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EXPLICIT AND IMPLICIT CRANDALL'S SCHEME FOR THE HEAT EQUATION WITH NONLOCAL NONLINEAR CONDITIONS

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Abstract. In this paper, explicit and implicit Crandall's formulas are applied for finding the solution of the one-dimensional heat equation with nonlinear nonlocal boundary conditions. The integrals in the boundary equations are approximated by the composite Simpson quadrature rule. Here nonlinear terms are approximated by Richtmyer's linearization method. Finally, some numerical examples are given to show the effectiveness of the proposed method.

1. Introduction

This paper is concerned with the numerical solution of the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad 0 < x < 1, \qquad 0 < t \le T, \tag{1.1}$$

with the initial condition

$$u(x,0) = \varphi(x), \qquad 0 < x < 1, \tag{1.2}$$

and the nonlinear nonlocal boundary conditions

$$u(0,t) = \int_0^1 p(x,t)\, u^{\gamma}(x,t)\, dx + E(t), \qquad 0 < t \le T, \tag{1.3}$$

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$$u\left(1,t\right)=\int_0^1 q\left(x,t\right)u^{\gamma}\left(x,t\right)dx+G\left(t\right), \qquad 0<t\leq T, \tag{1.4}$$

where $f,\ \varphi,\ p,\ q,\ G$ and E are known functions, $u\left(x,t\right)$ is unknown function to be determined.

This kind of nonlocal boundary-value problem (with $\gamma=1$) occur in many fields of science and engineering, especially in thermoelasticity [7,8,10] thermodynamics [9], heat conduction [5,6,17,28].

A lot of effort has been devoted in the past few years to the study of parabolic initial-boundary value problems which involve nonlocal boundary conditions of the type :

$$u\left(0,t\right)=\int_0^1 p\left(x,t\right)u\left(x,t\right)dx+E\left(t\right), \qquad 0<t\leq T,$$

$$u\left(1,t\right)=\int_0^1 q\left(x,t\right)u\left(x,t\right)dx+G\left(t\right), \qquad 0<t\leq T,$$

the numerical solution has been considered in several papers [1,2,4,11–13,15,16,18,19,21–24,26–28] . Much less effort is given to the problem with nonlocal nonlinear type boundary conditions (3) and (4). Recently, [3] proposed the implicit difference scheme for the solution of the heat equation with nonlinear nonlocal boundary condition. Therefore this work is aimed at producing a very efficient techniques for solving the heat equation with nonlinear nonlocal boundary condition.

The paper is organized as follows. The explicit and implicit Crandall's methods are employed for solving the one-dimensional heat equation with nonlinear nonlocal boundary value problem (1.1)-(1.4) in Section 2 and Section 3, respectively. To support our findings, we present results of numerical experiments in Section 4. The conclusion is given in Section 5.

In [29], the authors consider to solve one-dimensional diffusion equation with nonlinear nonlocal boundary conditions (1.1)-(1.4), by using the Forward time centered space (FTCS-NNC), Dufort–Frankel scheme (DFS-NNC), Backward time centered space (BTCS-NNC), Crank-Nicholson method (CNM-NNC).

Crandall's formulas

In order to describe our method, we introduce the following notation. First, we take a positive integers N and M. We divide the intervals $[0,1]$ and $[0,T]$ into M and N subintervals of equal lengths $h=1/M$ and $k=T/N$, respectively. By u_i^n, we denote the approximation to u at the i^{th} grid-point and n^{th} time step. The Grid point $(x_i,\ t_n)$ are given by $x_i=ih$, $i=0,\ 1,\ 2,\ \ldots,\ M$, $t_n=nk$, $n=0,\ 1,\ 2,\ \ldots,\ N$.
The notations $u_i^n,\ f_i^n,\ p_i^n,\ q_i^n,\ E^n$ and G^n are used for the finite difference approximations of $u(x_i,t_n)$, $f(x_i,t_n)$, $p(x_i,t_n)$, $q(x_i,t_n)$, $E(t_n)$ and $G(t_n)$, respectively.
If we consider the modified equivalent equation of the Crandall's formula, the scheme shown in [20]. This scheme for the equation (1.1) can be written as:

$$
\begin{aligned}
(1.5)\quad & \frac{u_i^{n+1}-u_i^n}{k} - \left[\left(\frac{1}{2}+\frac{1}{12r}\right)\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2}\right. \\
& \left. +\left(\frac{1}{2}-\frac{1}{12r}\right)\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{h^2}\right] \\
& = c_{00}f_{i-1}^{n+1}+c_{01}f_i^{n+1}+c_{00}f_{i+1}^{n+1}+c_{10}f_{i-1}^n+c_{11}f_i^n+c_{10}f_{i+1}^n,
\end{aligned}
$$

for $i=1,2,...,M-1$, $n=0,1,...,N$, and $r=k/h^2$, where c_{00}, c_{01}, c_{10}, c_{11} are coefficient to be determined . By using Taylor series in (1.5), we obtain the local truncation error :

$$
\begin{aligned}
& -\left(2c_{00}+c_{01}+2c_{10}+c_{11}-1\right)f+\frac{1}{12}\left[-6\left(2c_{00}+2c_{10}-\frac{1}{12}\right)f_{xx}\right. \\
& \left. -6\left(4c_{00}+2c_{01}-1\right)rf_t\right]h^2+O(h^4).
\end{aligned}
$$

