SOME RESULTS ABOUT A BOUNDARY VALUE PROBLEM ON MIXED CONVECTION

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Abstract. The purpose of this paper is to study the autonomous third order non linear differential equation \( f''' + ff'' + g(f') = 0 \) on \([0, +\infty[\) with \( g(x) = \beta x(x - 1) \) and \( \beta > 1 \), subject to the boundary conditions \( f(0) = a \in \mathbb{R}, f'(0) = b < 0 \) and \( f'(t) \to \lambda \in \{0, 1\} \) as \( t \to +\infty \). This problem arises when looking for similarity solutions to problems of boundary-layer theory in some contexts of fluids mechanics, as mixed convection in porous medium or flow adjacent to a stretching wall. Our goal, here is to investigate by a direct approach this boundary value problem as completely as possible, say study existence or non-existence and uniqueness solutions and the sign of this solutions according to the value of the real parameter \( \beta \).

1. Introduction

In fluid mechanics, the problems are usually governed by systems of partial differential equations. In modeling of boundary layer, this is sometimes possible, and in some cases, the system of partial differential
equations reduces to a systems involving a third order differential equation of the form
\[ f''' + ff'' + g(f') = 0, \] (1.1)
where the function \( g : \mathbb{R} \to \mathbb{R} \) is assumed to be locally Lipschitz.

If \( g(x) = 0 \), the equation is the Blasius equation (1907) see [6], [15]. The case \( g(x) = \beta(x^2 - 1) \) was first given by Falkner and Skan (1931) see [13]. The case \( g(x) = \beta x^2 \), this case occurs in the study of free convection (1966) see [3], [5], [7], [9], [12]. And for \( g(x) = \beta x - 1 \) is the mixed convection (2003) see [1], [2], [4], [8], [10], [11], [14], [16]. In this paper is to investigate this last case with \( \beta > 1 \). We consider the equation
\[ f''' + ff'' + \beta f'(f' - 1) = 0 \] (1.2)
And we associate to equation (1.2) the boundary value problem:
\[
\begin{cases}
    f''' + ff'' + \beta f'(f' - 1) = 0 \\
    f(0) = a, \ a \in \mathbb{R} \\
    f'(0) = b < 0 \\
    f''(t) \to \lambda \text{ as } t \to +\infty
\end{cases}
\]
where \( \lambda \in \{0, 1\} \) and \( \beta > 1 \). This problem arises in the study of mixed convection boundary layer near a semi-infinite vertical plate embedded in a saturated porous medium, with a prescribed power law of the distance from the leading edge for the temperature. The parameter \( \beta \) is a temperature power-law profile and \( b \) is the mixed convection parameter, namely \( b = \frac{Ra}{Pe} - 1 \), with \( Ra \) the Rayleigh number and \( Pe \) the Péclet number. The case where \( a \geq 0, b \geq 0, \beta > 0 \) and \( \lambda \in \{0, 1\} \) was treated by Aïboudi and al.see [1], and for \( a \in \mathbb{R}, b \leq 0, 0 < \beta < 1 \) see [2], and the results obtained generalize the ones of [11]. In [8], Brighi and Hoernel established some results about the existence and uniqueness of convex and concave solution of \((\mathcal{P}_{\beta:a,b,1})\) where \(-2 < \beta < 0 \) and \( b > 0 \). These results can be recovered from [10], where the general equation \( f''' + ff'' + g(f') = 0 \) is studied. In [16], some theoretical results can be found about the problem \((\mathcal{P}_{\beta:0,b,1})\) with \(-2 < \beta < 0 \), and \( b < 0 \). In [14] and [16], the method used by the authors allows them to prove the existence of a convex solution for the case \( a = 0 \) and seems difficult to generalize for \( a \neq 0 \). The problem \((\mathcal{P}_{\beta:a,b,\lambda})\) with \( \beta = 0 \) is the well known Blasius problem. In the following, we note by \( f_c \) a solution of the problem to the initial values below and by \([0,T_c)\) the right maximal interval of its existence:
\[
\begin{cases}
    f''' + ff'' + \beta f'(f' - 1) = 0 \\
    f(0) = a \\
    f'(0) = b \\
    f''(0) = c
\end{cases}
\]
\((\mathcal{P}_{\beta:a,b,c})\)
To solve the boundary value problem \((P_{\beta;a,b,\lambda})\) we will use the shooting method, which consists of finding the values of a reel parameter \(c\) for which the solution of (1.2) satisfying the initial conditions.

