2}}{2C} \tan \left( \frac{1}{2} (\sqrt{4AC - B^2}(k_1(x-m_1t) + k_2(y-m_2t) + k_3(z-m_3t) + d_0)) \right) \right]$$

$$- \frac{12C^2}{p_1} (p_2k_1^2 + p_3k_2^2 + p_4k_3^2) \left[ \frac{-B}{2C} + \frac{\sqrt{4AC - B^2}}{2C} \tan \left( \frac{1}{2} (\sqrt{4AC - B^2}(k_1(x-m_1t) + k_2(y-m_2t) + k_3(z-m_3t) + d_0)) \right) \right]^2. \quad (31)$$

Case 2: $A = 0$, $B \neq 0$, $C \neq 0$.

$$u(t, x, y, z) = \frac{1}{k_1p_1} (k_1m_1 + k_2m_2 + k_3m_3 - k_1(B^2 + 8AC)(p_2k_1^2 + p_3k_2^2 + p_4k_3^2))$$

$$- \frac{12BC}{p_1} (p_2k_1^2 + p_3k_2^2 + p_4k_3^2) \left[ \frac{-B}{C} \exp(B(k_1(x-m_1t) + k_2(y-m_2t) + k_3(z-m_3t)) + Bd_0) - 1 \right]$$

$$- \frac{12C^2}{p_1} (p_2k_1^2 + p_3k_2^2 + p_4k_3^2) \left[ \frac{-B}{C} \exp(B(k_1(x-m_1t) + k_2(y-m_2t) + k_3(z-m_3t)) + Bd_0) - 1 \right]^2. \quad (32)$$

Case 3: $A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$.

$$u(t, x, y, z) = \frac{1}{k_1p_1} (k_1m_1 + k_2m_2 + k_3m_3 - 2k_1(p_2k_1^2 + p_3k_2^2 + p_4k_3^2))$$

$$- \frac{3}{p_1} (p_2k_1^2 + p_3k_2^2 + p_4k_3^2) \left[ \frac{\tan(k_1(x-m_1t) + k_2(y-m_2t) + k_3(z-m_3t))}{1 \pm \sec(k_1(x-m_1t) + k_2(y-m_2t) + k_3(z-m_3t))} \right]^2. \quad (33)$$
**Case III:** We have two subcases

**Subcase(a)** We take \( \theta = k_1 \zeta_1 + k_2 \zeta_2 + k_3 \zeta_3 \), then equation of case (III) can be written in the form

\[
k_1 F' - p_1 (k_2 m_1 + k_3 m_2) F F' - (k_2 m_1 + k_3 m_2) \]

\[
(p_2 (k_2 m_1 + k_3 m_2)^2 + p_3 k_2^2 + p_4 k_3^2) F''' = 0. \tag{34}
\]

Substituting (29) into (34), equating to zero the coefficients of all powers of \( \phi(\theta) \) yields a set of algebraic equations for \( a_i \) and \( b_i \), i.e \( i = -1,1 \). By aid Maple we solve this algebraic equations, yields

\[
a_1 = b_{-1} = b_0 = 0, \quad B = 0, \quad a_0 \text{ is arbitrary}
\]

\[
k_1 = (k_2 m_1 + k_3 m_2) (a_0 p_1 + 8 A C (k_2 m_1 k_3 m_2 p_2 + k_1^2 (m_1^2 p_2 + p_3) + k_2^2 (m_2^2 p_3 + p_4)))
\]

\[
a_{-1} = \left[ - \frac{12 A^2}{p_1} (k_2 m_1 k_3 m_2 p_2 + k_1^2 (m_1^2 p_2 + p_3) + k_2^2 (m_2^2 p_3 + p_4)) \right] + k_3^2 (m_2^2 p_3 + p_4) b_1. \tag{35}
\]

We apply the related \( \phi(\theta) \) functions for this choice of \( A, B \) and \( C \).