In order to achieve the fourth order, it is necessary that coefficients $c_{00}, c_{01}, c_{10}, c_{11}$ would satisfy conditions

$$
\begin{aligned}
2c_{00}+c_{01}+2c_{10}+c_{11}-1 &= 0, \\
2c_{00}+2c_{10}-\frac{1}{6} &= 0, \\
4c_{00}+2c_{01}-1 &= 0.
\end{aligned}
$$

We have chosen $c_{00}=\frac{1}{12}$, $c_{10}=0$, $c_{01}=\frac{1}{3}$ and $c_{11}=\frac{1}{2}$ in our computation.

After some rearrangement, the equation (1.5) becomes :

$$
\begin{aligned}
(1.6)\quad & (1-6r)u_{i-1}^{n+1}+(10+12r)u_i^{n+1}+(1-6r)u_{i+1}^{n+1} \\
& = (1+6r)u_{i-1}^n+(10-12r)u_i^n+(1+6r)u_{i+1}^n \\
& +12k\left(\tfrac{1}{12}f_{i-1}^{n+1}+\tfrac{1}{3}f_i^{n+1}+\tfrac{1}{12}f_{i+1}^{n+1}+\tfrac{1}{2}f_i^n\right),
\end{aligned}
$$

for $i=1,2,...,M-1$, $n=0,1,...,N$.

2. The Explicit Crandall's Scheme (ECS)

If $r=\dfrac{1}{6}$, the Crandall's scheme (1.6) becomes explicite :

$$
\begin{aligned}
(2.1)\quad & 12u_i^{n+1} = 2u_{i-1}^n+8u_i^n+2u_{i+1}^n \\
& +12k\left(\frac{1}{12}f_{i-1}^{n+1}+\frac{1}{3}f_i^{n+1}+\frac{1}{12}f_{i+1}^{n+1}+\frac{1}{2}f_i^n\right),
\end{aligned}
$$

This scheme can be written as:

$$
\begin{aligned}
(2.2)\quad & u_i^{n+1} = \frac{1}{6}u_{i-1}^n+\frac{2}{3}u_i^n+\frac{1}{6}u_{i+1}^n \\
& +k\left(\frac{1}{12}f_{i-1}^{n+1}+\frac{1}{3}f_i^{n+1}+\frac{1}{12}f_{i+1}^{n+1}+\frac{1}{2}f_i^n\right)
\end{aligned}
$$

for $i = 1, 2, ..., M-1$, $n = 0, 1, ..., N$.

We still have to determinates two unknowns u_0 and u_{M+1}, for this we approximate integrals in (1.3) and (1.4) numerically by the composite Simpson quadrature formula (We have chosen this approximation since it is of the same, fourth-order of accuracy in space as the methods used for the interior part of the problem) which requires M to be even. Letting $M = 2m$, yields

$$\begin{aligned} u_0^{n+1} &= u\left(0, t^{n+1}\right) = \textstyle\int_0^1 p\left(x, t^{n+1}\right) u\left(x, t^{n+1}\right) dx \\ &= \frac{h}{3}\left(p_0^{n+1}\left(u_0^{n+1}\right)^{\gamma} + 4\textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1}\left(u_{2i-1}^{n+1}\right)^{\gamma}\right. \\ &\left. +2\textstyle\sum_{i=1}^{M/2-1} p_{2i}^{n+1}\left(u_{2i}^{n+1}\right)^{\gamma} + p_M^{n+1}\left(u_M^{n+1}\right)^{\gamma}\right) + E^{n+1} + o(h^4), \end{aligned} \tag{2.3}$$

$$\begin{aligned} u_M^{n+1} &= u\left(1, t^{n+1}\right) = \textstyle\int_0^1 q\left(x, t^{n+1}\right) u\left(x, t^{n+1}\right) dx \\ &= \frac{h}{3}\left(q_0^{n+1}\left(u_0^{n+1}\right)^{\gamma} + 4\textstyle\sum_{i=1}^{M/2} q_{2i-1}^{n+1}\left(u_{2i-1}^{n+1}\right)^{\gamma}\right. \\ &\left. +2\textstyle\sum_{i=1}^{M/2-1} q_{2i}^{n+1}\left(u_{2i}^{n+1}\right)^{\gamma} + q_M^{n+1}\left(u_M^{n+1}\right)^{\gamma}\right) + G^{n+1} + o(h^4). \end{aligned} \tag{2.4}$$

Thus, we can write

$$\begin{aligned} &3u_0^{n+1} - hp_0^{n+1}(u_0^{n+1})^{\gamma} - hp_M^{n+1}(u_M^{n+1})^{\gamma} \\ &= 4\textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1}(u_{2i-1}^{n+1}t)^{\gamma} + 2\textstyle\sum_{i=1}^{M/2-1} p_{2i}^{n+1}(u_{2i}^{n+1})^{\gamma} + 3E^{n+1}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &-hq_0^{n+1}(u_0^{n+1})^{\gamma} + 3u_M^{n+1} - hq_M^{n+1}(u_M^{n+1})^{\gamma} \\ &= 4\textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1}(u_{2i-1}^{n+1}t)^{\gamma} + 2\textstyle\sum_{i=1}^{M/2-1} p_{2i}^{n+1}(u_{2i}^{n+1})^{\gamma} + 3G^{n+1}. \end{aligned} \tag{2.6}$$

