STABILITY RESULT FOR A WEAKLY NONLINEARLY DAMPED POROUS SYSTEM WITH DISTRIBUTED DELAY

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Abstract. In this paper, we consider a one-dimensional porous system damped with a single weakly nonlinear feedback and distributed delay term. Without imposing any restrictive growth assumption near the origin on the damping term, we establish an explicit and general decay rate, using a multiplier method and some properties of convex functions in case of the same speed of propagation in the two equations of the system. The result is new and opens more research areas into porous-elastic system.
1. Introduction

In this paper, we consider the following porous system:

\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x + \mu_1 u_t + \int_{\tau_1}^{t_2} \mu_2(s) u_t(x, t - s) \, ds &= 0, \quad x \in (0, 1), \quad t > 0, \\
j\phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \alpha(t) g(\phi_t) &= 0, \quad x \in (0, 1), \quad t > 0, \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \\
\phi(x, 0) &= \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in (0, 1), \\
u_x(0, t) &= u_x(1, t), \quad \phi(0, t) = \phi(1, t) = 0 \\
u_t(x, -t) &= f_0(x, t) \quad \text{in} \quad (0, 1) \times (0, \tau_2)
\end{aligned}
\]  

(1.1)

Firstly, to deal with the delay term, we introduce the new variable \cite{17}

\[
z(x, \rho, s, t) = u_t(x, t - \rho s), \quad x \in (0, 1), \quad \rho \in (0, 1), \quad \rho \in (\tau_1, \tau_2), \quad t > 0
\]

Then we obtain

\[
sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0, \quad x \in (0, 1), \quad \rho \in (0, 1), \quad \rho \in (\tau_1, \tau_2), \quad t > 0
\]

Then problem (1.1) is equivalent to

\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x + \mu_1 u_t + \int_{\tau_1}^{t_2} \mu_2(s) z(x, 1, t, s) \, ds &= 0, \quad x \in (0, 1), \quad t > 0, \\
j\phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \alpha(t) g(\phi_t) &= 0, \quad x \in (0, 1), \quad t > 0, \\
sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) &= 0, \quad x \in (0, 1), \quad \rho \in (0, 1), \quad \rho \in (\tau_1, \tau_2), \quad t > 0 \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \\
\phi(x, 0) &= \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in (0, 1), \\
u_x(0, t) &= u_x(1, t), \quad \phi(0, t) = \phi(1, t) = 0 \\
z(x, \rho, s, 0) &= f_0(x, \rho s), \quad (x, \rho, s) \in (0, 1) \times (0, 1) \times (\tau_1, \tau_2)
\end{aligned}
\]  

(1.2)

In recent paper, Apalara in \cite{2} considered the following on-dimensional porous system damped with a single weakly nonlinear feedback

\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \quad x \in (0, 1), \quad t > 0, \\
j\phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \alpha(t) g(\phi_t) &= 0, \quad x \in (0, 1), \quad t > 0, \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \\
\phi(x, 0) &= \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in (0, 1), \\
u_x(0, t) &= u_x(1, t), \quad \phi(0, t) = \phi(1, t) = 0
\end{aligned}
\]

Without in pasing an explicit and general decay rate, he used a multiplier method and some proprieties of convex functions in case of the same speed of propagation in the both equation of the system. The same author, in \cite{3} considered a porous-elastic system with memory term acting only on the porous equation, with
the mixed boundary Neumann–Dirichlet conditions, he proved a general decay result, for which exponential
and polynomial decay results are special cases.

Back to system (1.1), it is to be noted that when \( \mu_1 = \mu_2 = 0 \) and replacing the term \( \alpha(t)g(\phi_t) \) by the
term \( \int_0^t g(t - s)u_{xx}(x,s)\,ds \) then (1.1) is equivalent to the well-known Timoshenko system of memory type
which is exponentially stable depending of the relaxation function \( g \) and provided that the wave speeds of
the system are equal (See [1,15]).

