GENERALIZED SPECTRUM AND NUMERICAL RANG OF MATRIX THE LORENTZIAN OSCILLATOR GROUP OF DIMENSION FOUR

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Abstract. In this paper, we find the spectrum, pseudo-spectrum and numerical rang of matrix of the metric $g_a$.

1. Introduction

Connected Lie groups that admit a bi-invariant Lorentzian metric were determined by the first of the authors in [14]. Among them, those that are solvable, non-commutative, and simply connected are called oscillator groups. This group has many properties useful both in geometry and physics.

We study here the geometry of these groups and their networks, i.e their discrete sub-groups co-compact. If $G$ is an oscillator group, its networks determine compact homogeneous Lorentz manifolds, on which $G$ acts by isometries.

Let $H_{2k+1} = \mathbb{R} \times \mathbb{C}^k$ be the Heisenberg group and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ $k$ be strictly positive real numbers. Let the additive group $\mathbb{R}$ act on $H_{2k+1}$ by the action:

$$\rho(t)(u, (z_j)) = (u, (e^{i\lambda_j t}z_j)).$$

The group $G_k(\lambda)$, a semi-direct product of $\mathbb{R}$ by $H_{2k+1}$ following $\rho$, is provided with a bi-invariant Lorentz metric. Here is how it is built:

$$g = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2k}$$
is the tangent space at the origin. Let us extend the usual scalar product of $\mathbb{R}^{2k}$ into a symmetric bilinear form over $\mathfrak{g}$ so that the plane $\mathbb{R} \times \mathbb{R}$ is hyperbolic and orthogonal to $\mathbb{R}^{2k}$. This form defines an invariant Lorentz metric on the left on $G_k(\lambda)$, it is also invariant on the right because the adjoint operators on $\mathfrak{g}$ are antisymmetric [15].

Groups $G_k(\lambda)$ are characterized [14] by:

**Theorem 1.1.** The groups $G_k(\lambda)$ are the only Lie group simply connected, resolvable and noncommutative which admit a bi-invariant Lorentz metric.

**Remark 1.1.** it is easy to see that the groups $G_1(\lambda)$ are isomorphic; the group $G_1 = G_1(1)$ is usually known as the oscillator group [20].

Since [1], [2] et [3] the oscillator group has been generalized to a dimension equal to an even number $2n$ with $n \geq 2$, plus this provides a known example of homogeneous space-time [6].

For $n = 2$, the oscillator group of dimension 4 admits a Lorentzian metric invariant on the left and on the right (bi-invariant). This bi-invariant metric has been generalized a family $g_a$, $-1 < a < 1$, invariant Lorentzian metrics on the left. For $a = 0$, the metric $g_0$ become or the only example of Lorentzian bi-invariant metric [7]

The researchers Giovani and Zaeim extracted three vectors fields from the oscillator group, which are: Killing vector field, Affine vector field, parallel vector field (see [4]).

and also Giovani and Zaeim classified the totally geodesic and parallel hypersurfaces of four-dimensional groups (see [3]).

Varah published an article entitled "On the separation of two matrices" in which he defined with standard 2 the pseudospectrum using the smallest singular value $\sigma_{\min}(zI - A)$ under the notion $\Lambda_\epsilon(A)$ see [23]. In the 1960s the pseudospectrum was studied in several by L. N. Trefethen [19], [21].

In recent years the study of the pseudospectrum has been very active, many contributions related to the pseudospectrum have been made by various researchers, for example, J. S. Baggett, A. Bottcher, M. Embree, L. N. Trefethen, L. Reichel, S.C.Reddy, T.A. Driscoll.

The pseudospectrum of a normal matrix $A$ consists of circles of radius $\epsilon$ around each eigenvalue. For non-normal matrices, the pseudospectrum takes different forms in the complex plane. in [19] The pseudospectrum of thirteen highly non-normal matrices is presented.