By applying the Taylor's expansion

$$\begin{aligned} \left(u_i^{n+1}\right)^{\gamma} &= (u_i^n)^{\gamma} + k\left((u_i^n)_t\right)^{\gamma} + ... \\ &= (u_i^n)^{\gamma} + k\gamma\left(u_i^n\right)^{\gamma-1}\left(\frac{u_i^{n+1} - u_i^n}{k}\right) + ... \\ &= (u_i^n)^{\gamma} + \gamma\left(u_i^n\right)^{\gamma-1}\left(u_i^{n+1} - u_i^n\right) + ... \end{aligned}$$

Hence to terms of order k,

$$\left(u_i^{n+1}\right)^{\gamma} \approx \gamma\left(u_i^n\right)^{\gamma-1}\left(u_i^{n+1}\right) + (1-\gamma)\left(u_i^n\right)^{\gamma}, \tag{2.7}$$

a result which replace the non-linear unknown $\left(u_i^{n+1}\right)^{\gamma}$ by approximation linear in u_i^{n+1} (the Richtmyer's linearization method [25]).

Substituting (2.7) for $i = 0$ and $i = M$ in (2.4) and (2.5), we have

$$\begin{aligned} &\left(3 - h\gamma p_0^{n+1}(u_0^n)^{\gamma-1}\right) u_0^{n+1} - hp_M^{n+1}\gamma(u_M^n)^{\gamma-1}u_M^{n+1} \\ &= 4\textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1}(u_{2i-1}^{n+1}t)^{\gamma} + 2\textstyle\sum_{i=1}^{M/2-1} p_{2i}^{n+1}(u_{2i}^{n+1})^{\gamma} \\ &+h(1-\gamma)p_0^{n+1}(u_0^n)^{\gamma} + h(1-\gamma)p_M^{n+1}(u_M^n)^{\gamma} + 3E^{n+1}, \end{aligned} \tag{2.8}$$

(2.9)
$$\begin{aligned} &-hq_0^{n+1}\gamma(u_0^n)^{\gamma-1}u_0^{n+1} + (3 - h\gamma q_M^{n+1}(u_M^n)^{\gamma-1})u_M^{n+1} \\ &= 4\textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1}(u_{2i-1}^{n+1}t)^\gamma + 2\sum_{i=1}^{M/2-1} p_{2i}^{n+1}(u_{2i}^{n+1})^\gamma \\ &+h(1-\gamma)q_0^{n+1}(u_0^n)^\gamma + h(1-\gamma)q_M^{n+1}(u_M^n)^\gamma + 3G^{n+1}, \end{aligned}$$

Hence we have:

(2.10)
$$u_0^{n+1} = \frac{z_1(3 - h\gamma q_M^{n+1}(u_M^n)^{\gamma-1}) + z_2 h p_M^{n+1}\gamma(u_M^n)^{\gamma-1})}{Y},$$

(2.11)
$$u_M^{n+1} = \frac{z_2(3 - h\gamma p_0^{n+1}(u_0^n)^{\gamma-1}) + z_1 h q_0^{n+1}\gamma(u_0^n)^{\gamma-1})}{Y},$$

where

(2.12)
$$\begin{aligned} z_1 &= 4\textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1}(u_{2i-1}^{n+1}t)^\gamma + 2\sum_{i=1}^{M/2-1} p_{2i}^{n+1}(u_{2i}^{n+1})^\gamma \\ &+h(1-\gamma)p_0^{n+1}(u_0^n)^\gamma + h(1-\gamma)p_M^{n+1}(u_M^n)^\gamma + 3E^{n+1}, \end{aligned}$$

(2.13)
$$\begin{aligned} z_2 &= 4\textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1}(u_{2i-1}^{n+1}t)^\gamma + 2\sum_{i=1}^{M/2-1} p_{2i}^{n+1}(u_{2i}^{n+1})^\gamma \\ &+h(1-\gamma)q_0^{n+1}(u_0^n)^\gamma + h(1-\gamma)q_M^{n+1}(u_M^n)^\gamma + 3G^{n+1}, \end{aligned}$$

and

(2.14)
$$\begin{aligned} Y &= (3 - h\gamma q_M^{n+1}(u_M^n)^{\gamma-1})(3 - h\gamma p_0^{n+1}(u_0^n)^{\gamma-1}) \\ &-h^2\gamma^2 q_0^{n+1}(u_0^n)^{\gamma-1}p_M^{n+1}(u_M^n)^{\gamma-1} \neq 0 \end{aligned}$$

$Y \neq 0$ for sufficiently small h.

