



## EXPONENTIAL STABILITY FOR A NONLINEAR TIMOSHENKO SYSTEM WITH DISTRIBUTED DELAY

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ABSTRACT. This paper is concerned with a nonlinear Timoshenko system modeling clamped thin elastic beams with distributed delay time. The distributed delay is defined on feedback term associated to the equation for rotation angle. Under suitable assumptions on the data, we establish the exponential stability of the system under the usual equal wave speeds assumption.

### 1. INTRODUCTION

In this work, we consider the following non linear Timoshenko system with distributed delay,

$$(1.1) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1 \psi_t \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) \psi_t(x, t - s) ds + f(\psi) = 0, \end{cases}$$

where  $t$  denotes the time variable and  $x$  the space variable along a beam of length 1 in its equilibrium configuration. Here,  $\varphi = \varphi(x, t)$  and  $\psi = \psi(x, t)$  denotes the transverse displacement of the beam and the rotation angle of its filament, respectively. The term  $\mu_1 \psi_t$  represents a frictional damping and  $f(\psi)$  is a forcing term. The coefficients,  $\rho_1, \rho_2, k$  are positive constants represent the density, the polar momentum

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of inertia of a cross section, shear modulus respectively, and  $b = EI$  where  $E$  is the young's modulus of elasticity,  $I$  is the moment of inertia cross-section.

System (1.1) is supplemented with the following initial conditions

$$(1.2) \quad \begin{aligned} \varphi(x, 0) = \varphi_0, \quad \varphi_t(x, 0) = \varphi_t, \quad \psi(x, 0) = \psi_0, \quad \psi_t(x, 0) = \psi_1 \\ \psi_t(x, -t) = f_0(x, t), \end{aligned}$$

and Dirichlet boundary conditions

$$(1.3) \quad \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t),$$

where  $x \in (0, 1)$ ,  $t \in (\tau_1, \tau_2)$ . The initial data  $(\varphi_0, \psi_0, \varphi_1, \psi_1, f_0)$  belongs to a suitable functional spacial.

This type of problems (without delay), has been considered, first in [15] where  $\mu_1 = \mu_2 = f = 0$ . The stability of this problems has received much attention in last years, we can find in the literature many results about different stability of Timoshenko systems depending, in particular, on the weights  $\mu_1$  and  $\mu_2$  (see [14])

Recently also a great consideration ha been addressed to time delay effects. On such problems, it was showed that a small delay acted on a boundary control, or internal can destabilize a system which is uniformly asymptotically stable in the absence of delays. See for instance ([5]).

In [13] S. Nicaise and C. Pignotti examined a system of wave equation with initial feedback

$$(1.4) \quad \begin{cases} u_{tt} - u_{xx} + \mu_0 u_t + \int_{\tau_1}^{\tau_2} a(x)\mu(s)u_t(t-s)ds \\ u = 0 \quad \text{on} \quad \Gamma_0(0, \alpha) \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1(0, \alpha) \\ u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \\ u_t(x, -t) = f_0(x, -t) \quad \text{in} \quad \Omega(0, \tau_2) \end{cases}$$

where  $a \in L^2(\Omega)$  is a function chosen with some assumptions. They proved that the above system is exponentially stable under the condition

$$\mu_0 > \|a\|_{\alpha} \int_{\tau_1}^{\tau_2} \mu(s)ds$$

Similarly result was obtained by the authors when the distributed delay acted on the part of boundary. In [11] Mustapha considered a Timoshenko system of thermoelasticity of type III with distributed delay and establish the stability for the case of equal and non equal speeds of wave propagation .Appalara [1]

investigated a thermo-elastic system of Timoshenko type with second sound and distributed delay

$$(1.5) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \gamma_1 \varphi_t + \int_{\tau_1}^{\tau_2} \gamma_2(s) \varphi_t(x, t-s) = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x = 0 \\ \rho_3 \theta_t + q_x + \delta \psi_{tx} = 0 \\ \tau q_t + \beta q + \theta_x = 0. \end{cases}$$

in  $(0, 1)(0, \alpha)$ , this system is exponentially stable regardless the speeds of wave propagation. The same author studied in [2] a one dimensional Timoshenko system with linear frictional damping and a distributed delay acting on the displacement equation, he showed that dissipation through the frictional damping is strong enough to uniformly stabilize the system. for other results about different types of time delay (discrete and continuous delay) we refer the reader to see ([1-4, 8, 10]).

