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MEROMORPHIC STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. The main purpose of this article is to introduce a class of meromorphic functions associated with the symmetric points in circular domain. We investigate the necessary and sufficient conditions, distortions theorem for this class. Furthermore, we obtain closure and convolutions properties, radii of starlikeness and partial sum results for these functions.

1. Introduction

Denoted by $\mathcal{M}$, the class of functions f which are analytic in the $\mathfrak{D}^* = \mathfrak{D} \setminus \{0\}$, where $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ and having the following series expansion form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n \, z^n, \quad z \in \mathfrak{D}^*. \tag{1.1}$$

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We say that an analytic function $f_1(z)$ is subordinate to $f_2(z)$ in $\mathfrak{D}$, symbolically represented as $f_1(z) \prec f_2(z)$, if there exists an analytic function $w(z)$ with conditions $|w(z)| < 1$ and $w(0) = 1$ such that $f_1(z) = f_2(w(z))$. Moreover, if $f_2(z)$ is univalent, then we have the following equivalency from [1] and [2],

$$f_1(0) = f_2(0) \text{ and } f_1(\mathfrak{D}) \subseteq f_2(\mathfrak{D}).$$

For two functions $f_1(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1}\ z^n$ and $f_2(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,2}\ z^n$ in $\mathfrak{D}^*$ the convolution or Hadamard product is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^n.$$

A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphic starlike functions of order α if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha,\ z \in \mathfrak{D}^*\ 0 \le \alpha < 1. \tag{1.2}$$

For some recent investigation of meromorphic functions see [3–13]. Motivated by aforementioned and recent work of [14], we define the functions' class as below:

Let $-1 \le B < A \le 1$. Then the function f is in the class $\mathcal{MS}^{**}[A, B]$ if it satisfies

$$-\frac{2zf'(z)}{f(z) - f(z)} \prec \frac{1 + Az}{1 + Bz}, (z \in \mathfrak{D}^*), \tag{1.3}$$

or equivalently

$$\left|\frac{\frac{2zf'(z)}{f(z)-f(-z)} + 1}{B\frac{2zf'(z)}{f(z)-f(-z)} + A}\right| < 1\ \ (z \in \mathfrak{D}^*). \tag{1.4}$$

2. Coefficient Inequalities

Theorem 2.1. *Let $f \in \mathcal{M}$ and assumed as in (1.1) then $f \in \mathcal{MS}^{**}[A, B]$, if and only if*

$$\sum_{n=1}^{\infty}\left((1 + B)\, n + (1 + A)\frac{1 - (-1)^n}{2}\right)|a_n| \le A - B. \tag{2.1}$$

This inequality is sharp.

Proof. Let us assume that condition (2.1) holds. To show $f \in \mathcal{MS}^{**}[A, B]$, we only need to show the inequality (1.4) holds. For this consider

$$\begin{aligned}
\left|\frac{\frac{2zf'(z)}{f(z)-f(-z)} + 1}{B\frac{2zf'(z)}{f(z)-f(-z)} + A}\right| &= \left|\frac{zf'(z) + \frac{f(z)-f(-z)}{2}}{Bzf'(z) + A\frac{f(z)-f(-z)}{2}}\right| \\
&= \left|\frac{\sum_{n=1}^{\infty}\left(n+\frac{1-(-1)^n}{2}\right)a_n}{(B-A)+\sum_{n=1}^{\infty}\left(Bn+A\frac{1-(-1)^n}{2}\right)a_n}\right| \\
&\le \frac{\sum_{n=1}^{\infty}\left(n+\frac{1-(-1)^n}{2}\right)a_n}{(B-A)-\sum_{n=1}^{\infty}\left(Bn+A\frac{1-(-1)^n}{2}\right)a_n} < 1.
\end{aligned}$$