3. The Implicit Crandall's Scheme (ICS)

If $r \neq \dfrac{1}{6}$, the Crandall's scheme (1.6) is implicite :

(3.1)
$$\begin{aligned} &(1-6r)u_{i-1}^{n+1} + (10+12r)u_i^{n+1} + (1-6r)u_{i+1}^{n+1} \\ &= (1+6r)u_{i-1}^n + (10-12r)u_i^n + (1+6r)u_{i+1}^n \\ &+12k\left(\frac{1}{12}f_{i-1}^{n+1} + \frac{1}{3}f_i^{n+1} + \frac{1}{12}f_{i+1}^{n+1} + \frac{1}{2}f_i^n\right), \end{aligned}$$

for $i = 1, 2, ..., M-1$, $n = 0, 1, ..., N$.

Equation (3.1) presents $M-1$ linear equations in $M+1$ unknowns $u_0, u_1, \cdots, u_M$. In order to solve the linear system, we need two more equations, that we can obtain approximating the integrals in (1.3) and

(1.4) numerically by the fourth-order Simpson's composite formula (which requires M to be even). Letting $M = 2m$, yields

$$
\begin{aligned}
u_0^{n+1} &= u\left(0, t^{n+1}\right) = \int_0^1 p\left(x, t^{n+1}\right) u\left(x, t^{n+1}\right) dx \\
&= \frac{h}{3}\Big(p_0^{n+1}\left(u_0^{n+1}\right)^\gamma + 4\sum_{i=1}^{M/2} p_{2i-1}^{n+1}\left(u_{2i-1}^{n+1}\right)^\gamma \\
&\quad +2\sum_{i=1}^{M/2-1} p_{2i}^{n+1}\left(u_{2i}^{n+1}\right)^\gamma + p_M^{n+1}\left(u_M^{n+1}\right)^\gamma\Big) + E^{n+1} + o(h^4),
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
u_M^{n+1} &= u\left(1, t^{n+1}\right) = \int_0^1 q\left(x, t^{n+1}\right) u\left(x, t^{n+1}\right) dx \\
&= \tfrac{h}{3}\Big(q_0^{n+1}\left(u_0^{n+1}\right)^\gamma + 4\sum_{i=1}^{M/2} q_{2i-1}^{n+1}\left(u_{2i-1}^{n+1}\right)^\gamma \\
&\quad +2\sum_{i=1}^{M/2-1} q_{2i}^{n+1}\left(u_{2i}^{n+1}\right)^\gamma + q_M^{n+1}\left(u_M^{n+1}\right)^\gamma\Big) + G^{n+1} + o(h^4).
\end{aligned} \tag{3.3}
$$

By applying the Richtmyer's linearization method (2.7) in (3.2) and (3.3), we get

$$
\begin{aligned}
u_0^{n+1} = {} & \tfrac{h}{3}\Big(p_0^{n+1}\Big(\gamma\left(u_0^n\right)^{\gamma-1}\left(u_0^{n+1}\right) + (1-\gamma)\left(u_0^n\right)^\gamma\Big) \\
& +4\sum_{i=1}^{M/2} p_{2i-1}^{n+1}\Big(\gamma\left(u_{2i-1}^n\right)^{\gamma-1}\left(u_{2i-1}^{n+1}\right) + (1-\gamma)\left(u_{2i-1}^n\right)^\gamma\Big) \\
& +2\sum_{i=1}^{M/2-1} p_{2i}^{n+1}\Big(\gamma\left(u_{2i}^n\right)^{\gamma-1}\left(u_{2i}^{n+1}\right) + (1-\gamma)\left(u_{2i}^n\right)^\gamma\Big)\Big) \\
& +\tfrac{h}{3} p_M^{n+1}\Big(\gamma\left(u_M^n\right)^{\gamma-1}\left(u_M^{n+1}\right) + (1-\gamma)\left(u_M^n\right)^\gamma\Big) + E^{n+1},
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
u_M^{n+1} = {} & \tfrac{h}{3}\Big(q_0^{n+1}\Big(\gamma\left(u_0^n\right)^{\gamma-1}\left(u_0^{n+1}\right) + (1-\gamma)\left(u_0^n\right)^\gamma\Big) \\
& +4\sum_{i=1}^{M/2} q_{2i-1}^{n+1}\Big(\gamma\left(u_{2i-1}^n\right)^{\gamma-1}\left(u_{2i-1}^{n+1}\right) + (1-\gamma)\left(u_{2i-1}^n\right)^\gamma\Big) \\
& +2\sum_{i=1}^{M/2-1} q_{2i}^{n+1}\Big(\gamma\left(u_{2i}^n\right)^{\gamma-1}\left(u_{2i}^{n+1}\right) + (1-\gamma)\left(u_{2i}^n\right)^\gamma\Big)\Big) \\
& +\tfrac{h}{3} q_M^{n+1}\Big(\gamma\left(u_M^n\right)^{\gamma-1}\left(u_M^{n+1}\right) + (1-\gamma)\left(u_M^n\right)^\gamma\Big) + G^{n+1}.
\end{aligned} \tag{3.5}
$$