Messaoudi and Fareh [16] investigated the following system:

\[
\begin{align*}
\rho u_{tt} & = \mu u_{xx} + b\phi_x - \beta \theta_x, \quad \text{in} \ (0,1) \times (0,\infty), \\
\delta \phi_{tt} & = \alpha \phi_{xx} - bu_x + \xi \phi + m\theta + \tau \phi_t, \quad \text{in} \ (0,1) \times (0,\infty), \\
c \phi_t & = -q_x - \beta u_{tx} - m\phi_t, \quad \text{in} \ (0,1) \times (0,\infty), \\
\tau_0 q_t - q + k\theta_x & = 0, \quad \text{in} \ (0,1) \times (0,\infty),
\end{align*}
\]

and established, using the energy method, an exponential decay result. For more results on the subject,
we refer the reader to [5,10,11,19].

Concerning the weight of the delay, we assume that
\[
\int_{\tau_1}^{\tau_2} |\mu_2(s)| \,ds < \mu_1
\]
and establish the well-posedness as well as the exponential stability results of the energy \( E(t) \), defined by
\[
E(t) = \frac{1}{2} \int_0^1 \left[ \rho u_t^2 + \mu u_{xx}^2 + \delta \phi^2 + \delta \phi_x^2 + j \phi_t^2 + 2b\phi u_x \right] \,dx \\
+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| \,dz \,dx \quad (1.3)
\]

2. Preliminaries

In this section, we present some materials needed in the proof of our result. We assume \( \alpha \) and \( g \) satisfy
the following hypotheses:

(H1) \( \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \) is a non-increasing differentiable function;

(H2) \( g : \mathbb{R} \to \mathbb{R} \) is a non-decreasing \( C^0 \)-function such that there exist positive constants \( c_1, c_2, \eta \) and
\( G \in C^1 ([0,\infty)) \), with \( G(0) = 0 \), and \( G \) is linear or strictly convex \( C^2 \)-function on \( (0,\eta] \) such that
\[
\begin{align*}
\left\{ \begin{array}{l}
s^2 + g^2(s) \leq G^{-1}(sg(s)) \text{ for all } |s| \leq \eta \\
c_1 |s| \leq |g(s)| \leq c_2 |s| \text{ for all } |s| \geq \eta
\end{array} \right.
\end{align*}
\]

Remark 2.1. Hypothesis (H2) implies that \( sg(s) > 0 \) for all \( s \neq 0 \).

* According to our knowledge, hypothesis (H2) with \( \eta = 1 \) was first introduced by Lasiecka and Tataru [13].

They established a decay result, which depends on the solution of an explicit nonlinear ordinary differential
equation. Furthermore, they proved that the monotonicity and continuity of \( g \) guarantee the existence of the
function \( G \) defined in (H2).
For completeness purpose we state, without proof, the existence and regularity result of system (1.1). First, we introduce the following spaces:

\[
\mathcal{H} = H^1_+ (0, 1) \times L^2_+ (0, 1) \times H^1 (0, 1) \times L^2 (0, 1) \times \tau_1 \tau_2 ,
\]

and

\[
\tilde{H} = \phi_0 \in \left[ H^2_+ (0, 1) \cap H^1_+ (0, 1) \right] \times H^1_+ (0, 1) \times \left[ H^2 (0, 1) \cap H^1 (0, 1) \right] \\
\times H^1_+ (0, 1) \times L^2 (0, 1) \times \tau_1 \tau_2 ,
\]

where

\[
L^2_+ (0, 1) = \left\{ \psi \in L^2 (0, 1) : \int_0^1 \psi (x) \, dx = 0 \right\}, \\
H^1_+ (0, 1) = H^1_+ (0, 1) \times L^2_+ (0, 1), \\
H^2_+ (0, 1) = \left\{ \psi \in H^2 (0, 1) : \psi_x (0) = \psi_x (1) = 0 \right\}.
\]

For \( U = (u, u_t, \phi, \phi_t, z) \), we have the following existence and regularity result:

**Proposition 2.1.** Assume that (H1) and (H2) are satisfied. Then for all \( U_0 \in \mathcal{H} \), the system (1.1) has a unique global (weak) solution

\[
u \in C \left( \mathbb{R}^+ ; H^1_+ (0, 1) \right) \cap C^1 \left( \mathbb{R}^+ ; L^2_+ (0, 1) \right), \quad \phi \in C \left( \mathbb{R}^+ ; H^1 (0, 1) \right) \cap C^1 \left( \mathbb{R}^+ ; L^2 (0, 1) \right).
\]