B. Feng and H. L. Pelier [6] considered a following non linear Timoshenko system with constant delay and forcing term:

$$(1.6) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2(s) \psi_t(x, t-\tau) + f(\psi) = 0, \end{cases}$$

and obtained an exponential stability under equal wave speeds.

Recently S. A. Messaoudi, B. Said-Houari [10] established the stability of a thermoelastic Timoshenko system of type III with past history and distributed delay for the cases of equal and non equal speeds of wave propagation respectively.

In the present work, we extend the result of Feng and Pelier, [6] where constant delay is replaced by distributed delay.

## 2. PRELIMINARIES

In this section we present the some assumptions needed later to prove our results. As in [12], we introduce the following new dependent variable

$$z(x, \rho, s, t) = \psi_t(x, t - \rho s), \quad x \in (0, 1), \quad \rho \in (0, 1), \quad t, s \in (\tau_1, \tau_2).$$

Then, the above variable  $z$  satisfies

$$sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0, \quad (x, \rho, s, t) \in (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, +\infty).$$

Therefore, the problem (1.1) is equivalent to

$$(2.1) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, & x \in (0, 1), t > 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1 \psi_t \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds + f(\psi) = 0, & x \in (0, 1), t > 0, \\ sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0, & \rho \in (0, 1), s \in (\tau_1, \tau_2), t > 0, \end{cases}$$

with the following initial and boundary conditions

$$(2.2) \quad \begin{cases} \varphi(x, 0) = \varphi_0, \varphi_t(x, 0) = \varphi_1, & x \in (0, 1), \\ \psi(x, 0) = \psi_0, \psi_t(x, 0) = \psi_1, & x \in (0, 1), \\ z(x, \rho, s, 0) = f_0(x, \rho s), & x \in (0, 1), \rho \in (0, 1), s \in (0, \tau_2), \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, & t > 0. \end{cases}$$

Concerning the weight of the delay, we only assume that

$$(2.3) \quad \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds < \mu_1.$$

In addition, we give some hypothesis on the forcing term  $f(\psi(x, t))$ . We assume that  $f : IR \rightarrow IR$  satisfies the following condition

$$(2.4) \quad |f(\psi^1) - f(\psi^2)| \leq k_0 \left( |\psi^1|^\theta - |\psi^2|^\theta \right) |\psi^1 - \psi^2|$$

for all  $\psi^1, \psi^2 \in IR$ , where  $k_0 > 0, \theta > 0$ . Also

$$(2.5) \quad 0 \leq \tilde{f}(\psi) \leq f(\psi)\psi, \text{ for all } \psi \in IR,$$

with

$$\tilde{f}(y) = \int_0^y f(s) ds.$$

We introduce the Hilbert space,

$$\mathcal{H} = H_0^1(0, 1) \times L^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L^2((0, 1) \times (0, 1) \times (\tau_1, \tau_2))$$

For  $U = (\varphi, u, \psi, v, z)^T, (\tilde{\varphi}, \tilde{u}, \tilde{\psi}, \tilde{v}, \tilde{z})^T$  equipped with the scalar product

$$\begin{aligned} \langle u, \tilde{u} \rangle_H &= \int_0^1 \left[ \rho_1 u \tilde{u} + \rho_2 v \tilde{v} + k(\varphi_x + \psi) (\tilde{\varphi}_x + \tilde{\psi}) + b\psi_x \tilde{\psi}_x \right] dx \\ &+ \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| \int_0^1 z(x, \rho, s, t) \tilde{z}(x, \rho, s, t) d\rho ds dx. \end{aligned}$$