Thus, we can write (3.4) and (3.5) as follows

$$
\begin{aligned}
& \Big(\gamma h p_0^{n+1}\left(u_0^n\right)^{\gamma-1} - 3\Big) u_0^{n+1} + 4\gamma h\sum_{i=1}^{M/2} p_{2i-1}^{n+1}\left(u_{2i-1}^n\right)^{\gamma-1}\left(u_{2i-1}^{n+1}\right) \\
& +2\gamma h\sum_{i=1}^{M/2-1} p_{2i}^{n+1}\left(u_{2i}^n\right)^{\gamma-1}\left(u_{2i}^{n+1}\right) + \gamma h p_M^{n+1}\left(u_M^n\right)^{\gamma-1}\left(u_M^{n+1}\right) \\
& = (\gamma-1) h p_0^{n+1}\left(u_0^n\right)^\gamma + 4(\gamma-1) h\sum_{i=1}^{M/2} p_{2i-1}^{n+1}\left(u_{2i-1}^n\right)^\gamma \\
& +2(\gamma-1) h\sum_{i=1}^{M/2-1} p_{2i}^{n+1}\left(u_{2i}^n\right)^\gamma + (\gamma-1) h p_M^{n+1}\left(u_M^n\right)^\gamma - 3E^{n+1},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
& \gamma h q_0^{n+1}\left(u_0^n\right)^{\gamma-1} u_0^{n+1} + 4\gamma h\sum_{i=1}^{M/2} q_{2i-1}^{n+1}\left(u_{2i-1}^n\right)^{\gamma-1}\left(u_{2i-1}^{n+1}\right) \\
& +2\gamma h\sum_{i=1}^{M/2-1} q_{2i}^{n+1}\left(u_{2i}^n\right)^{\gamma-1}\left(u_{2i}^{n+1}\right) + \Big(\gamma h q_M^{n+1}\left(u_M^n\right)^{\gamma-1} - 3\Big)\left(u_M^{n+1}\right) \\
& = (\gamma-1) h q_0^{n+1}\left(u_0^n\right)^\gamma + 4(\gamma-1) h\sum_{i=1}^{M/2} q_{2i-1}^{n+1}\left(u_{2i-1}^n\right)^\gamma \\
& +2(\gamma-1) h\sum_{i=1}^{M/2-1} q_{2i}^{n+1}\left(u_{2i}^n\right)^\gamma + (\gamma-1) h q_M^{n+1}\left(u_M^n\right)^\gamma - 3G^{n+1},
\end{aligned} \tag{3.7}
$$

then, we have

$$
a_0^n u_0^{n+1} + a_1^n u_1^{n+1} + a_2^n u_2^{n+1} + \ldots + a_{M-1}^n u_{M-1}^{n+1} + a_M^n u_M^{n+1} = L_M^n, \tag{3.8}
$$

where

$$\text{(3.9)} \qquad \begin{cases} a_0^n = \gamma h p_0^{n+1} \left(u_0^n\right)^{\gamma-1} - 3 \\ a_M^n = \gamma h p_M^{n+1} \left(u_M^n\right)^{\gamma-1} \\ a_{2i+1}^n = 4\gamma h p_{2i+1}^{n+1} \left(u_{2i+1}^n\right)^{\gamma-1} & i = 0, 1, 2, \cdots, \frac{M}{2} - 1, \\ a_{2i}^n = 2\gamma h p_{2i}^{n+1} \left(u_{2i}^n\right)^{\gamma-1} & i = 1, 2, \cdots, \frac{M}{2} - 1, \end{cases}$$

and

$$\text{(3.10)} \qquad \begin{aligned} L_M^n = (\gamma-1)\, h p_0^{n+1} \left(u_0^n\right)^{\gamma} + 4(\gamma-1)\, h \textstyle\sum_{i=1}^{M/2} p_{2i-1}^{n+1} \left(u_{2i-1}^n\right)^{\gamma} \\ +2(\gamma-1)\, h \textstyle\sum_{i=1}^{M/2-1} p_{2i}^{n+1} \left(u_{2i}^n\right)^{\gamma} + (\gamma-1)\, h p_M^{n+1} \left(u_M^n\right)^{\gamma}, \end{aligned}$$

and also

$$\text{(3.11)} \qquad b_0^n u_0^{n+1} + b_1^n u_1^{n+1} + b_2^n u_2^{n+1} + ... + b_{M-1}^n u_{M-1}^{n+1} + b_M^n u_M^{n+1} = K_M^n,$$

where

$$\text{(3.12)} \qquad \begin{cases} b_0^n = \gamma h q_0^{n+1} \left(u_0^n\right)^{\gamma-1} \\ b_M^n = \gamma h q_M^{n+1} \left(u_M^n\right)^{\gamma-1} - 3 \\ b_{2i+1}^n = 4\gamma h q_{2i+1}^{n+1} \left(u_{2i+1}^n\right)^{\gamma-1} & i = 0, 1, 2, \cdots, \frac{M}{2} - 1, \\ b_{2i}^n = 2\gamma h q_{2i}^{n+1} \left(u_{2i}^n\right)^{\gamma-1} & i = 1, 2, \cdots, \frac{M}{2} - 1, \end{cases}$$

and

$$\text{(3.13)} \qquad \begin{aligned} K_M^n = (\gamma-1)\, h q_0^{n+1} \left(u_0^n\right)^{\gamma} + 4(\gamma-1)\, h \textstyle\sum_{i=1}^{M/2} q_{2i-1}^{n+1} \left(u_{2i-1}^n\right)^{\gamma} \\ +2(\gamma-1)\, h \textstyle\sum_{i=1}^{M/2-1} q_{2i}^{n+1} \left(u_{2i}^n\right)^{\gamma} + (\gamma-1)\, h q_M^{n+1} \left(u_M^n\right)^{\gamma}, \end{aligned}$$