Moreover, if \( U_0 \in \tilde{H} \), then the solution satisfies

\[
u \in L^\infty \left( \mathbb{R}^+ ; H^2_+ (0, 1) \cap H^1_+ (0, 1) \right) \cap W^{1,\infty} \left( \mathbb{R}^+ ; H^1_+ (0, 1) \right) \cap W^{2,\infty} \left( \mathbb{R}^+ ; L^2_+ (0, 1) \right), \\
\phi \in L^\infty \left( \mathbb{R}^+ ; H^2 (0, 1) \cap H^1_0 (0, 1) \right) \cap W^{1,\infty} \left( \mathbb{R}^+ ; H^1_0 (0, 1) \right) \cap W^{2,\infty} \left( \mathbb{R}^+ ; L^2 (0, 1) \right)
\]

**Remark 2.2.** This result can be proved using the theory of maximal nonlinear monotone operators (see [8]).

### 3. Technical Lemmas

In this section, we state and prove our stability results for the energy of system (1.1) by using the multiplier technique. To achieve our goal, we need the following lemmas.

**Lemma 3.1.** Let \( (u, \phi, z) \) be the solution of (1.2), then we have

\[
E'(t) \leq -m_e \int_0^1 u_t^2 \, dx - \int_0^1 \alpha (t) \phi \phi_t \, dx \leq 0
\]
Proof. Multiplying (1.2)₁ and (1.2)₂ by \( u_t, \phi_t \) respectively, and integrating over \((0, 1)\), using integration by parts and the boundary conditions, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( \rho u_t^2 + \mu u_x^2 + \xi \phi^2 + \delta \phi_x^2 + j \phi_t^2 + 2b \phi u_x \right) dx = \tag{3.2}
\]

\[
- \int_0^1 \alpha(t) \phi_t g(\phi_t) dx - \mu_1 \int_0^1 u_t^2 dx - \int_0^1 u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds dx
\]

Multiplying (1.2)₃ by \(|\mu_2(s)|z\), integrating the product over \((0, 1) \times (0, 1) \times (\tau_1, \tau_2)\), and recall that \(z(x, 0, s, t) = u_t\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, t, s) ds dx = \tag{3.3}
\]

\[- \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, t, s) ds dx \]

A combination of (3.2) and (3.3) gives

\[
E'(t) = - \int_0^1 \alpha(t) \phi_t g(\phi_t) dx - \mu_1 \int_0^1 u_t^2 dx - \int_0^1 u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds dx
\]

with

\[- \int_0^1 u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds dx \leq \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 u_t^2 dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, t, s) ds dx\]

then

\[
E'(t) \leq - \int_0^1 \alpha(t) \phi_t g(\phi_t) dx - \left( \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \right) \int_0^1 u_t^2 dx
\]

taking \(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| = m_e\) we obtain (3.1). \(\square\)

Lemma 3.2. Assume that (H₁) and (H₂) hold. Then, for all \(U_0 \in \mathcal{H}\), the functional

\[
F_1(t) = j \int_0^1 \phi_t \phi dx + \frac{b \rho}{\mu} \int_0^1 \phi \int_0^x u_t(y) dy dx \tag{3.4}
\]

satisfies, for any \(\varepsilon_1 > 0\)

\[
F_1'(t) \leq \left( j + \frac{\varepsilon_1 b \rho}{\mu} \right) \int_0^1 \phi_t^2 dx - j \delta \int_0^1 \phi_x^2 dx + bj \varepsilon_1 \int_0^1 u_x^2 dx + \frac{b \rho}{4 \varepsilon_1} \int_0^1 u_t^2 dx + \left( j \alpha(0) \varepsilon_1 + \frac{bj}{4 \varepsilon_1} - \varepsilon j \right) \int_0^1 \phi_x^2 dx + \frac{j \alpha(0)}{4 \varepsilon_1} \int_0^1 \phi \phi_t dx \tag{3.5}
\]