We introduce two new dependent variables  $\varphi_t = u$  and  $\psi_t = v$ , then the system (2.1)-(2.2) can be written as

$$(2.6) \quad \begin{cases} \frac{\partial U}{\partial t} = AU + F, & t > 0 \\ U(x, 0) = U^0(x) = (\varphi^0, \varphi^1, \psi^0, \psi^1, f_0)^T, \end{cases}$$

and

$$(2.7) \quad AU = \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_{xx} + \psi_x) \\ v \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{\mu_1}{\rho_2}v - \frac{\mu_1}{\rho_2} \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, \rho, s, t) ds \\ -\frac{1}{\tau}z_\rho(x, \rho, s, t) \end{pmatrix},$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-1}{\rho^2}f(\psi) \\ 0 \end{pmatrix}$$

with the domain

$$D(A) = \left\{ (\varphi, u, \psi, v, z)^T \in H : v = z(x, 0, s, t) \text{ in } (0, 1) \right\},$$

where

$$H = (H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1) \times (H^2(0, 1) \cap H_0^1(0, 1)) \\ \times H_0^1(0, 1) \times L^2((0, 1) \times (0, 1) \times (\tau_1, \tau_2)).$$

Clearly,  $D(A)$  is dense in  $H$ , we have the following existence and uniqueness result (see [6]).

**Theorem 2.1.** *Let  $U_0 \in \mathcal{H}$  and assume that(2.4)-(2.5) and  $\mu_2 < \mu_1$  hold. Then, there exists a unique solution  $U \in C(\mathbb{R}^+, \mathcal{H})$  of problem (2.1). Moreover, if  $U_0 \in D(A)$ , then  $U \in C(\mathbb{R}^+, D(A)) \cap C(\mathbb{R}^+, \mathcal{H})$ .*

### 3. STABILITY RESULT

In this section, we use the energy method to show that the solution of problem (2.1)–(2.2) decays exponentially, below we shall give the stability result.

**Theorem 3.1.** *Assume that(2.4)-(2.5) and  $\mu_2 < \mu_1$  hold. Assume that  $\frac{\rho_1}{\rho_2} = \frac{k}{b}$  also holds. Then, with respect to mild solutions, there exist  $\varpi_1 > 0$  and  $\varpi_2 > 0$  such that*

$$(3.1) \quad E(t) \leq \varpi_1 e^{-\varpi_2 t}, t \geq 0.$$

To achieve our goal we state and prove the following lemmas.

**Lemma 3.1.** *The energy functional  $E(t)$  of problem (2.1)–(2.2), defined by*

$$(3.2) \quad E(t) = \frac{1}{2} \int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + \frac{1}{2} \int_0^1 \left\{ K(\varphi_x + \psi)^2 + b\psi_x^2 \right\} dx \\ + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z(x, \rho, s, t) ds d\rho dx + \int_0^1 \tilde{f}(\psi) dx$$

satisfies

$$(3.3) \quad \frac{dE(t)}{dt} \leq -m_1 \int_0^1 \psi_t^2 dx \leq 0,$$

where  $m_1 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds$ .

*Proof.* Multiplying the first equation in (2.1) by  $\varphi_t$ , the second equation by  $\psi_t$ , integrating over  $(0, 1)$  and summing them up we get

$$(3.4) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + \frac{1}{2} \frac{d}{dt} \int_0^1 \{K(\varphi_x + \psi)^2 + b\psi_x^2\} dx \\ & = -\mu_1 \int_0^1 \psi_t^2 dx - \mu_1 \int_0^1 f(\psi) \psi_t dx - \int_0^1 \int_{\tau_1}^{\tau_2} \psi_t \mu_2(s) z(x, 1, s, t) ds dx. \end{aligned}$$

Multiplying the third equation of (2.1) by  $|\mu_2(s)| z(x, \rho, s, t)$  and integrating over  $(0, 1) \times (0, 1) \times (\tau_1, \tau_2)$ , we obtain

$$(3.5) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds dp dx \\ & + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\ & - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 0, s, t) ds dx = 0, \end{aligned}$$

by summing (3.5), (3.4) and using the fact that  $z(x, 0, s, t) = \varphi_t(x, t)$ , we have

$$(3.6) \quad \begin{aligned} \frac{dE(t)}{dt} & = - \left( \mu_1 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_0^1 \psi_t^2 dx \\ & - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\ & - \int_0^1 \psi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds dx. \end{aligned}$$