Combining (3.8), (3.11), with (3.1) yields an $(M+1)\times(M+1)$ linear system of equations. We write the system in the matrix form

$$\text{(3.14)} \qquad A^{n+1} U^{n+1} = B^{n+1},$$

which

$$A^{n+1} = \begin{pmatrix} a_0^n & a_1^n & a_2^n & a_3^n & & a_{M-2}^n & a_{M-1}^n & a_M^n \\ 1-6r & 10+12r & 1-6r & 0 & & \cdots & & 0 \\ 0 & 1-6r & 10+12r & 1-6r & & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & & & 1-6r & 10+12r & 1-6r \\ b_0^n & b_1^n & b_2^n & & & b_{M-2}^n & b_{M-1}^n & b_M^n \end{pmatrix},$$

$$U^{n+1} = \begin{pmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{M-1}^{n+1} \\ u_M^{n+1} \end{pmatrix},$$

$$B^{n+1} = \begin{pmatrix} (\gamma-1)\,h\left(p_0^{n+1}\,(u_0^n)^\gamma + 4\sum_{i=1}^{M/2} p_{2i-1}^{n+1}\,(u_{2i-1}^n)^\gamma + 2\sum_{i=1}^{M/2-1} p_{2i}^{n+1}\,(u_{2i}^n)^\gamma + p_M^{n+1}\,(u_M^n)^\gamma\right) - 3E^{n+1} \\ (1+6r)\,u_0^n + (10-12r)u_1^n + (1+6r)\,u_2^n + 12k(\frac{1}{12}f_0^{n+1} + \frac{1}{3}f_1^{n+1} + \frac{1}{12}f_2^{n+1} + \frac{1}{2}f_1^n) \\ \vdots \\ (1+6r)\,u_{M-2}^n + (10-12r)u_{M-1}^n + (1+6r)\,u_M^n + 12k(\frac{1}{12}f_{M-2}^{n+1} + \frac{1}{3}f_{M-1}^{n+1} + \frac{1}{12}f_M^{n+1} + \frac{1}{2}f_{M-1}^n) \\ (\gamma-1)\,h\left(q_0^{n+1}\,(u_0^n)^\gamma + 4\sum_{i=1}^{M/2} q_{2i-1}^{n+1}\,(u_{2i-1}^n)^\gamma + 2\sum_{i=1}^{M/2-1} q_{2i}^{n+1}\,(u_{2i}^n)^\gamma + q_M^{n+1}\,(u_M^n)^\gamma\right) - 3G^{n+1} \end{pmatrix}$$

where $a_0^n, a_1^n, a_2^n, ..., a_{M-1}^n, a_M^n$ and $b_0^n, b_1^n, b_2^n, ..., b_{M-1}^n, b_M^n$ are the coefficients in (3.9) and (3.12), respectively. This procedure is unconditionally von Neumann stable [14] for all $r > 0$.

Theorem 3.1. *the ICM scheme has a unique solution for sufficiently small h .*

Proof. It is easy to see that $|10 + 12r| > |2 + 12r|$. the matrix (3.14) is diagonally dominant, if

$$|a_0^n| > \sum_{i=1}^{M} |a_i^n| \quad \text{and} \quad |b_M^n| > \sum_{i=0}^{M-1} |b_i^n|$$

i.e

$$h\sum_{i=0}^{M} \gamma\omega_i |p_i^{n+1}\,(u_i^n)^{\gamma-1}| < 1, \quad \text{and} \quad h\sum_{i=0}^{M} \gamma\omega_i |q_i^{n+1}\,(u_i^n)^{\gamma-1}| < 1 \tag{3.15}$$

where $\omega_0 = \omega_M = \frac{1}{3}$, $\omega_{2i} = \frac{2}{3}, i = 1, ..., M-1$ and $\omega_{2i+1} = \frac{4}{3}, i = 0, ..., M-1$. As (3.15) is true for sufficiently small h, the existence and uniqueness of the solution of ICS scheme are proved. □

4. Numerical experiments

To test the above algorithms described in Section 2-3 , we use two examples with known analytical solutions as follows:

Example 4.1. *We consider the following problem (Test given in paper [3])*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{-2\left(x^2 + t + 1\right)}{(t+1)^3}, \qquad 0 < x < 1, \qquad 0 < t \le T, \tag{4.1}$$

subject to the initial condition

$$u\,(x, 0) = x^2, \qquad 0 \le x < 1, \tag{4.2}$$

and the nonlinear nonlocal boundary conditions

$$u(0,t) = \int_0^1 xu^2(x,t)\,dx - \frac{1}{6(t+1)^4}, \qquad 0 < t \le T, \tag{4.3}$$

$$u(1,t) = \int_0^1 xu^2(x,t)\,dx + \frac{6x^2+12t+5}{6(t+1)^4}, \qquad 0 < t \le T, \tag{4.4}$$

The functions $f, \varphi,\ p,\ q,\ E$ and G are chosen so that the function

$$u(x,t) = \left(\frac{x}{t+1}\right)^2. \tag{4.5}$$

is the exact solution solution of the problem(1.1)-(1.4).