## 2. Preliminaries

At the moment we consider on $G_\lambda$ a family parametre of left-invariant Lorentzian metrics $g_a$. With respect to coordinates $(x_1, x_2, x_3, x_4)$, this metric $g_a$ is explicitly given by

$$g_a = adx_1^2 + 2ax_3dx_1dx_2 + (1 + ax_3^2)dx_2^2 + dx_3^2 + 2dx_1dx_4 + 2x_3dx_2dx_4 + adx_4^2,$$
with \(-1 < a < 1\).

Note that for \(a = 0\) and \(\lambda = 1\) we have the bi-invariant metric on the oscillator group \(G_1\) [7]. In all other cases, \(g_a\) is only invariant on the left.

The matrix of the metric \(g_a\) is given by

\[
A_a = \begin{pmatrix}
a & ax_3 & 0 & 1 \\
ax_3 & 1 + ax_3^2 & 0 & x_3 \\
0 & 0 & 1 & 0 \\
1 & x_3 & 0 & a
\end{pmatrix}.
\]

Numerical rang

**Definition 2.1.** Let \(A\) be an \(n \times n\) complex matrix. Then the numerical range of \(A\), \(W(A)\), is defined to be

\[
W(A) = \left\{ \frac{x^*Ax}{x^*x}, \ x \in \mathbb{C}^n, \ x \neq 0 \right\}.
\]

where \(x^*\) denotes the conjugate transpose of the vector \(x\).

**Proposition 2.1.** Based on the definition of the numerical range, one can now fairly easily deduce the following basic properties; for details see primarily [9], Chapter 1 but also [8].

1– For any \(A \in M_n(\mathbb{C})\) and for any \(a, b \in \mathbb{C}\), \(W(aA + bI_n) = aW(A) + b\).

2– For any \(A, B \in M_n(\mathbb{C})\), \(W(A + B) \subseteq W(A) + W(B)\).

3– For any \(A \in M_n(\mathbb{C})\), \(W(A)\) contains the convex hull of the eigenvalues of \(A\). If \(A\) is normal, i.e., \(A^*A = AA^*\), then \(W(A)\) equals the convex hull of \(\sigma(A)\).

4– For any \(A \in M_n(\mathbb{C})\), \(W(A) \subset \mathbb{R}\) if and only if \(A\) is Hermitian, i.e., \(A^* = A\), in this case, the endpoints of \(W(A)\) coincide with the minimum and the maximum eigenvalues of \(A\). Furthermore, \(W(A)\) is a line segment in the complex plane if and only if the matrix \(A\) is normal and has collinear eigenvalues; or equivalently, if and only if \(A = aH + bI\) for some \(a, b \in \mathbb{C}\) and an Hermitian matrix \(H\).

3. Eigenvalues and Pseudo-spectrum of matrix \(A_a\)

**Proposition 3.1.** The eigenvalues of the matrix \(A_a\) are:

\[
\begin{align*}
\lambda_1 &= 1, \\
\lambda_2 &= \frac{2}{3}a + \frac{1}{3}ax_3^2 - \frac{1}{2}S + \frac{1}{2} + \frac{1}{2} - \sqrt{3}i \left( S + \frac{C}{S} \right), \\
\lambda_3 &= \overline{\lambda_2}, \\
\lambda_4 &= \frac{2}{3}a + \frac{1}{3}ax_3^2 + S - \frac{C}{S} + \frac{1}{2},
\end{align*}
\]

with

\[
S = \sqrt[3]{{M} + \sqrt{N} - \frac{8}{27}}.
\]
and

\[ M = 2a + a^2 - \frac{1}{27}a^3 + \frac{1}{6}x^3 + \frac{11}{18}ax^2 + \frac{1}{6}ax^4 \]

\[-\frac{1}{18}a^2x^3 - \frac{1}{18}a^3x^3 + \frac{1}{9}a^2x^4 + \frac{1}{18}a^3x^4 + \frac{1}{27}a^3x^6 \]