Now for other part let suppose $f \in \mathcal{MS}^{**}[A,B]$. We are to show that the inequality (2.1), holds true. Consider

$$\begin{aligned}\left|\frac{\frac{2zf'(z)}{f(z)-f(-z)}+1}{B\frac{2zf'(z)}{f(z)-f(-z)}+A}\right| &= \left|\frac{zf'(z)+\frac{f(z)-f(-z)}{2}}{Bzf'(z)+A\frac{f(z)-f(-z)}{2}}\right| \\ &= \left|\frac{\sum_{n=1}^{\infty}\left(n+\frac{1-(-1)^n}{2}\right)a_n}{(A-B)+\sum_{n=1}^{\infty}\left(Bn+A\frac{1-(-1)^n}{2}\right)a_n}\right| ,\end{aligned}$$

since the $\Re(z) \leq |z|$, we have

$$\Re\left\{\frac{\sum_{n=1}^{\infty}\left(n+\frac{1-(-1)^n}{2}\right)a_n z^{n-1}}{(A-B)+\sum_{n=1}^{\infty}\left(Bn+A\frac{1-(-1)^n}{2}\right)a_n z^{n-1}}\right\} < 1 \ , \tag{2.2}$$

now if we choose the value of z on real axis then $\frac{2zf'(z)}{f(z)-f(-z)}$ is real. Letting $z \to 1^-$ on real axis and some simple calculation in (2.2), lead us to (2.1). □

Theorem 2.2. *Let $f \in \mathcal{M}$ and assumed as in (1.1) then $f \in \mathcal{MS}^{**}[A,B]$, if and only if*

$$z\left[f(z) * \left(\frac{(1-2z)\left(1+Be^{i\theta}\right)}{z(1-z)^2} - \frac{z\left(1+Ae^{i\theta}\right)}{1-z^2}\right)\right] \neq 0, \quad (z \in \mathfrak{D}). \tag{2.3}$$

Proof. It is easy to verify the relations

$$f(z) * \frac{z}{1-z^2} = \frac{f(z)-f(-z)}{2} \text{ and } f(z) * \left[\frac{1}{z(1-z)^2} - \frac{2}{(1-z)^2}\right] = -zf'(z). \tag{2.4}$$

To prove (2.3), if $f \in \mathcal{MS}^{**}[A,B]$ then we write (1.3), by using definition of subordination as

$$-\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+Aw(z)}{1+Bw(z)}, \tag{2.5}$$

which is equivalent to

$$-\frac{2zf'(z)}{f(z)-f(-z)} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}, \ z \in \mathfrak{D}, \ \theta \in [0,2\pi],$$

which implies that

$$-zf'(z)\left(1+Be^{i\theta}\right) - \frac{f(z)-f(-z)}{2}\left(1+Ae^{i\theta}\right) \neq 0. \tag{2.6}$$

Using the relation (2.4), (2.6) become

$$z\left[f(z) * \left(\frac{(1-2z)\left(1+Be^{i\theta}\right)}{z(1-z)^2} - \frac{z\left(1+Ae^{i\theta}\right)}{1-z^2}\right)\right] \neq 0, \text{ for } z \in \mathfrak{D}.$$

Conversely, suppose that the condition (2.3) hold, it follows that $zf(z) \neq 0$ for all $z \in \mathfrak{D}$. Hence $\Phi(z) = -\frac{2zf'(z)}{f(z)-f(-z)}$ is analytic in $\mathfrak{D}$ with $\Phi(0) = 1$. Since

$$-\frac{2zf'(z)}{f(z)-f(-z)} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}. \tag{2.7}$$

If we denote

$$\Psi(z) = \frac{1+Az}{1+Bz},$$

the relation (2.7), show that $\Phi(\mathfrak{D}) \cap \Psi(\mathfrak{D}) = \emptyset$. Therefore the simply connected domain $\Phi(\mathfrak{D})$ is contained in connected component of $\mathbb{C} \diagdown \Psi(\partial\mathfrak{D})$. The univalence of "$\Phi$" togather with the fact $\Psi(0) = \Phi(0) = 1$, this show that $\Phi \prec \Psi$ which shows that $f \in \mathcal{MS}^{**}[A, B]$. □

Theorem 2.3. *The class $\mathcal{MS}^{**}[A, B]$ is closed under convex combination.*

Proof. Let $f_i(z) \in \mathcal{MS}^{**}[A, B]$, such that

$$f_i(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n, \ i \in \mathbb{N}.$$

Then by equation (2.1), we have

$$\sum_{n=1}^{\infty} \left((1+B)n + (1+A) \frac{1-(-1)^n}{2} \right) |a_{n,i}| \leq A - B.$$

For $\sum_{i=1}^{\infty} \delta_i = 1$, $0 \leq \delta \leq 1$, we have

$$\sum_{i=1}^{\infty} \delta_i f_i(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \delta_i a_{n,i} \right) z^n.$$