Proof. Differentiating \(F_1(t)\), taking into account (1.2) using integrating by parts, and Young’s inequality, we obtain

\[
F_1'(t) \leq j \int_0^1 \phi_t^2 dx - j \delta \int_0^1 \phi_x^2 dx + bj \varepsilon_1 \int_0^1 u_x^2 dx + \frac{c \rho b j}{4 \varepsilon_1} \int_0^1 \phi_x^2 dx - \varepsilon j c_p \int_0^1 \phi_t^2 dx
\]

\[- j \int_0^1 \alpha(t) \phi_t g(\phi_t) dx + \frac{b \rho}{\mu} \int_0^1 \phi_t \int_0^x u_t(y) dy dx\]

\[+ \frac{b \rho}{\mu} \int_0^1 \phi \frac{d}{dt} \left( \int_0^x u_t(y) dy \right) dx\]
By Cauchy-Schwartz inequality, it is clear that
\[
\int_0^1 \left( \int_0^x u_t (y) dy \right)^2 dx \leq \int_0^1 \left( \int_0^1 u_t dx \right)^2 dx \leq \int_0^1 u_t^2 dx
\]
then
\[
F'_1 (t) \leq j \int_0^1 \phi_t^2 dx - j \int_0^1 \phi_t^2 dx + b j \epsilon_1 \int_0^1 u_{x}^2 dx + \frac{c p b j}{4 \epsilon_1} \int_0^1 \phi_t^2 dx - \xi j c_p \int_0^1 \phi_t^2 dx
\]
\[
- j \int_0^1 \alpha (t) \phi_t (\phi_t) dx
\]
\[
+ \frac{\epsilon_1 b p}{\mu} \int_0^1 \phi_t^2 dx + \frac{b p}{4 \epsilon_1 \mu} \int_0^1 \left( \int_0^x u_t (y) dy \right)^2 dx
\]
\[
+ \frac{b p}{\mu} \int_0^1 \phi_t \frac{d}{dt} \left( \int_0^x u_t (y) dy \right) dx
\]
thus we obtain
\[
F'_1 (t) \leq \left( j + \frac{\epsilon_1 b p}{\mu} \right) \int_0^1 \phi_t^2 dx - j \int_0^1 \phi_t^2 dx + b j \epsilon_1 \int_0^1 u_{x}^2 dx + \frac{b p}{4 \epsilon_1 \mu} \int_0^1 u_t^2 dx
\]
\[
+ \left( j \alpha (t) \epsilon_1 + \frac{b j}{4 \epsilon_1} - \xi j \right) \int_0^1 \phi_t^2 dx + \frac{j \alpha (t)}{4 \epsilon_1} \int_0^1 g^2 (\phi_t) dx
\]
\[
\square
\]

**Lemma 3.3.** Assume that (H1), (H2) and (3.8) hold. Then, for all \( U_0 \in \mathcal{H} \), the functional
\[
F_2 (t) = b \int_0^1 \phi_x u_t dx + b \int_0^1 \phi_t u_x dx
\]
(3.6)
satisfies, for any \( \epsilon_2 > 0 \)
\[
F'_2 (t) \leq \left( \frac{b^2}{\rho} + \frac{\epsilon_2 b \mu_1}{\mu} + \frac{b n_0}{2 \rho} \right) \int_0^1 \phi_t^2 dx - \left( \frac{b^2}{j} - \frac{b \xi}{4 \epsilon_2 j} - \frac{b}{j} \alpha (t) \right) \int_0^1 u_t^2 dx
\]
\[
+ \frac{b \mu_1}{4 \epsilon_2 \rho} \int_0^1 u_t^2 dx + \frac{b \xi}{j} \int_0^1 \phi_t^2 dx + \frac{b}{j} \alpha (t) \int_0^1 g^2 (\phi_t) dx
\]
\[
+ \frac{1}{2} \frac{b}{\rho} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2 (s)| \phi_t^2 (x, 1, s, t) ds dx
\]
(3.7)