\[ N = \frac{4}{27}a^3 - \frac{4}{27}a^2 - \frac{1}{27}a^4 - \frac{8}{27}x^2 - \frac{13}{108}x^4 - \frac{2}{27}x^6 - \frac{2}{9}ax^2 + \frac{1}{54}ax^4 - \frac{1}{54}ax^6 + \frac{7}{27}a^2x^2 \]

\[ + \frac{4}{27}a^3x^3 + \frac{7}{36}a^2x^4 - \frac{1}{9}a^4x^3 + \frac{1}{18}a^2x^6 - \frac{11}{108}a^4x^4 + \frac{1}{27}a^3x^6 + \frac{1}{54}a^5x^4 - \frac{1}{108}a^2x^8 \]

\[-\frac{1}{108}a^6x^4 - \frac{1}{54}a^5x^3 + \frac{1}{54}a^4x^8 - \frac{1}{54}a^6x^6 - \frac{1}{108}a^6x^8 \]

\[ C = \frac{2}{9}a - \frac{1}{9}a^2 - \frac{1}{3}x^2 - \frac{2}{9}ax^3 - \frac{1}{9}a^2x^3 - \frac{1}{9}a^2x^4 - \frac{4}{9} \]

**Proof.** We have

\[ \det(A - \lambda I_4) = (1 - \lambda)(-\lambda^3 + L\lambda^2 + K\lambda + (a^2 - 1)), \]

with

\[ L = (1 + 2a + ax^3), \]

\[ K = (-a^2 - 2a - a^2x^3 + x^3 + 1), \]

so, \( \det(A - \lambda I_4) = 0, \) If and only if either \( \lambda_1 = 1 \) or

\[ -\lambda^3 + L\lambda^2 + K\lambda + (a^2 - 1) = 0. \]

According to the CARDAN method we find,

\[ z^3 + pz + q = 0, \]

such as

\[ z = \lambda - \frac{L}{3}, \ z \in \mathbb{C}, \]

(3.1)
and
\[ p = -(\frac{1}{3}L^2 + K) = -\frac{1}{3}(4 + a^2x_4^3 + ax_3^2 + a^2 - 2a + a^2x_3^2 + 3x_3^2), \]
\[ q = -\frac{1}{27}(-16 + 2a^3x_3^6 + 6a^2x_3^4 + 33ax_3^2 - 2a^3 + 6a^2 - 3a^3x_3^2 + 12a + 3a^3x_3^4 - 3a^2x_3^2 + 9x_3^2). \]

Then the **CARDAN** method he says that the 3 solutions are:
\[ z_k = j^k \sqrt{\frac{1}{2} \left( -q + \sqrt{\frac{-\Delta}{27}} \right)} + j^{-k} \sqrt{\frac{1}{2} \left( -q - \sqrt{\frac{-\Delta}{27}} \right)}, \quad 0 \leq k \leq 2 \]
such as,
\[ \Delta = -4p^3 - 27q^2, \]
\[ j = e^{i\frac{2\pi}{3}}. \]

So, according to (3.1) we find,
\[ \lambda_k = z_k + \frac{L}{3}, \quad 0 \leq k \leq 2 \]
\[ \square \]

**Pseudo-spectrum of** $A_a$: since $A$ is symmetrical therefore $A_a$ is normal, therefore pseudo-spectrum noted by $\Lambda_{\epsilon}(A_a)$ given by:
\[ \Lambda_{\epsilon}(A_a) = \{ z \in \mathbb{C} : |z - \lambda_i| \leq \epsilon \} \text{ with } i \in \{1, \ldots, 4\}. \]

### 3.1. Numerical rang of matrix $A_a$,

**Proposition 3.2.** The numerical rang of matrix $A_a$ check the following relation:
\[ \frac{x^*A_ax}{x^*x} \leq (1 + |a|)(1 + |x_3|) + |ax_3^2| \]