In Table 1 and Table 2 we present results with $h = 0.05, 0.005$ using the Crandall's scheme discussed in section 2-3 together with the results from [3] for $x = 0.1$ and $t = 0.01, 0.02, 0.03, ..., 0.1$. Table 3 gives the maximum errors of the numerical solutions experimental order of convergence. The maximum error is defined as follows

$$Er(h,k) = \|u - u_{hk}\|_\infty = \max_{0\le k\le N}\{\max_{0\le i\le M}|u(x_i,t_k) - u_i^k|\},$$

and the experiment order convergence is calculated using the formula :

$$Order = \frac{\ln\left(Er\left(h_{i-1},k_{i-1}\right)/Er\left(h_i,k_i\right)\right)}{\ln\left(h_{i-1}/h_i\right)}.$$

t_i	exact	ECS	ICS	from [3]
0.01	0.0098029	0.00980407	0.0098037	0.0093
0.02	0.0096116	0.00961317	0.0096126	0.0091
0.03	0.0094259	0.00942762	0.0094270	0.0090
...	...	...	...	...
0.1	0.0082644	0.00826636	0.0082657	0.0079

TABLE 1. Some numerical results at $x = 0.1$ with $h = 0.05$ for Example 1

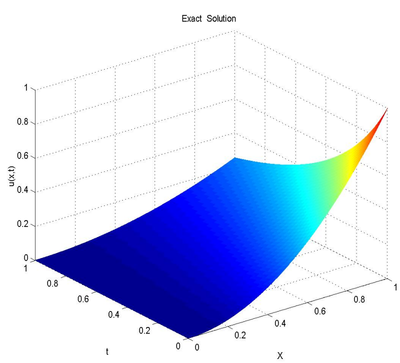
Figs 1-2 illustrates the exact solution and an approximate solution of Example 1 by ECS and ICS, respectively. We plot the errors $e_i^n = u(x_i,t_n) - u_i^n$, $i = 0,1,2,...,M$, $n = 0,1,...,N$, for the schemes ECS and ICS in Figure 3.

t_i	*exact*	*ECS*	*ICS*	*from [3]*
0.01	*0.0098029*	*0.0098029*	*0.0098029*	*0.0098*
0.02	*0.0096116*	*0.0096116*	*0.0096116*	*0.0096*
0.03	*0.0094259*	*0.0094259*	*0.0094259*	*0.0094*
...	...	...	...	...
0.1	*0.0082644*	*0.0082644*	*0.0082644*	*0.0083*

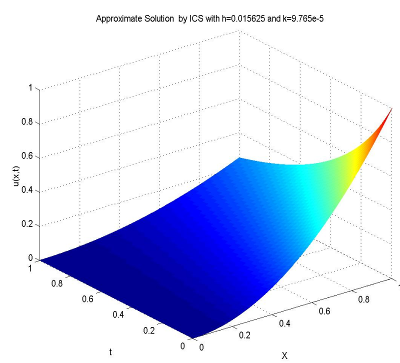
TABLE 2. Some numerical results at $x = 0.1$ with $h = 0.005$ for Example 1

M	*ECS*	*order*	*ICS*	*order*
12	$1.700987 \cdot 10^{-6}$		$1.356601 \cdot 10^{-6}$	
24	$1.064666 \cdot 10^{-7}$	*3.9978*	$8.483043 \cdot 10^{-8}$	*3.9992*
48	$6.658956 \cdot 10^{-9}$	*3.9989*	$5.302550 \cdot 10^{-9}$	*3.9998*
96	$4.163736 \cdot 10^{-10}$	*3.9993*	$3.314515 \cdot 10^{-10}$	*3.9998*

TABLE 3. The maximum errors and experiment order of convergence for Example 1 .



(a)



(b)

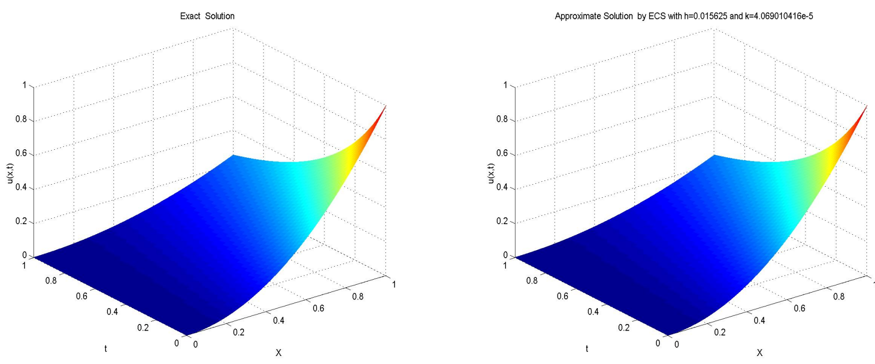
FIGURE 1. (a) Exact and (b) Approximate Solution by ICS for Example 1.