Using the cases in Appendix A wherein \( A = 1, C = 1 \), yields

\[
u(t, x, y, z) = a_0 - \frac{12}{p_1} (k_2 m_1 k_3 m_2 p_2 + k_1^2 (m_1^2 p_2 + p_3) + k_2^2 (m_2^2 p_3 + p_4))
\]

\[
\frac{1}{\tan^2 (k_1 t + k_2 (y - m_1 x) + k_3 (z - m_2 x))} \tag{36}
\]

where \( k_1 = (k_2 m_1 + k_3 m_2) (a_0 p_1 + 8 A C (k_2 m_1 k_3 m_2 p_2 + k_1^2 (m_1^2 p_2 + p_3) + k_2^2 (m_2^2 p_3 + p_4))) \).

For \( A = \frac{1}{2}, C = \frac{1}{2} \), we get the following solutions

\[
u(t, x, y, z) = a_0 + \frac{3}{p_1} (k_2 m_1 k_3 m_2 p_2 + k_1^2 (m_1^2 p_2 + p_3) + k_2^2 (m_2^2 p_3 + p_4))
\]

\[
\left[ \frac{1 \pm \text{sech}(k_1 t + k_2 (y - m_1 x) + k_3 (z - m_2 x))}{\tanh(k_1 t + k_2 (y - m_1 x) + k_3 (z - m_2 x))} \right]^2 \tag{37}
\]

where \( k_1 = (k_2 m_1 + k_3 m_2) (a_0 p_1 - 2 (k_2 m_1 k_3 m_2 p_2 + k_1^2 (m_1^2 p_2 + p_3) + k_2^2 (m_2^2 p_3 + p_4))) \)

**Subcase(b)** Using the scaling transformation to case III

\[
F = e^t \tilde{F}, \quad \zeta_1 = e^{e_1} \tilde{\zeta}_1, \quad \zeta_2 = e^{e_2} \tilde{\zeta}_2, \quad \zeta_3 = e^{e_3} \tilde{\zeta}_3. \tag{38}
\]

Substituting from Eq.(38) into case III we have \( \frac{d^2}{dt^2} \epsilon_1 = -2 \epsilon_2 = -2 \epsilon_3 = \epsilon \).

Then, the characteristic equation can be written as

\[
- \frac{2}{3} \frac{d \zeta_1}{\zeta_1} = - \frac{2}{3} \frac{d \zeta_2}{\zeta_2} = - \frac{2}{3} \frac{d \zeta_3}{\zeta_3} = \frac{dF}{F}. \tag{39}
\]
We get the similarity variables
\[ \eta_1 = \frac{\zeta_2}{\zeta_1}, \quad \eta_2 = \frac{\zeta_3}{\zeta_1}, \quad F = \frac{f}{\zeta_1^2} \] (40)

By substituting into case III we get
\[
\frac{2}{3} f + \frac{1}{3} \eta_1 f_{\eta_1} + \frac{3}{3} \eta_2 f_{\eta_2} + m_1 p_1 f f_{\eta_1} + m_2 p_1 f f_{\eta_2} \\
+ m_1 (m_1^2 p_2 + p_3) f_{\eta_1,\eta_1,\eta_1} + m_2 (3m_1^2 p_2 + p_3) f_{\eta_1,\eta_1,\eta_2} \\
+ m_1 (3m_2^2 p_2 + p_4) f_{\eta_1,\eta_2,\eta_2} + m_2 (2m_2^2 p_2 + p_4) f_{\eta_2,\eta_2,\eta_2} = 0.
\] (41)

By using \( \theta = k_1 \eta_1 + k_2 \eta_2 \), (41) can be written in form
\[
\frac{2}{3} f + \frac{1}{3} \theta f' + p_1 (m_1 k_1 + m_2 k_2) f f'
+ m_1 k_1^2 (m_1^2 p_2 + p_3) f''' + m_2 k_2 k_2 (3m_1^2 p_2 + p_3) f'''
+ m_1 k_2^2 k_1 (3m_2^2 p_2 + p_4) f''' + m_2 k_2^2 (m_2^2 p_2 + p_4) f''' = 0.
\] (42)