Using 2.1, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \left((1+B)n + (1+A) \frac{1-(-1)^n}{2} \right) \delta_i |a_{n,i}| \right) \\ \leq \ & \sum_{i=1}^{\infty} \delta_i \left\{ \sum_{n=1}^{\infty} \left((1+B)n + (1+A) \frac{1-(-1)^n}{2} \right) |a_{n,i}| \right\} \\ \leq \ & (A-B) \sum_{i=1}^{\infty} \delta_i = A - B. \end{aligned}$$

Hence $\mathcal{MS}^{**}[A, B]$ is convex. □

Theorem 2.4. *Let $f \in \mathcal{MS}^{**}[A, B]$, $|z| = r$. Then*

$$\frac{1}{r} - \frac{A-B}{A+B+2} r \leq |f(z)| \leq \frac{1}{r} + \frac{A-B}{A+B+2} r. \tag{2.8}$$

Proof. As

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \\ &\leq \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| |r|^n \\ &\leq \frac{1}{r} + \frac{A-B}{A+B+2} r. \end{aligned}$$

Where we have used Theorem 2.1, on similar argument we have

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \\ &\geq \frac{1}{r} - \sum_{n=1}^{\infty} |a_n| \, |r|^n \\ &\geq \frac{1}{r} - \frac{A-B}{A+B+2} r. \end{aligned}$$

Thus prove the result. □

Theorem 2.5. *Let $f(z) \in \mathcal{MS}^{**}[A,B]$, $|z| = r$. Then*

$$\frac{1}{r^2} - \frac{2(A-B)}{A+B+2} r \leq |f'(z)| \leq \frac{1}{r^2} + \frac{2(A-B)}{A+B+2} r. \tag{2.9}$$

Proof. As

$$\begin{aligned} |f'(z)| &= \left| -\frac{1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1} \right| \\ &\leq \frac{1}{r^2} + \sum_{n=1}^{\infty} |a_n| \, |r|^{n-1} \\ &\leq \frac{1}{r^2} + \frac{2(A-B)}{A+B+2} r. \end{aligned}$$

Where we have used Theorem 2.1, and

$$\begin{aligned} |f'(z)| &= \left| -\frac{1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1} \right| \\ &\geq \frac{1}{r^2} - \sum_{n=1}^{\infty} |a_n| \, |r|^n \\ &\geq \frac{1}{r^2} - \frac{2(A-B)}{A+B+2} r. \end{aligned}$$

Thus prove the result. □

Theorem 2.6. *Let $f(z) \in \mathcal{MS}^{**}[A,B]$ of the form (1.1), and $h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{MS}^{**}(A,B)$ with $|b_n| \leq 1$, then $f(z) * h(z) \in \mathcal{MS}^{**}[A,B]$.*

Proof. Since by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \left((1+B) n + (1+A) \frac{1-(-1)^n}{2} \right) |a_n| \leq A - B.$$

Since

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left((1+B)\,n+(1+A)\,\frac{1-(-1)^n}{2}\right)|a_n b_n| \\
=\ &\sum_{n=1}^{\infty}\left((1+B)\,n+(1+A)\,\frac{1-(-1)^n}{2}\right)|a_n|\,|b_n| \\
\leq\ &\sum_{n=1}^{\infty}\left((1+B)\,n+(1+A)\,\frac{1-(-1)^n}{2}\right)|a_n| \leq A-B.
\end{aligned}$$

Thus $f(z) * h(z) \in \mathcal{MS}^{**}[A,B]$. □

Theorem 2.7. *If $f \in \mathcal{MS}^{**}[A,B]$. Then $f \in \mathcal{MS}^{*}(\alpha)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{(1-\alpha)\left((1+B)n+(1+A)\frac{1-(-1)^n}{2}\right)}{(n+\alpha)(A-B)}\right)^{\frac{1}{n+1}}. \tag{2.10}$$

Proof. Let $f \in \mathcal{MS}^{**}[A,B]$. To prove $f \in \mathcal{MS}^{*}(\alpha)$, we only need to show

$$\left|\frac{zf'(z)+f(z)}{zf'(z)-(1-2\alpha)\,f(z)}\right| < 1.$$

Using (1.1) along with some simple computation yields

$$\sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}\,|a_n|\,|z|^{n+1} \leq 1. \tag{2.11}$$

As f is in the class $\mathcal{MS}^{**}[A,B]$ so we have from (2.1),

$$\sum_{n=1}^{\infty}\frac{(1+B)\,n+(1+A)\,\frac{1-(-1)^n}{2}}{A-B}\,|a_n| \leq 1.$$