Now, using Young's inequality, we arrive at

$$(3.7) \quad \begin{aligned} & - \int_0^1 \psi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds dx \\ & \leq \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \psi_t^2 dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx. \end{aligned}$$

Inserting (3.7) in (3.6) and using (2.3), we have (3.2) and (3.3). The proof is complete. □

**Lemma 3.2.** *Let  $(\varphi, \psi, z)$  be the solution of (2.1)–(2.2). Then, the functional*

$$(3.8) \quad I_1(t) := - \int_0^1 (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) dx - \frac{\mu_1}{2} \int_0^1 \psi^2 dx.$$

satisfies

$$(3.9) \quad \begin{aligned} \frac{dI_1(t)}{dt} \leq & - \int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + c_0 \int_0^1 \psi_x^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \frac{\mu_1}{4} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx, \end{aligned}$$

*Proof.* Differentiating  $I_1(t)$ , we obtain

$$\begin{aligned} \frac{dI_1(t)}{dt} = & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_1 \int_0^1 \varphi \varphi_{tt} dx - \rho_2 \int_0^1 \psi_t^2 dx \\ & - \rho_2 \int_0^1 \psi \psi_{tt} dx - \mu_1 \int_0^1 \psi \psi_t dx, \end{aligned}$$

and using (2.1)<sub>1</sub>, (2.1)<sub>2</sub>, we get

$$(3.10) \quad \begin{aligned} \frac{dI_1(t)}{dt} = & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + b \int_0^1 \psi_x^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \int_0^1 f(\psi) \psi dx + \int_0^1 \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds dx. \end{aligned}$$

Applying Young's and Poincaré inequalities, we have

$$(3.11) \quad \begin{aligned} & \int_0^1 \psi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, s, t) ds dx \\ & \leq \mu_1 \int_0^1 \psi_x^2 dx + \frac{\mu_1}{4} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx, \end{aligned}$$

$$(3.12) \quad \begin{aligned} \int_0^1 |f(\psi) \psi| dx & \leq \int_0^1 |\psi|^\theta |\psi| |\psi| dx \\ & \leq \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\psi\| \\ & \leq c_1 \int_0^1 \psi_x^2 dx. \end{aligned}$$

By substituting (3.11), (3.12) in (3.10), we obtain (3.9). □

Now, let  $w$  be the solution of

$$(3.13) \quad -w_{xx} = \psi_x, \quad w(0) = w(1) = 0,$$

then we get

$$w(x, t) = - \int_0^x \psi(y, t) dy + x \left( \int_0^1 \psi(y, t) dy \right).$$

We have the following inequalities.

**Lemma 3.3.** *The solution of (3.13) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx \text{ and } \int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

*Proof.* We multiply equation (3.13) by  $w$ , integrate by parts and use the Cauchy–Schwarz inequality to obtain

$$(3.14) \quad \int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

Next, we differentiate (3.13) with respect to  $t$  and by the same procedure as above, we obtain

$$(3.15) \quad \int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma (3.3). □

**Lemma 3.4.** *Let  $(\varphi, \psi, z)$  be the solution of (2.1)–(2.2). Then, for any  $\varepsilon_2 > 0$ , the functional*

$$(3.16) \quad I_2(t) := \int_0^1 \left( \rho_2 \psi_t \psi + \rho_1 \varphi_t w + \frac{\mu_1}{2} \psi^2 \right) dx,$$

*satisfies*

$$(3.17) \quad \begin{aligned} \frac{dI_2(t)}{dt} \leq & -\frac{b}{2} \int_0^1 \psi_x^2 dx + \left( \frac{\rho_1}{4\varepsilon_2} + \rho_2 \right) \int_0^1 \psi_t^2 dx + \rho_1 \varepsilon_2 \int_0^1 \varphi_t^2 dx \\ & + \frac{\mu_1}{4\varepsilon_2} \int_0^1 \left( \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds \right) dx - \int_0^1 \tilde{f}(\psi) dx \end{aligned}$$