**Proof.** Simple computations give
\[
F'_2 (t) = \frac{b^2}{\rho} \int_0^1 \phi_t^2 dx - \frac{b^2}{j} \int_0^1 u_t^2 dx
\]
\[
+ \frac{b \mu_1}{\rho} \int_0^1 u_{xx} \phi_x dx - \frac{b \mu_1}{\rho} \int_0^1 \phi_x u_t dx
\]
\[
+ \frac{b \delta}{j} \int_0^1 \phi_{xx} u_t dx - \frac{b \xi}{j} \int_0^1 \phi u_{x} dx
\]
\[
- \frac{b}{\rho} \int_0^1 \phi_t \int_{\tau_1}^{\tau_2} \mu_2 (s) u_t (x, 1, t, s) ds dx - \frac{b}{j} \int_0^1 \alpha (t) u_{x} g (\phi_t) dx
\]
taking into account the fact that
\[
\frac{\mu}{\rho} = \frac{\delta}{j}
\] (3.8)
and using Young’s inequality
\[
F_2'(t) \leq \left( \frac{b_2^2}{\rho} + \varepsilon_2 \frac{b\mu_1}{\rho} \right) \int_0^1 \phi_2^2 dx + \left( \frac{b\varepsilon_2}{4\varepsilon_2 j} - \frac{b_2^2}{j} + \frac{b}{j} \alpha(t) \right) \int_0^1 u_2^2 dx \\
+ \frac{b\mu_1}{4\varepsilon_2 \rho} \int_0^1 u_1^2 dx + \varepsilon_2 \frac{b\xi}{j} \int_0^1 \phi_1^2 dx + \frac{b}{j} \alpha(t) \int_0^1 g^2(\phi) dx \\
- \frac{b}{\rho} \int_0^1 \phi \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds dx
\]
\[
- \frac{b}{\rho} \int_0^1 \phi_2 \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds dx \leq \frac{b}{2\rho} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \phi_2^2 dx + \frac{b}{2\rho} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx
\]
\[
F_2'(t) \leq \left( \frac{b_2^2}{\rho} + \varepsilon_2 \frac{b\mu_1}{\rho} + \frac{b\mu_1}{4\varepsilon_2 \rho} \right) \int_0^1 \phi_2^2 dx - \left( \frac{b_2^2}{j} + \frac{b\xi}{4\varepsilon_2 j} - \frac{b}{j} \alpha(t) \right) \int_0^1 u_2^2 dx \\
+ \frac{b\mu_1}{4\varepsilon_2 \rho} \int_0^1 u_1^2 dx + \varepsilon_2 \frac{b\xi}{j} \int_0^1 \phi_1^2 dx + \frac{b}{j} \alpha(t) \int_0^1 g^2(\phi) dx \\
+ \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, t, s) ds dx
\]
with \( \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds = n_0 \)
\[
F_3(t) = -\rho \int_0^1 u_1 u dx
\] (3.9)
satisfies, for any \( \varepsilon_3 > 0 \)
\[
F_3'(t) = + \left( \mu + \frac{n_0 c_p}{2} + c_p b\varepsilon_3 \right) \int_0^1 u_2^2 dx \\
+ \frac{b}{4\varepsilon_3} \int_0^1 \phi_2^2 dx - \left( \rho - \frac{\mu_1}{4\varepsilon_3} \right) \int_0^1 u_1^2 dx \\
+ \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, t, s) ds dx
\] (3.10)

**Proof.** A simple differentiation of \( F_3(t) \), using the first equation in (1.2), give
\[
F_3'(t) = -\rho \int_0^1 u_1^2 dx + \mu \int_0^1 u_2^2 dx \\
+ c_p b\varepsilon_3 \int_0^1 u_2^2 dx + \frac{b}{4\varepsilon_3} \int_0^1 \phi_2^2 dx \\
+ \mu_1 c_p b\varepsilon_3 \int_0^1 u_2^2 dx + \frac{\mu_1}{4\varepsilon_3} \int_0^1 u_2^2 dx \\
+ \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| u u_t(x, 1, t, s) ds dx
\]
\[
\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| u u_t(x, 1, t, s) ds dx \leq \frac{c_p}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 u_2^2 dx + \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, t, s) ds dx
\]
Lemma 3.5. The functional

\[ F_4(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-\rho z} |\mu_2(s)| z^2 (x, \rho, t, s) dsd\rho dx \]  \hspace{1cm} (3.11)

satisfies, for some positive constant \( m_1 \), the following estimate

\[
F'_4(t) \leq -m_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2 (x, 1, t, s) dsdx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 u_t^2 dx \\
- m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2 (x, \rho, t, s) dsd\rho dx 
\]  \hspace{1cm} (3.12)