Example 4.2. *The second test example to be solved is*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \left(1+\pi^2\right)\exp\left(t\right)\cos\left(\pi x\right), \qquad 0 < x < 1, \qquad 0 < t \le T, \tag{4.6}$$

with the initial condition

$$u\left(x,0\right) = \cos\left(\pi x\right), \qquad 0 < x < 1, \tag{4.7}$$



(a) (b)

Figure 2. (a) Exact and (b) Approximate Solution by ECS for Example 1.

and the nonlinear nonlocal boundary conditions

$$u(0,t) = \int_0^1 \sin(\pi x)\, u^3(x,t)\, dx + \exp(t), \qquad 0 < t \le T, \tag{4.8}$$

$$u(1,t) = \int_0^1 \sin(\pi x)\, u^3(x,t)\, dx - \exp(t), \qquad 0 < t \le T. \tag{4.9}$$

The analytic solution is

$$u(x,t) = \cos(\pi x)\exp(t). \tag{4.10}$$

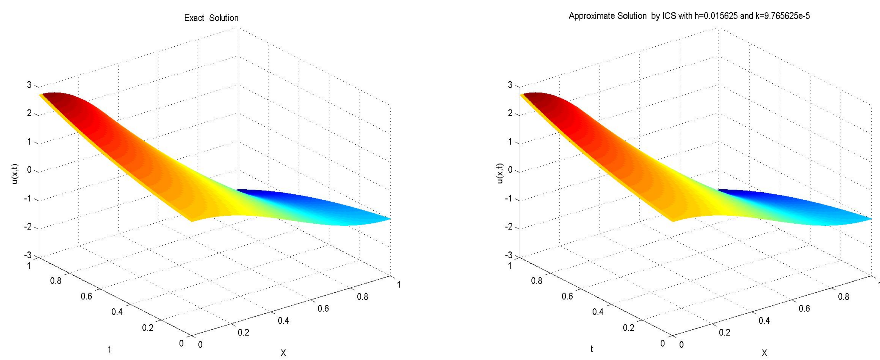
In Table 4 and Table 5 we present results with $h = 0.05, 0.005$ *and* $r = 0.4$ *using the finite difference formulate discussed in section 2-3 for* $x = 0.1$ *and* $t = 0.01, 0.02, 0.03, ..., 0.1$. *Figures 3-4 illustrates the exact solution and an approximate solution of Example 2 by ECS and ICS, respectively.*

t_i	*exact*	*ECS*	*ICS*
0.01	*0.96061479*	*0.96061492*	*0.96061503*
0.02	*0.97026913*	*0.97026933*	*0.97026948*
0.03	*0.98002050*	*0.98002074*	*0.98002092*
...	...	...	...
0.1	*1.05108000*	*1.05108034*	*1.05108060*

Table 4. Some numerical results at $x = 0.1$ for $h = 0.05$ for Example 2

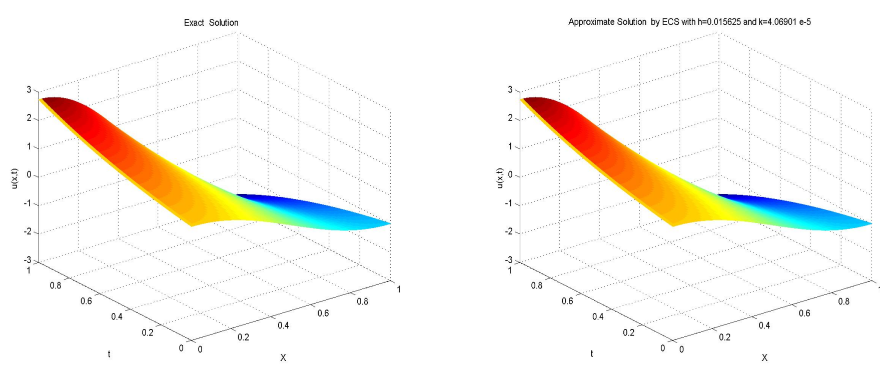
t_i	*exact*	*ECS*	*ICS*
0.01	*0.96061479*	*0.96061479*	*0.96061479*
0.02	*0.97026913*	*0.97026913*	*0.97026913*
0.03	*0.98002050*	*0.98002050*	*0.98002050*
...	...	...	...
0.1	*1.05108000*	*1.05108000*	*1.05108000*

Table 5. Some numerical results at $x = 0.1$ for $h = 0.005$ for Example 2



(a) (b)

Figure 3. (a) Exact and (b) Approximate Solution by ICS for Example 2.



(a) (b)

Figure 4. (a) Exact and (b) Approximate Solution by ECS for Example 2.

5. Conclusion

In this paper new techniques were applied to the one-dimensional heat equation with nonlinear nonlocal boundary conditions. The numerical results obtained by using the methods described in this article give acceptable results and suggests convergence to the exact solution when h goes to zero. The ECS method is explicit and require less computational time than the implicit Crandall's scheme, but the disadvantage of this discretization is the restriction in choosing the value of r $\left(r = \frac{1}{6}\right)$.

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