Now inequality (2.11) will be true, if the following holds

$$\sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}\,|a_n|\,|z|^{n+1} < \sum_{n=1}^{\infty}\frac{(1+B)n+(1+A)\frac{1-(-1)^n}{2}}{A-B}\,|a_n|,$$

which implies that

$$|z|^{n+1} < \frac{(1-\alpha)\left((1+B)n+(1+A)\frac{1-(-1)^n}{2}\right)}{(n+\alpha)(A-B)},$$

and so

$$|z| < \left(\frac{(1-\alpha)\left((1+B)n+(1+A)\frac{1-(-1)^n}{2}\right)}{(n+\alpha)(A-B)}\right)^{\frac{1}{n+1}} = r_1,$$

we get the required condition. □

Theorem 2.8. *If* $f_0(z) = \frac{1}{z}$ *and for* $n \geq 1$

$$f_n(z) = \frac{1}{z} + \frac{A-B}{(1+B)n + (1+A)\frac{1-(-1)^n}{2}} z^n.$$

Then $f \in \mathcal{MS}^{**}[A,B]$ *if and only if*

$$f(z) = \sum_{n=0}^{\infty} \delta_n f_n(z), \tag{2.12}$$

where $\delta_n \geq 0$ *and* $\sum_{n=1}^{\infty} \delta_n = 1$.

Proof. Let $f(z)$ be expressed in the form (2.12), then

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \delta_n \frac{A-B}{(1+B)n + (1+A)\frac{1-(-1)^n}{2}} z^n,$$

and for above function, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[(1+B)n + (1+A)\frac{1-(-1)^n}{2} \right] \\ & \times \delta_n \frac{A-B}{(1+B)n + (1+A)\frac{1-(-1)^n}{2}} \\ = \; & (A-B)(1-\delta_0) \leq A-B. \end{aligned}$$

Thus by Theorem 2.1, $f(z) \in \mathcal{MS}^{**}[A,B]$.

Conversely, let $f(z) \in \mathcal{MS}^{**}[A,B]$, since by Theorem 2.1, we have

$$|a_n| \leq \frac{A-B}{(1+B)n + (1+A)\frac{1-(-1)^n}{2}}, \; n \geq 1,$$

we set

$$\delta_n = \frac{(1+B)n + (1+A)\frac{1-(-1)^n}{2}}{A-B} |a_n|, \; n \geq 1,$$

and

$$\delta_0 = 1 - \sum_{n=1}^{\infty} \delta_n,$$

so, it follows that

$$f(z) = \sum_{n=0}^{\infty} \delta_n f_n(z).$$

Hence proof is complete. □

3. Partial Sums

Silverman [17] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [17] and Cho and Owa [15](also see [16, 18]) we will investigate the ratio of a function of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{3.1}$$

to its sequence of partial sums

$$f_1(z) = \frac{1}{z} \text{ and } f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \tag{3.2}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} \left((1+B)\, n + (1+A)\, \frac{1-(-1)^n}{2} \right) |a_n| \leq A - B.$$

For the sake of brevity we rewrite it as

$$\sum_{n=1}^{\infty} d_n |a_n| \leq A - B, \tag{3.3}$$

where

$$d_n(A,B) := \left((1+B)\, n + (1+A)\, \frac{1-(-1)^n}{2} \right) \tag{3.4}$$

More precisely we will determine sharp lower bounds for $\Re\{f(z)/f_k(z)\}$ and $\Re\{f_k(z)/f(z)\}$. In this connection we make use of the well known results that$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0 \quad (z \in \mathfrak{D}^*)$ if and only if $\omega(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|\omega(z)| \leq |z|$. Unless otherwise stated, we will assume that f is of the form (1.1) and its sequence of partial sums is denoted by $f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n$.