2. ON BLASIUS EQUATION

In this section, we recall some results about subsolutions and supersolutions of the Blasius equation. Recall that the so-called Blasius equation is the third order ordinary differential equation \(f''' + ff'' = 0\) i.e Eq.(1.1) with \(g = 0\). Let us notice that, for any \(\tau \in \mathbb{R}\), the function \(h_\tau : t \mapsto \frac{3}{t-\tau}\) is a solution of Blasius equation on each \((-\infty, \tau)\) and \((\tau, +\infty)\). Let \(I \subset \mathbb{R}\) be an interval and \(f : I \rightarrow \mathbb{R}\) be a function.

**Definition 2.1.** We say that \(f\) is a subsolution (resp. a supersolution) of the Blasius equation \(f''' + ff'' = 0\) if \(f\) is of class \(C^3\) and if \(f''' + ff'' \leq 0\) on \(I\) (resp. \(f''' + ff'' \geq 0\) on \(I\)).

**Proposition 2.1.** Let \(t_0 \in \mathbb{R}\). There does not exist no positive concave subsolution of the Blasius equation on the interval \([t_0, +\infty)\).

**Proof.** See [10], Proposition 2.11.

**Proposition 2.2.** Let \(t_0 \in \mathbb{R}\). There does not exist no positive convex supersolution of the Blasius equation on the interval \([t_0, +\infty)\).

**Proof.** See [10], Proposition 2.5.

3. PRELIMINARY RESULTS

**Proposition 3.1.** Let \(f\) be a solution of the equation (1.2) on some maximal interval \(I = (T_-, T_+)\) and \(\beta > 1\).

1. If \(F\) is any anti-derivative of \(f\) on \(I\), then \((f''e^F)' = -\beta f'(f' - 1)e^F\).
2. Assume that \(T_+ = +\infty\) and that \(f'(t) \rightarrow \lambda \in \mathbb{R}\) as \(t \rightarrow +\infty\). If moreover \(f\) is of constant sign at infinity, then \(f''(t) \rightarrow 0\) as \(t \rightarrow +\infty\).
3. If \(T_+ = +\infty\) and if \(f'(t) \rightarrow \lambda \in \mathbb{R}\) as \(t \rightarrow +\infty\), then \(\lambda = 0\) or \(\lambda = 1\).
4. If \(T_+ < +\infty\), then \(f''\) and \(f'\) are unbounded near \(T_+\).
5. If there exists a point \(t_0 \in I\) satisfying \(f''(t_0) = 0\) and \(f'(t_0) = \mu\), where \(\mu = 0\) or \(1\) then for all \(t \in I\), we have \(f(t) = \mu(t - t_0) + f(t_0)\).
6. If \(f'(t) \rightarrow 0\) as \(t \rightarrow +\infty\), then \(f(t)\) does not tend to \(-\infty\) or \(+\infty\) as \(t \rightarrow +\infty\).

**Proof.** The first item follows immediately from equation (1.2). For the proof of items 2-5, see [3], and item 6 see [1].
Proposition 3.2. Let us suppose that $f$ be a solution of equation (1.2) on the maximal interval $I = (T_-, T_+)$

1. Let $H_1 = f'' + f(f' - 1)$ then $H_1' = (1 - \beta)f'(f' - 1)$, for all $t \in I$;

2. Let $H_2 = 3f'' + \beta f^2(2f' - 3)$ then $H_2' = -6f f''$, for all $t \in I$;

3. Let $H_3 = 2f'' - f^2 + (2f' - \beta)f^2$ then $H_3' = 2(2 - \beta)f f''$, for all $t \in I$;

4. Let $H_4 = f'' + ff'$ then $H_4' = (1 - \beta)f^2 + \beta f'$, for all $t \in I$;

5. Let $H_5 = f' + \frac{1}{2}f^2$ then $H_5' = H_4 = f'' + ff'$, for all $t \in I$.

Proof. This statement follows immediately from equation (1.2). \qed

4. The boundary value problem $(P_{\beta;a,b,\lambda})$

Let the boundary value problem $(P_{\beta;a,b,\lambda})$, we are interested here in a concave, convex and convex-concave solutions of a problem $(P_{\beta;a,b,\lambda})$ and there sign. We used shooting method to find these solutions, this method consists of finding the values of a parameter $c \in \mathbb{R}$ for which the solution of $(P_{\beta;a,b,c})$ satisfying the initial conditions $f'(0) = a, f'(0) = b$ and $f''(0) = c$, exists up to infinity and is such that $f'(t) \to \lambda$ as $t \to +\infty$. Define the following sets:

$C_0 = \{c \leq 0 : f''_c \leq 0 \text{ on } [0, T_c]\}$,

$C_1 = \{c > 0 : f'_c \leq 0 \text{ and } f''_c \geq 0 \text{ on } [0, T_c]\}$,

$C_2 = \{c > 0 : \exists t_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f'_c < 0 \text{ on } (0, t_c), f''_c > 0 \text{ on } (t_c, t_c + \varepsilon_c) \text{ and } f''_c > 0 \text{ on } (0, t_c + \varepsilon_c)\}$,

$C_3 = \{c > 0 : \exists s_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f''_c > 0 \text{ on } (0, t_c), f''_c < 0 \text{ on } (s_c, s_c + \varepsilon_c) \text{ and } f'_c < 0 \text{ on } (0, s_c + \varepsilon_c)\}$. 
Remark 4.1. It is easy to prove that $C_0$, $C_1$, $C_2$ and $C_3$ are disjoint nonempty open subsets of $\mathbb{R}$, $C_0 = [ -\infty, 0]$ and $C_1 \cup C_2 \cup C_3 = ]0, +\infty[$ (see Appendix A of [10] with $g(x) = \beta x(x - 1)$ and $\beta > 0$).

Lemma 4.1. Let $\beta > 0$. If $c \in C_0$, then $T_c < +\infty$. Moreover, $f_c$ is concave solution, decreasing and $f'_c(t) \to -\infty$ as $t \to T_c$.

Proof. If $c \in C_0$, the result follows from proposition 3.1 item 1, we have $f''_c(t) < 0$ and $f''_c(t) < 0$ for all $t \in [0, +\infty)$, then $f_c$ is a no positive concave subsolution of the Blasius equation on $[0, +\infty)$ if $a < 0$, and on $[t_0, +\infty)$ such that $f_c(t_0) = 0$ if $a > 0$, with $f'_c(t) \to -\infty$ as $t \to T_c$. If we assume that $T_c = +\infty$, This leads to a contradiction with proposition 3.1, then $T_c < +\infty$. □

Lemma 4.2. Let $\beta > 0$. Then $f_c$ is a convex solution of the boundary value problem $(\mathcal{P}_{\beta,a,b,0})$ if and only if $c \in C_1$.

Proof. See Appendix A of [10] with $g(x) = \beta x(x - 1)$ and $\beta > 0$. □

Lemma 4.3. Let $\beta > 0$. If $c \in C_3$, then $T_c < +\infty$. Moreover, $f_c$ is convex-concave, decreasing and $f'_c(t) \to -\infty$ as $t \to T_c$.

Proof. See [2], lemma 5.3. □

Remark 4.2. From proposition 3.1 items 1,3 and 5, if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $(\mathcal{P}_{\beta,a,b,c})$:

1. $f_c$ is convex and $f'_c(t) \to +\infty$ as $t \to T_c$ (with $T_c \leq +\infty$);
2. there exists a point $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$ and $f'_c(t_0) > 1$;
3. $f_c$ is a convex solution of $(\mathcal{P}_{\beta,a,b,1})$.

The next proposition shows that the case (1) cannot hold.

Proposition 4.1. Let $\beta > 0$. There does not exit $c \geq 0$, such that $f_c$ is convex and $f'_c(t) \to +\infty$ as $t \to T_c$ on its right maximal interval of existence $[0, T_c)$.

Proof. Assume that $f_c$ is convex on its right maximal interval of existence $[0, T_c)$ and $f'_c(t) \to +\infty$ as $t \to T_c$.

There exist $t_0 \in [0, T_0)$, which the function $H_2$ is decreasing for $t > t_0$, this is a contradiction as $t \to T_c$. □

Proposition 4.2. Let $\beta > 1$. If there exist $t_0 \in [0, T_c)$ such that $f'_c(t_0) = 0$ and $f''_c(t_0) < 0$, then for all $t > t_0$, $f''_c(t) < 0$ and $f'_c(t) \neq 0$.