To find the solution for the ODE corresponding to this case, we assume that this solution takes the following form
\[ f = a_0 + a_1 \theta + a_2 \theta^2 + \frac{b_1}{\theta} + \frac{b_2}{\theta^2}, \] (43)
where \( a_0, a_1, a_2, b_1 \) and \( b_2 \) are arbitrary constants. Substituting from (43) into (42) and collecting the various powers of \( \theta \) then equating them to zero, we get system of algebraic equations in the constants \( a_0, a_1, a_2, b_1 \) and \( b_2 \). Solving this system with the aid of Maple program, we get the following solutions:
\[ a_0 = a_2 = b_1 = b_2 = 0, \]
\[ k_1 = \frac{-(1 + a_1 p_1 m_2 k_2)}{a_1 p_1 m_1} \] (44)

Then, we have obtained the following new exact solution for (1)
\[ u(t, x, y, z) = \frac{a_2}{t} (k_1 (y - m_1 x) + k_2 (z - m_2 x)). \] (45)

**Cases IV:** We take the transformation \( \theta = k_1 \zeta_1 + k_2 \zeta_2 + k_3 \zeta_3 \), we get
\[
(k_2 m_1 + k_3 m_2) F' - p_1 k_1 FF' \\
- 3k_1 (p_2 k_1^2 + p_3 k_2^2 + p_4 k_3^2) F''' = 0.
\] (46)

To obtain the solution for the ODE corresponding to this case, substituting (29) into (46), equating to zero the coefficients of all powers of \( \phi(\theta) \) yields a set of algebraic equations for \( a_0, a_i, b_i, i=1, 2 \). By aid
Maple we solve this algebraic equations, yields

\[ a_0 = \frac{1}{12p_1k_1A^2}(a_{-1}k_1p_1(B^2 + 8AC) + 12b_1A^2(m_1k_2 + m_2k_3)) \]

\[ p_2 = -\frac{1}{12k_1A^2}(p_1a_{-1}b_1 + 12A^2(p_3k_2^2 + p_4k_3^2)), \quad a_0 = \frac{a_{-1}B}{p_1} \]

\[ b_{-1} = b_0 = 0. \quad (47) \]

Using choices for A, B and C, then we obtain the following exact solutions of (1)

\[ u(t, x, y, z) = \frac{1}{3p_1k_1}(2a_{-1}k_1p_1 + 3b_1(m_1k_2 + m_2k_3)) \]

\[ + \frac{a_{-1}}{b_1 (\tan(k_1x + k_2(y-m_1t) + k_3(z-m_2t)) \pm \sec(k_1x + k_2(y-m_1t) + k_3(z-m_2t))^2,} \quad (48) \]

where \( A = \frac{1}{2}, B = 0, C = \frac{1}{2} \) and \( p_2 = -\frac{1}{3k_1}(\frac{p_1a_{-1}}{b_1} + 3(p_3k_2^2 + p_4k_3^2)). \)

\[ u(t, x, y, z) = \frac{1}{3p_1k_1}(-2a_{-1}k_1p_1 + 3b_1(m_1k_2 + m_2k_3)) \]

\[ + \frac{a_{-1}}{b_1 (\tanh(k_1x + k_2(y-m_1t) + k_3(z-m_2t)) \pm \text{sech}(k_1x + k_2(y-m_1t) + k_3(z-m_2t))^2,} \quad (49) \]

where \( A = \frac{1}{2}, B = 0, C = -\frac{1}{2} \) and \( p_2 = -\frac{1}{3k_1}(\frac{p_1a_{-1}}{b_1} + 3(p_3k_2^2 + p_4k_3^2)). \)

4. Conclusion

Symmetry analysis and modified Exp method were successfully used to obtain new solitary wave solutions for ZK equation. The solutions have physical structures and depend on the real parameters. Finally, new type solutions of Riccati were obtained in family 1-4.
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References