*Proof.* By differentiation  $I_2(t)$ , we obtain and by using (2.1)<sub>1</sub>, (2.1)<sub>2</sub>, we have

$$(3.18) \quad \begin{aligned} \frac{dI_2(t)}{dt} = & \rho_2 \int_0^1 \psi_t^2 dx - b \int_0^1 \psi_x^2 dx + \rho_1 \int_0^1 \varphi_t w_t dx - k \int_0^1 \psi^2 dx + k \int_0^1 w_x^2 dx \\ & - \int_0^1 f(\psi) \psi dx - \int_0^1 \psi \left( \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx \end{aligned}$$

Using Young’s inequality and (3.15), we have

$$(3.19) \quad \begin{aligned} \rho_1 \int_0^1 \varphi_t w_t dx & \leq \rho_1 \varepsilon_2 \int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\varepsilon_2} \int_0^1 w_t^2 dx \\ & \leq \rho_1 \varepsilon_2 \int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\varepsilon_2} \int_0^1 \psi_t^2 dx \end{aligned}$$

Using Young’s, Cauchy–Schwarz, Poincaré inequalities, we get

$$(3.20) \quad \begin{aligned} & - \int_0^1 \psi \left( \int_{\tau_1}^{\tau_2} |\mu_2(s) z(x, 1, s, t)| ds \right) dx \\ & \leq \delta_1 \int_0^1 \psi_x^2 dx + \frac{\mu_1}{4\delta_1} \int_0^1 \left( \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds \right) dx \end{aligned}$$

Cauchy–Schwarz and Poincaré’s inequalities, give

$$(3.21) \quad \begin{aligned} \int_0^1 |f(\psi) \psi| dx & \leq \int_0^1 |\psi|^\theta |\psi| |\psi| dx \\ & \leq \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\psi\| \\ & \leq c_1 \int_0^1 \psi_x^2 dx. \end{aligned}$$



By substituting (3.19), (3.20), (3.21) in (3.18), recalling (3.14), (3.15), (2.5) and letting  $\delta_1 = \frac{b}{2}$ , we obtain (3.17). The proof is now complete.  $\square$

**Lemma 3.5.** *Let  $(\varphi, \psi, z)$  be the solution of (2.1)–(2.2). Then, the functional*

$$(3.22) \quad I_3(t) := \rho_2 \int_0^1 \psi_t(\varphi_x + \psi) + \rho_2 \int_0^1 \psi_x \varphi_t dx,$$

satisfies

$$(3.23) \quad \begin{aligned} \frac{dI_3(t)}{dt} &\leq b[\psi_x \varphi_x]_0^1 dx + \left(\rho_2 + \frac{\mu_1^2}{k}\right) \int_0^1 \psi_t^2 dx - \frac{k}{4} \int_0^1 (\varphi_x + \psi)^2 dx \\ &+ c_1 \int_0^1 \psi_x^2 dx + \frac{\mu_1}{k} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx - \int_0^1 \tilde{f}(\psi) dx \\ &+ \left(\frac{\rho_2 k - \rho_1 b}{\rho_1}\right) \int_0^1 \psi_x (\varphi_x + \psi)_x dx, \end{aligned}$$

where  $c_1$  is a positive constant.

*Proof.* By differentiation  $I_3(t)$  and using (2.1)<sub>1</sub>, (2.1)<sub>2</sub>, we obtain

$$(3.24) \quad \begin{aligned} \frac{dI_3(t)}{dt} &= b[\psi_x \varphi_x]_0^1 + \rho_2 \int_0^1 \psi_t^2 dx - k \int_0^1 (\varphi_x + \psi)^2 dx - \mu_1 \int_0^1 \psi_t(\varphi_x + \psi) dx \\ &- \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s) (\varphi_x + \psi) z(x, 1, s, t) ds dx - \int_0^1 f(\psi)(\varphi_x + \psi) dx, \end{aligned}$$