Proof. With

\[ sz_t (x, \rho, t, s) + z_\rho (x, \rho, t, s) = 0 \text{ in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty) \]  \hspace{1cm} (3.13)

\[ z_t (x, \rho, t, s) = -\frac{1}{s} z_\rho (x, \rho, t, s) \]

Differentiating \( F_4(t) \), and using the equation (3.13), we obtain

\[
F'_4(t) = 2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-\rho z} |\mu_2(s)| z z_t (x, \rho, t, s) dsd\rho dx \\
= - \frac{\partial}{\partial \rho} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\rho z} |\mu_2(s)| z^2 (x, \rho, t, s) dsd\rho dx \\
- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-\rho z} |\mu_2(s)| z^2 (x, \rho, t, s) dsd\rho dx 
\]

\[
F'_4(t) = - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| [e^{-\rho z} z^2 (x, 1, t, s) - z^2 (x, 0, t, s)] dsdx \\
- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-\rho z} |\mu_2(s)| z^2 (x, \rho, t, s) dsd\rho dx 
\]

Using the fact that \( z(x, 0, t, s) = u_t \) and \( e^{-s} \leq e^{-\rho} \leq 1 \), for all \( \rho \in [0, 1] \), we obtain

\[ F'_4(t) \leq - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} z^2 (x, 1, t, s) dsdx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 u_t^2 dx \\
- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s} |\mu_2(s)| z^2 (x, \rho, t, s) dsd\rho dx 
\]

Because \(-se^{-s}\) is an increasing function, we have \(-se^{-s} \leq -se^{-\tau_2}\), for all \( s \in [\tau_1, \tau_2] \)
Finally, setting \( m_1 = e^{-\tau_2} \), with \( \int_{\tau_1}^{\tau_2} |\mu_2(s)| < \mu_1 \), we obtain

\[
F'_1(t) \leq -m_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s) ds dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 u_t^2 dx
- m_1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho |\mu_2(s)| z^2(x,\rho,t,s) ds d\rho dx
\]

**Lemma 3.6.** Suppose (H1), (H2), and Eq. (3.8) hold. Let \( U_0 \in \mathcal{U} \). Then, for \( N, N_1, N_2, N_3 > 0 \) sufficiently large, the Lyapunov functional defined by

\[
\mathcal{L}(t) := NE(t) + N_1 F_1(t) + N_2 F_2(t) + F_3(t) + N_4 F_4(t)
\]

satisfies, for some positive constants \( d_1, d_2 \) and \( k_1 \)

\[
d_1 \mathcal{L}(t) \leq E(t) \leq d_2 \mathcal{L}(t), \ \forall t \geq 0 \quad (3.14)
\]

and

\[
\mathcal{L}'(t) \leq -k_1 E(t) + c \int_0^1 \left( \phi_i^2 + g^2(\phi_i) \right) dx, \ \forall t \geq 0 \quad (3.15)
\]