Proof. Let $f_c$ is convex on its right maximal interval of existence $[0, T_c)$, suppose there exist $t_1 > t_0$ such that $f'_c(t_1) = 0$, hence the function $H_4$ is decreasing on $[t_0, t_1]$, therefore $H_4(t_0) > H_4(t_1)$, we have $f''_c(t_0) > f''_c(t_1)$, which yields a contradiction. □
5. The $a < 0$ Case

**Proposition 5.1.** Let $\beta > 1$, the boundary value problem $(P_{\beta,a,b,1})$ has no convex solution.

*Proof.* Let $f_c$ is convex on maximal interval of existence $[0, T_c)$, such that $f'_c(t) \to 0$ as $t \to T_c$, then there exist $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$, the function $H_1$ is creasing for all $t > t_0$, therefore $H_1(t) > H_1(t_0)$ for $t > t_0$, we have $f''_c(t) - f''_c(t_0) > -f_c(t)(f'_c(t) - 1) > 0$, we obtain a contradiction for $t$ large enough because $f''_c(t) \to 0$ and $f_c(t) > 0$. □

**Proposition 5.2.** The boundary value problem $(P_{\beta,a,b,0})$ has no negative convex-concave solution.

*Proof.* Let $f_c$ is convex-concave on maximal interval of existence $[0, T_c)$, such that $f'_c(t) \to 0$ as $t \to T_c$, then there exist $t_0 \in [0, T_c)$ such that $f'_c(t_0) = 0$, the function $H_2$ is creasing for all $t > t_0$, we have $3f''_c(t_0) < H_2(t)$ for all $t > t_0$, $H_2(t) \to 0$ as $t \to +\infty$, a contradiction. □

**Remark 5.1.** If the boundary value problem $(P_{\beta,a,b,0})$ has a convex-concave solution, then this solution changes the sign.

**Lemma 5.1.** If $c \in C_1$, then there exist $c_*$ such that $0 < c < c_*$, $T_c = +\infty$, and the solution $f_c$ is negative on $[0, +\infty)$.

*Proof.* Let $f_c$ is solution on maximal interval of existence $[0, T_c)$, if $c \in C_1$, then $T_c = +\infty$, the function $H_2$ is creasing on $[0, T_c)$, it follows that $3c^2 + \beta b^2(2b - 3) < 0$, we obtain $c < -b\sqrt{\frac{3(3-2b)}{3}}$, and the solution $f_c$ is negative because $a < 0$ and $f'_c < 0$. □

**Lemma 5.2.** If $c \in C_3$, then there exist $c_*$ such that $0 < c < c_*$, $T_c < +\infty$ and the solution $f_c$ is negative on $[0, T_c)$.

*Proof.* If $c \in C_3$, then $f'_c \to -\infty$, and $T_c < +\infty$, other results same proof that lemma 5.1. □

**Remark 5.2.** It follows from lemma 5.1 and lemma 5.2, there exist $c_0 > 0$ such that $c > c_0$, $C_2 \neq \emptyset$ and here the solution $f_c$ is convex-concave.

**Lemma 5.3.** Let $1 < \beta < 2$, if $c \in C_2$ and $f_c$ is a no positive solution on maximal interval of existence $[0, T_c)$, then for all $t \in [0, T_c)$ we have $f_c(t) \leq \max \left\{a, \frac{b}{\sqrt{\beta}}\right\}$, $T_c < +\infty$ and $f'_c(t) \to -\infty$ as $t \to T_c$.

*Proof.* Let $c \in C_2$ and $f_c$ is a no positive solution on maximal interval of existence $[0, T_c)$. From the proposition 3.1, 4.2 and 5.2, there exist $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$.

Moreover, the function $H_3$ is decreasing on $[0, T_c)$, we have $H_3(0) > H_3(t)$, it follows that,

$-b^2 > 2ac - b^2 + (2b - \beta)a^2 > 2f_c(t_0)f''_c(t_0) - \beta f^2_2(t_0) > -\beta f^2_2(t_0)$, we get $f_c(t_0) < \frac{b}{\sqrt{\beta}}$ for all $t \in [0, T_c)$, the conclusion follows from that, for all $t \in [0, T_c)$, if $a < \frac{b}{\sqrt{\beta}}$ we have $f_c(t) \leq f_c(t_0)$ and if $a > \frac{b}{\sqrt{\beta}}$ we have $f_c(t) \leq a$ with $T_c < +\infty$, and $f'_c \to -\infty$ as $t \to T_c$. □
Lemma 5.4. If \( c \in C_2 \), and if there exist \( t_1 \in [0, T_c) \) such that \( f''_c(t_1) = 0 \) and \( f_c(t_1) < 0 \), then \( f'_c(t_1) > \frac{\beta}{2} \).