By using Young's inequality, we have

$$(3.25) \quad \mu_1 \int_0^1 |\psi_t(\varphi_x + \psi)| dx \leq \frac{k}{4} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\mu_1^2}{k} \int_0^1 \psi_t^2 dx.$$

Using Young's and Cauchy Schwarz inequalities, we get

$$(3.26) \quad \begin{aligned} &\int_0^1 (\varphi_x + \psi) \int_{\tau_1}^{\tau_2} |\mu_2(s) z(x, 1, s, t)| ds dx \\ &\leq \frac{k}{4} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\mu_1^2}{k} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx. \end{aligned}$$

Young's, Cauchy Schwarz and Poincaré inequalities lead to

$$(3.27) \quad \begin{aligned} \int_0^1 f(\psi) \varphi_x dx &\leq \|\varphi_x\| \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \\ &\leq \frac{\delta_0}{2b^2} \int_0^1 \varphi_x^2 dx + \frac{b^2}{2\delta_0 \lambda_1} \int_0^1 \psi_x^2 dx \\ &\leq \frac{\delta_0}{2b^2} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\delta_0}{2b^2} \int_0^1 \psi^2 dx + \frac{b^2}{2\delta_0 \lambda_1} \int_0^1 \psi_x^2 dx \\ &\leq \frac{\delta_0}{2b^2} \int_0^1 (\varphi_x + \psi)^2 dx + \left(\frac{\delta_0}{2\lambda_1 b^2} + \frac{b^2}{2\delta_0 \lambda_1}\right) \int_0^1 \psi_x^2 dx. \end{aligned}$$

Inserting (3.25)–(3.27) in (3.24) and letting  $\delta_0 = \frac{1}{2}kb^2$ , we obtain (3.23).  $\square$

Next, in order to handle the boundary terms, appearing in (3.23), we define the function

$$q(x) = -4x + 2, \quad x \in (0, 1)$$

So, we have the following result.

**Lemma 3.6.** *Let  $(\varphi, \psi, z)$  be the solution of (2.1)–(2.2), then for any  $\varepsilon_1 > 0$ , the following estimate holds*

$$\begin{aligned} b[\psi_x \varphi_x]_0^1 &\leq -\frac{b\rho_2}{4\varepsilon_1} \frac{d}{dt} \int_0^1 q\psi_t\psi_x dx - \frac{\rho_1\varepsilon_1}{k} \frac{d}{dt} \int_0^1 q\varphi_t\varphi_x dx + 3\varepsilon_1 \int_0^1 \varphi_x^2 dx \\ &+ \left(\frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1}\right) \int_0^1 \psi_t^2 dx + \left(\frac{k^2\varepsilon_1^2}{4} + \frac{\varepsilon_1}{4b^2}\right) \int_0^1 (\varphi_x + \psi)^2 dx \\ &+ \frac{b}{4\varepsilon_1} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) dx \\ (3.28) \quad &+ \left(\frac{b^2}{2\varepsilon_1^2} + \frac{1}{4\lambda_1 b^2} + \frac{b^2}{8\varepsilon_1^2 \lambda_1} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1\right) \int_0^1 \psi_x^2 dx \end{aligned}$$

*Proof.* By using Young’s inequality, we easily see that, for  $\varepsilon_1 > 0$ ,

$$(3.29) \quad b[\psi_x \varphi_x]_0^1 \leq \varepsilon_1 [\varphi_x^2(1) + \varphi_x^2(0)] + \frac{b^2}{4\varepsilon_1} [\psi_x^2(1) + \psi_x^2(0)],$$

we need the following fact

$$\frac{d}{dt} \int_0^1 b\rho_2 q\psi_t\psi_x dx = b\rho_2 \int_0^1 q\psi_{tt}\psi_x dx + b\rho_2 \int_0^1 q\psi_t\psi_{xt} dx.$$