**Proof.** We have
\[ W(A) = \left\{ \frac{x^*A_ix}{x^*x} : x \in \mathbb{C}^4, x \neq 0 \right\} \]
we put $x = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$, with $z_i = r_ie^{i\theta_i}$. We have
\[ x^*A_ax = a |z_1|^2 + a |z_4|^2 + |z_2|^2 + |z_3|^2 + ax_3(z_1\overline{z_2} + z_2\overline{z_1}) + x_3(z_2\overline{z_1} + z_4\overline{z_3}) + (z_1\overline{z_4} + z_4\overline{z_1}) + a |z_2|^2 x_3^2, \]
so,
\[
x^* A x \overline{x^* x} = 1 + \left( a - 1 \right) \left( |z_1|^2 + |z_4|^2 \right) + a x_3 \sum_{i=1}^{4} |z_i|^2 + x_3 \sum_{i=1}^{4} |z_i|^2 + \sum_{i=1}^{4} \frac{|z_i|^2}{|z_i|^2} + ax_3^2 |z_2|^2.
\]

We have
\[
(3.2) \quad \frac{|z_j|^2}{\sum_{i=1}^{4} |z_i|^2} \leq 1, \quad \forall j \in \{1, \ldots, 4\}.
\]

and
\[
(3.3) \quad \frac{z_i \overline{z_4} + z_4 \overline{z_i}}{\sum_{i=1}^{4} |z_i|^2} \leq 1, \quad \forall i, j \in \{1, \ldots, 4\},
\]

So from (3.2) and (3.3) we find
\[
\left| x^* A x \overline{x^* x} \right| \leq 1 + |ax_3| + |x_3| + |a| + ax_3^2.
\]

It had to be proven. \(\square\)

**Example 3.1.**  1) For \( a = 0 \) and \( x_3 = 0 \),

\[
A_0^0 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

so

\[
g_0^0(x^*, x) = 1 - \frac{r_1^2 + r_4^2 - 2r_1r_4 \cos(\theta_1 - \theta_4)}{r_1^2 + r_2^2 + r_3^2 + r_4^2} \leq 1,
\]

moreover \( 1 \in W(A_0^0) \)

On the other hand, we have

\[
- \frac{r_1^2 + r_4^2 - 2r_1r_4 \cos(\theta_1 - \theta_4)}{r_1^2 + r_2^2 + r_3^2 + r_4^2} \geq -2,
\]

therefore

\[
g_0^0(x^*, x) \overline{x^* x} \geq -1,
\]

moreover \(-1 \in W(A_0^0)\). So \( W(A_0^0) = [-1, 1] \)

2) For \( a = 0 \) and \( x_3 = 0.5 \),

\[
A_0^{0.5} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0.5 \\
0 & 0 & 1 & 0 \\
1 & 0.5 & 0 & 0
\end{pmatrix},
\]
so
\[
\frac{g_{0.5}^0(x^*, x)}{x^*x} = g_0^0(x^*, x) + \frac{r_2 r_4 \cos(\theta_2 - \theta_4)}{r_1^2 + r_2^2 + r_3^2 + r_4^2}.
\]

We have,
\[
\frac{r_2 r_4 \cos(\theta_2 - \theta_4)}{r_1^2 + r_2^2 + r_3^2 + r_4^2} \leq \frac{1}{2},
\]
and
\[
\frac{g_{0.5}^0(x^*, x)}{x^*x} \geq -\frac{5}{4}
\]
so
\[
-\frac{5}{4} \leq \frac{g_{0.5}^0(x^*, x)}{x^*x} \leq \frac{3}{2}.
\]

but \(-\frac{5}{4} \) and \(\frac{3}{2}\) does not belong to \(W(A_{0.5}^0)\), so we get \(W(A_{0.5}^0) \subset \left[ -\frac{5}{4}, \frac{3}{2} \right] \).

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References