Proof. If \( c \in C_2 \), there exist \( t_0 \in [0, T_c) \) such that \( f'_c(t_0) = 0 \), \( f_c(t_0) < 0 \), and there exist \( t_1 > t_0 \) such that \( f''_c(t_1) = 0 \), we suppose \( f_c(t_1) < 0 \) and \( f'_c(t_1) < \frac{\beta}{2} \), the function \( H_2 \) is creasing on \([0, t_1)\), we have \( 3f''_c(t_1) < \beta f'_{c_1}^2(1) (2f'_c(t_1) - 3) \), we obtain a contradiction. \( \square \)

Remark 5.3. Thanks to the previous lemma, if we have \( f'_c(t_1) < \frac{\beta}{2} \) and \( f_c \) is convex-concave solution on maximal interval of existence \([0, T_c)\), then \( f_c \) changes the sign.

Lemma 5.5. For \( 1 < \beta < 2 \) and \( b < -1 \), if \( c \in C_2 \) and if there exist \( t_0 \in [0, T_c) \) such that \( f_c(t_0) = 0 \), then \( f'_c(t_0) > 1 \).

Proof. Let \( f_c \) is convex-concave solution on maximal interval of existence \([0, T_c)\), \( 1 < \beta < 2 \) and \( b < -1 \), if there exist \( t_0 \in [0, T_c) \) such that \( f_c(t_0) = 0 \), the function \( H_3 \) is decreasing on \([0, t_0)\), we have \( H_3(0) > H_3(t_0) \), therefore \(-b^2 > -f''_{c_1}(t_0)\), and we obtain \( f'_c(t_0) > -b > 1 \). \( \square \)

Lemma 5.6. If \( c \in C_2 \) and there exist \( t_1 \in [0, T_c) \) such that \( f''_c(t_1) = 0 \) and if \( f_c(t_1) < 0 \), then \( f'_c(t_1) > \frac{-\beta}{1-\beta} \).

Proof. If \( c \in C_2 \), there exist \( t_0 \in [0, T_c) \) such that \( f'_c(t_0) = 0 \) and \( f''_c(t_0) > 0 \), there exist \( t_1 > t_0 \) such that \( f''_c(t_0) = 0 \), we suppose \( f'_c(t_1) < \frac{-\beta}{1-\beta} \), the function \( H_4 \) is creasing on \([t_0, t_1)\), we have \( f''_c(t_0) < f_c(t_1)f'_c(t_1) \), this is a contradiction. \( \square \)

Theorem 5.1. Let \( \beta > 1 \), \( a < 0 \) and \( b < 0 \).

1. The boundary value problem \((P_{\beta,a,b,0})\) has as least one negative convex solution on \([0, +\infty)\).
2. The boundary value problem \((P_{\beta,a,b,1})\) has no convex solution on \([0, +\infty)\).
3. The boundary value problem \((P_{\beta,a,b,\infty})\) has no convex solution on \([0, T_c)\).

Proof. The first result follows from remark 4.1 and lemma 4.2, the second result follows from proposition 5.1 and the third result follows from proposition 4.1. \( \square \)

6. The \( a > 0 \) case

Let \( a, b \in \mathbb{R} \) with \( b < 0 \) and \( a > 0 \). We assume \( \beta > 1 \), and \( f_c \) be a solution of the initial value problem \((P_{\beta,a,b,c})\) on the right maximal interval of existence \([0, T_c), c > 0\).