On the other hand

$$\begin{aligned} b\rho_2 \int_0^1 q\psi_{tt}\psi_x dx &= b^2 \int_0^1 q\psi_{xx}\psi_x dx - kb \int_0^1 q(\varphi_t + \psi)\psi_x dx \\ &- b \int_0^1 \int_{\tau_1}^{\tau_2} q\psi_x \mu_2(s) z(x, 1, s, t) ds dx - b \int_0^1 qf(\psi)\varphi_x dx \\ &\leq -b^2 [\psi_x^2(1) + \psi_x^2(0)] + 2b^2 \int_0^1 \psi_x^2 dx + (k^2\varepsilon^2 + \frac{\varepsilon}{b^2}) \int_0^1 (\varphi_x + \psi)^2 dx \\ &+ \left(\frac{b^2}{\varepsilon^2} + \frac{\varepsilon}{2\lambda_1 b^2} + \frac{b^2}{2\varepsilon\lambda_1} + \mu_1 b\right) \int_0^1 \psi_x^2 dx \\ (3.30) \quad &+ b \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx. \end{aligned}$$

Therefore

$$b\rho_2 \int_0^1 q\psi_t\psi_{xt} dx = 2\rho_2 b \int_0^1 \psi_t^2 dx$$

Similarly

$$\begin{aligned} \frac{d}{dt} \int_0^1 \rho_1 q\varphi_t\varphi_x dx &= \int_0^1 q(\varphi_t + \psi)\varphi_x dx + \int_0^1 \rho_1 q\varphi_t\varphi_{xt} dx \\ &\leq -k [\varphi_x^2(1) + \varphi_x^2(0)] + 3k \int_0^1 \varphi_x^2 dx \\ &+ k \int_0^1 \psi_x^2 dx + 2\rho_1 \int_0^1 \psi_t^2 dx \end{aligned}$$

which, along with (3.29)-(3.30), gives us (3.28). The proof is now complete.  $\square$

**Lemma 3.7.** *Let  $(\varphi, \psi, z)$  be the solution of (2.1)-(2.2). Then, for  $\eta_1 > 0$ , the functional*

$$(3.31) \quad F_4(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx,$$

satisfies

$$(3.32) \quad \begin{aligned} F_4'(t) &\leq -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \\ &\quad - \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx + \mu_1 \int_0^1 \psi_t^2 dx. \end{aligned}$$

*Proof.* Differentiating  $F_4(t)$  and using (2.1)<sub>3</sub>, we obtain

$$\begin{aligned} F_4'(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z(x, \rho, s, t) z_\rho(x, \rho, s, t) ds d\rho dx \\ &= - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| \frac{\partial}{\partial \rho} [z^2(x, \rho, s, t)] ds d\rho dx \end{aligned}$$

Integration by parts gives and using the fact that  $z(x, 0, s, t) = \psi_t$  and  $e^{-s} \leq e^{-s\rho} \leq 1$ , we get for all  $\rho \in [0, 1]$

$$\begin{aligned} F_4'(t) &\leq - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t) ds dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \psi_t^2 dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

Since  $-e^{-s}$  is an increasing function, we have  $-e^{-s} \leq -e^{-\tau_2}$  for all  $s \in [\tau_1, \tau_2]$ . Finally, setting  $\eta_1 = e^{-\tau_2}$  and recalling (2.3), we obtain (3.32).  $\square$

Now, we define the Lyapunov functional  $L(t)$  by

$$(3.33) \quad L(t) := NE(t) + \frac{1}{8}I_1(t) + N_1I_2(t) + I_3(t) + N_2I_4(t),$$

where  $N_1, N_2$  and  $N$  are positive constants.

**Lemma 3.8.** *Let  $(\varphi, \psi, z)$  be the solution of (2.1)-(2.2). Then, there exists two positive constants  $\beta_1$  and  $\beta_2$  such that the Lyapunov functional  $L(t)$  satisfies*

$$(3.34) \quad \beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \forall t \geq 0,$$

and

$$(3.35) \quad L'(t) \leq -\lambda_1 E(t) + \left( \frac{\rho_2 k - \rho_1 b}{\rho_1} \right) \int_0^1 \psi_x (\varphi_x + \psi)_x dx.$$