Theorem 3.1. *Let $f \in \mathcal{MS}^{**}[A,B]$ be given by (1.1)satisfies condition (2.1),then*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{d_{k+1}(A,B) + B - A}{d_{k+1}(A,B)} \qquad (z \in \mathfrak{D}^*) \tag{3.5}$$

where

$$d_n(A,B) \geq \begin{cases} A-B, & if \quad n = 1,2,3,\ldots,k \\ d_{k+1}(A,B), & if \quad n = k+1, k+2, \ldots. \end{cases} \tag{3.6}$$

The result (3.5) is sharp with the function given by

$$f(z) = \frac{1}{z} + \frac{A-B}{d_{k+1}(A,B)} z^{k+1}. \tag{3.7}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}(A,B)}{A-B}\left[\frac{f(z)}{f_k(z)} - \frac{d_{k+1}(A,B)+B-A}{d_{k+1}(A,B)}\right]$$

$$= \frac{1+\sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(A,B)}{A-B}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{1+\sum\limits_{n=1}^{k} a_n z^{n+1}}. \tag{3.8}$$

It suffices to show that $|w(z)| \le 1$. Now, from (3.8) we can write

$$w(z) = \frac{\left(\frac{d_{k+1}(A,B)}{A-B}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{2+2\sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(A,B)}{A-B}\right)\sum\limits_{k=n+1}^{\infty} a_n z^{n+1}}.$$

Hence we obtain

$$|w(z)| \le \frac{\left(\frac{d_{k+1}(A,B)}{A-B}\right)\sum\limits_{k=n+1}^{\infty} |a_n|}{2-2\sum\limits_{n=1}^{k} |a_n| - \left(\frac{d_{k+1}(A,B)}{A-B}\right)\sum\limits_{n=k+1}^{\infty} |a_n|}.$$

Now $|w(z)| \le 1$ if

$$2\left(\frac{d_{k+1}(A,B)}{A-B}\right)\sum_{n=k+1}^{\infty} |a_n| \le 2 - 2\sum_{n=1}^{k} |a_n|$$

or, equivalently,

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(A,B)}{A-B}\sum_{n=k+1}^{\infty} |a_n| \le 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(A,B)}{A-B}\sum_{n=k+1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} \frac{d_n(A,B)}{A-B} |a_n|$$

which is equivalent to

$$\begin{aligned} &\sum_{n=1}^{k} \left(\frac{d_n(A,B)+B-A}{A-B}\right)|a_n| \\ &+ \sum_{n=k+1}^{\infty} \left(\frac{d_n(A,B)-d_{k+1}(A,B)}{A-B}\right)|a_n| \\ \ge\ & 0. \end{aligned} \tag{3.9}$$

To see that the function given by (3.7) gives the sharp result, we observe that for $z = re^{i\pi/k}$

$$\begin{aligned}\frac{f(z)}{f_k(z)} &= 1 + \frac{A-B}{d_{k+1}(A,B)} z^n \to 1 - \frac{A-B}{d_{k+1}(A,B)} \\ &= \frac{d_{k+1}(A,B) + B - A}{d_{k+1}(A,B)} \quad \text{when } r \to 1^-.\end{aligned}$$

which shows the bound (3.5) is the best possible for each $k \in \mathbb{N}$. □

We next determine bounds for $f_k(z)/f(z)$.

Theorem 3.2. *If f of the form (3.2) satisfies the condition (2.1), then*

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{d_{k+1}(A,B)}{d_{k+1}(A,B) + B - A} \qquad (z \in \mathfrak{D}^*), \tag{3.10}$$

where

$$d_k(A,B) \geq \begin{cases} A - B, & if \quad k = 1,2,3,\ldots,n \\ d_{k+1}(A,B), & if \quad k = n+1, n+2, \ldots. \end{cases} . \tag{3.11}$$

The result (3.10) is sharp with the function given by (3.7).

Proof. We write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \frac{d_{k+1}(A,B) + B - A}{A-B}\left[\frac{f_k(z)}{f(z)} - \frac{d_{k+1}(A,B)}{d_{k+1}(A,B) + B - A}\right] \\ &= \frac{1 + \sum_{n=1}^{k} a_n z^{n+1} - \left(\frac{d_{k+1}(A,B)}{A-B}\right) \sum_{n=k+1}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{\infty} a_n z^{n+1}},\end{aligned}$$

where

$$|w(z)| \leq \frac{\left(\frac{d_{k+1}(A,B)+B-A}{A-B}\right) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2\sum_{n=1}^{k} |a_n| - \left(\frac{d_{k+1}(A,B)+B-A}{A-B}\right) \sum_{n=k+1}^{\infty} |a_n|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(A,B)}{A-B} \sum_{n=k+1}^{\infty} |a_n| \leq 1.$$

Make use of (2.1) to get (3.9). Finally, equality holds in (3.10) for the extremal function $f(z)$ given by (3.7). □

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