Before that, and in order to complete the study, let us divide the sets \( C_2 \) and \( C_3 \) into the following two
subsets:
\[ C_{2.1} = \{ c \in C_2; f'_c > 0 \text{ on } [t_c, T_c) \}, \]
\[ C_{2.2} = \{ c \in C_2; \exists s_c > t_c \text{ s.t. } f'_c > 0 \text{ on } [t_c, s_c) \text{ and } f'_c(s_c) = 0 \}, \]
\[ C_{3.1} = \{ c \in C_3; f_c(s_c) < 0 \}, \]
\[ C_{3.2} = \{ c \in C_3; f_c(s_c) > 0 \}. \]

**Proposition 6.1.** If \( c \in C_1 \cup C_2 \cup C_{3.1} \), then \( c > -ab \)

**Proof.** If \( c \in C_1 \), \( T_c = +\infty \), \( f'_c(t) \to 0 \) as \( t \to +\infty \), the function \( H_4 \) is decreasing on \([0, +\infty)\), we have \( c + ab > 0 \), if \( c \in C_2 \cup C_{3.1} \), there exist \( t_0 \in [0, T_c) \) such that \( f'_c(t_0) = 0 \) or \( f_c(t_0) = 0 \), we have \( c + ab \geq f''_c(t_0) > 0 \). \( \square \)

**Remark 6.1.** If \( c \leq -ab \) then \( c \in C_{3.2} \) and \( T_c < +\infty \). Thus \( C_{3.2} \neq \emptyset \) and the convex part of the solution \( f_c \) is positive.

**Proposition 6.2.** If \( c \in C_1 \cup C_{2.1} \) and \( b > -\frac{1}{2}a^2 \), then \( T_c = +\infty \) and the solution \( f_c \) is positive.

**Proof.** Let \( f_c \) solution of the initial value problem \((P_{\beta; a, b, c})\) on the right maximal interval of existence \([0, T_c)\), \( c > 0 \), if \( c \in C_1 \cup C_{2.1} \), thanks to propositions 3.1 and 4.1 it follows that \( T_c = +\infty \), no we suppose there exist \( t_0 \in [0, T_c) \) such that \( f_c(t_0) = 0 \), the function \( H_4 \) is decreasing for all \( t > 0 \), we have \( H_4(t_0) = f''_c(t_0) \), therefore \( H_5 \) is creasing on \([0, t_0)\), we obtain \( b + \frac{1}{2}a^2 < f'_c(t_0) < 0 \), this is a contradiction. \( \square \)

**Remark 6.2.** If \( c \in C_{2.2} \) and \( b > -\frac{1}{2}a^2 \), the solution \( f_c \) is positive on \([0, t_0)\), \( t_0 \) is the point such that \( t_0 > s_c \) with \( f_c(t_0) = 0 \) and \( s_c \) be as in definition of \( C_{2.2} \).

**Lemma 6.1.** Let \( \beta > 1 \) and \( -\frac{1}{2}a^2 < b < 0 \).

If \( f_c \) be solution of the initial value problem \((P_{\beta; a, b, c})\), on the right maximal interval of existence \([0, T_c)\) and if there exist \( t_0 \in [0, T_c) \) such that \( f_c(t_0) = 0 \) and \( f'_c(t_0) < 0 \) then \( f''_c(t_0) < 0 \).

**Proof.** For contradiction, let us that \( t_0 \in [0, T_c) \) with \( f_c(t_0) = 0 \) and \( f'_c(t_0) < 0 \), the function \( H_4 \) is decreasing on \([0, t_0)\) and \( H_4(t_0) = f''_c(t_0) > 0 \) then for all \( t \in [0, t_0) \), \( H_4 > 0 \), and \( H_5 \) is creasing on \([0, t_0)\), we have \( b + \frac{1}{2}a^2 < f'_c(t_0) < 0 \), this is a contradiction. \( \square \)

**Proposition 6.3.** Let \( 1 < \beta < 2 \), \( b > -\frac{1}{2}a^2 \) and \( c \in C_{2.2} \). For all \( t \in [0, t_0) \), one has \( f_c(t) < \sqrt{\frac{b^2 + (2b - \beta)a^2}{\beta}} \).

**Proof.** Let \( c \in C_{2.2} \) and \( s_c \) be as in the definition of \( C_{2.2} \), the function \( H_3 \) is creasing on \([0, s_c)\), we have:
\[ -b^2 + (2b - \beta)a^2 < 2ac - b^2 + (2b - \beta)a^2 < 2f_c(s_c)f'_c(s_c) - \beta f''_c(s_c) < -\beta f'_c(s_c), \]
which implies that \( f_c(s_c) < \sqrt{\frac{b^2 + (\beta - 2b)\alpha^2}{\beta}} \). From the proposition 4.2, the conclusion follows from that , for all \( t \in [0, T_c) \), we have \( f_c(t) \leq f_c(s_c) \).

\[ \square \]

**Lemma 6.2.** If \( c \in C_1 \cup C_{2.1} \) and \( b > -\frac{1}{2}a^2 \). Then \( T_c = +\infty \) and there exist \( c_* > 0 \) such that \( c > c_* \).