*Proof.* Let

$$L(t) := NE(t) + \frac{1}{8}I_1(t) + N_1I_2(t) + I_3(t) + N_2I_4(t),$$

then

$$\begin{aligned} |L(t) - NE(t)| &\leq \frac{\rho_1}{8} \int_0^1 |\varphi\varphi_t| dx + \frac{\rho_2}{8} \int_0^1 |\psi\psi_t| dx + \frac{\mu_1}{16} \int_0^1 \psi^2 dx \\ &\quad + N_1\rho_2 \int_0^1 |\psi_t\psi| dx + N_1\rho_1 \int_0^1 |\varphi_t w| dx + N_1 \frac{\mu_1}{2} \int_0^1 \psi^2 dx \\ &\quad + \rho_2 \int_0^1 |\psi_t(\varphi_x + \psi)| dx + \rho_2 \int_0^1 |\psi_x\varphi_t| dx \\ &\quad + N_2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

Exploiting Young's, Poincaré and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned} |L(t) - NE(t)| &\leq C \int_0^1 \left( \psi_x^2 + \psi_t^2 + \varphi_t^2 + (\varphi_x + \psi)^2 + \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, 1, s, t) ds d\rho \right) dx \\ &\quad + \int_0^1 \tilde{f}(\psi) dx \\ &\leq CE(t), \end{aligned}$$

Now, combining (3.3), (3.9), (3.17), (3.23) and (3.32), we get

By differentiating  $L(t)$ , exploiting (3.3), (3.9), (3.17), (3.23), (3.28), (3.32) and setting  $\varepsilon_2 = \frac{\rho_1}{16N_1}$ , we get

$$\begin{aligned} \frac{dL(t)}{dt} &= - \left( Nm_1 - N_1(4N_1 + \rho_2) - N_2\mu_1 - \left( \rho_2 + \frac{\mu_1^2}{k} \right) - \left( \frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1} \right) + \rho_2 \right) \int_0^1 \psi_t^2 dx \\ &\quad - \left( \frac{b}{2}N_1 - \frac{c_0}{8} - \left( \frac{b^2}{2\varepsilon_1^2} + \frac{1}{4b^2} + \frac{b^2}{8\varepsilon_1^2} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1 \right) \right) \int_0^1 \psi_x^2 dx \\ &\quad - \left( \frac{k}{8} - \varepsilon_1 \left( k^2\varepsilon_1 + \frac{1}{b^2} \right) \right) \int_0^1 (\varphi_x + \psi)^2 dx \\ &\quad - N_2\beta \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \\ &\quad - \left( N_2\beta - 4N_1^2\mu_1 - \frac{\mu_1}{32} - \frac{\mu_1}{k} - \frac{b}{4\varepsilon_1} \right) \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\ &\quad - \frac{\rho_1}{16} \int_0^1 \varphi_t^2 dx - (N_1 + 1) \int_0^1 \tilde{f}(\psi) dx, + \left( \frac{\rho_2 k - \rho_1 b}{\rho_1} \right) \int_0^1 \psi_x (\varphi_x + \psi)_x dx. \end{aligned}$$

First, we choose  $\varepsilon_1$  small enough such that

$$\frac{k}{8} - \varepsilon_1 \left( k^2\varepsilon_1 + \frac{1}{b^2} \right) > 0.$$

After, we take  $N_1$  large so that

$$\frac{b}{2}N_1 - \frac{c_0}{8} - \left( \frac{b^2}{2\varepsilon_1^2} + \frac{1}{4b^2} + \frac{b^2}{8\varepsilon_1^2} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1 \right) > 0.$$

Then, we select  $N_2$  large to satisfies

$$N_2\beta - 4N_1^2\mu_1 - \frac{\mu_1}{32} - \frac{\mu_1}{k} - \frac{b}{4\varepsilon_1} > 0.$$

By finally choose  $N$  large enough (even larger so that 1.1 remains valid) such that

$$Nm_1 - N_1 \left( \frac{\rho_1}{4\varepsilon_2} + \rho_2 \right) - N_2\mu_1 - \left( \rho_2 + \frac{\mu_1^2}{k} \right) - \left( \frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1} \right) + \rho_2 > 0,$$

we obtain (3.35). The proof is complete.  $\square$

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