**Proof.** Let \( c \in C_1 \cup C_{2.1}, \) and \( b > -\frac{1}{2}a^2 \). By the definition of \( C_1 \) and \( C_{2.1} \), thanks to proposition 6.2, we have \( T_c = +\infty \) and \( f'_c \) is bounded. Otherwise the function \( H_2 \) is decreasing for \( t > 0 \), we obtain \( 3c^2 + \beta b^2(2b - 3) > 0 \), which implies that \( c > -b\sqrt{\frac{\beta(3-2b)}{3}} \). \[ \square \]

**Remark 6.3.** There exist \( c_* > 0 \), if \( c < c_* \), then there exist \( t_0 \in [0, T_c) \) such that \( f_c(t_0) = 0, f'_c(t_0) < 0 \) and \( f''_c(t_0) < 0 \) say that \( c \in C_{2.2} \cup C_{3.2} \), since if \( c \in C_{2.1} \) then \( T_c = +\infty \).

Let us divide the set \( C_{2.1} \) into the following two subsets:

\[ C_{2.1.1} = \{ c \in C_{2.1}; f'_c(t) \to 0 \text{ as } t \to +\infty \}, \]

\[ C_{2.1.2} = \{ c \in C_{2.1}; f'_c(t) \to 1 \text{ as } t \to +\infty \}. \]

**Proposition 6.4.** Let \( 1 < \beta < 2, \) if \( c \in C_1 \cup C_3 \cup C_{2.2} \cup C_{2.1.1} \). Then there exist \( c_* > 0 \) such that \( c < c_* \).

**Proof.** Let \( f_c \) solution of the initial value problem \( (P_{\beta:a,b,c}) \) on the right maximal interval of existence \( [0, T_c) \), either there exist \( t_0 \in [0, T_c) \) such that \( f_c(t_0) = 0 \) or \( f'_c(t_0) = 0 \) if \( T_c < +\infty \), and if \( T_c = +\infty \), we have \( f'_c(t) \to 0 \text{ as } t \to +\infty \), from proposition 3.1 item 6, it follows that the function \( H_3 \) is creasing on \([0, t_0)\) or \([0, +\infty)\), we get \( 2ac - b^2 + (2b - \beta)a^2 < 0 \), which implies that \( c < \frac{b^2 + (\beta - 2b)a^2}{2a} \). \[ \square \]

**Remark 6.4.** From proposition 6.4 there exist \( c_* > 0 \), such that for \( c \geq c_* \), then \( c \in C_{2.1.2} \). Thus \( C_{2.1.2} \neq \emptyset \).

**Corollary 6.1.** If \( 1 < \beta < 2, a > 0, b < 0 \) and \( b > -\frac{1}{2}a^2 \), then the problem \( (P_{\beta:a,b,1}) \) has as least one positive convex or positive convex-concave solution on \([0, +\infty)\).

**Proof.** This follows immediately from remark 6.4, lemma 6.2 and proposition 6.3. \[ \square \]

**Theorem 6.1.** Let \( \beta > 1, a > 0 \) and \( b < 0 \).

1. The boundary value problem \( (P_{\beta:a,b,0}) \) has as least one convex solution on \([0, +\infty)\) if in addition we have \( b > -\frac{1}{2}a^2 \) it will be no negative convex solution.

2. The boundary value problem \( (P_{\beta:a,b,-\infty}) \) has infinity convex-concave solutions on the maximal interval of existence \([0, T_c)\) with \( T_c < +\infty \), if in addition we have \( b > -\frac{1}{2}a^2 \) the convex part of these solutions will be no negative.

3. The boundary value problem \( (P_{\beta:a,b,\infty}) \) has no convex solution on \([0, T_c)\).
Proof. The first result follows from remark 4.1, lemma 4.2 and proposition 6.2, the second result follows from proposition 3.1, proposition 4.2 and remark 6.1, and the third result follows from proposition 4.1. □

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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