with

\[
\mathcal{L}'(t) \leq \left[ \frac{b_\rho}{4 \xi_1 \mu} N_1 - Nm_e + N_3 \mu_1 + \frac{b \mu_1}{4 \xi_2 \rho} N_2 - (\rho - \mu_1) \right] \int_0^1 u_t^2 dx
+ \left( N_1 \left( j + \frac{\varepsilon_1 b \rho}{\mu} \right) \right) \int_0^1 \phi_t^2 dx
+ \left( b j \varepsilon_1 N_1 + \left( \mu + \frac{\varepsilon_0}{2} + \varepsilon_3 \right) N_2 \left( \frac{b^2}{j} - \frac{b c}{4 \xi_2 j} - \frac{b}{j} \alpha(t) \right) \right) \int_0^1 \phi_t^2 dx
+ \left( N_2 \frac{1}{2 \rho} \left( 2b^2 + 2 \varepsilon_2 b \mu_1 + b m_0 \right) - j \delta N_1 + \frac{b}{4 \xi_3} \right) \int_0^1 \phi^2 dx
+ \left( \varepsilon_2 b \frac{b c}{j} N_2 + N_1 \left( j \alpha(t) \varepsilon_1 + \frac{b j}{4 \xi_3} - \varepsilon_3 \right) \right) \int_0^1 \phi^2 dx
+ \left( \frac{b \alpha(t)}{4 \xi_3} + \frac{b j}{j} \alpha(t) N_2 \right) \int_0^1 g^2(\phi_i) dx
+ \left( \frac{1}{2} \left( \frac{b N_2}{\rho} + 1 \right) - m_1 N_3 \right) \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx
- m_1 N_3 \int_0^1 \int_{\tau_1}^{\tau_2} \rho |\mu_2(s)| z^2(x,\rho,t,s) ds d\rho dx - N \int_0^1 \alpha(t) \phi_i g(\phi_i) dx
\]

At this point, we have to choose our constants very carefully. First, choosing \( \varepsilon_3 \ll 1 \), and \( \varepsilon_1, \varepsilon_2 \) small enough such that

\[
\varepsilon_1 \leq \frac{b \rho N_1}{4 \mu (Nm_e - N_3 \mu_1)}, \quad \varepsilon_2 \leq \frac{b \mu_1 N_2}{4 \rho}
\]
Moreover, we pick $N_i$ for $i = 1, 2, 3$ large enough so that

$$N_2 \geq \frac{bj\varepsilon_1 N_1 + (\mu + \frac{n_0}{2} + b\varepsilon_3)}{b - \frac{b\xi}{4\varepsilon_2j} - \frac{b}{j} \alpha(t)}$$

and

$$N_3 \geq \left(\frac{bN_2}{\rho} + 1\right)$$

After that, we can choose $N$ large enough such that

$$N > \frac{1}{m_c} \left[\frac{b\rho N_1}{4\varepsilon_1\mu} + N_3\mu_1 + \frac{N_2b\mu_1}{4\varepsilon_2\rho} - \left(\mu_1 - \frac{\mu_1}{4\varepsilon_3}\right)\right].$$

Consequently, there exists a positive constant $\eta_1$ such that (3.15) becomes

$$\frac{d}{dt} \mathcal{L}(t) \leq -c_1 \int_0^1 \left(u_t^2 + u_x + \varphi_x^2 + \phi_t^2\right) dx + c_2 \int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right) dx - c_3 \int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z^2(x, 1, s, t) ds dx.$$  \hfill (3.16)

In this section, we state and prove our stability result.

4. Stabilty Result

Theorem 4.1. Suppose (H1), (H2), and (3.8) hold. Let $U_0 \in \mathcal{H}$, there exist positive constants $a_1, a_2, a_3$ and $\eta_0$ such that the solution of (1.2) satisfies

$$E(t) \leq a_1 G_1^{-1} \left(a_2 \int_0^t \alpha(s) ds + a_3\right), \quad t \geq 0,$$  \hfill (4.1)

where

$$G_1^{-1} = \int_t^1 \frac{1}{G_0(s)} ds \quad \text{and} \quad G_0(s) = tG'(\eta_0 t).$$

Remark 4.1. $G_1$ strictly decreases and is convex on $(0, 1]$ and $\lim_{t \to 0} G_1(t) = +\infty$.

Proof. We multiply (3.15) by $\alpha(t)$ to get

$$\alpha(t) \mathcal{L}'(t) \leq -k_1 \alpha(t) E(t) + c\alpha(t) \int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right) dx.$$  \hfill (4.2)

Now, we discuss two cases:

**Case I:** $G$ is linear on $[0, \eta]$. In this case, using (H2) and Eq.(3.1), we deduce that

$$\alpha(t) \mathcal{L}'(t) \leq -k_1 \alpha(t) E(t) + c\alpha(t) \int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right) dx = -k_1 \alpha(t) E(t) - cE'(t),$$

which can be rewritten as

$$(\alpha(t) \mathcal{L}(t) + cE(t))' - \alpha'(t) \mathcal{L}(t) \leq -k_1 \alpha(t) E(t).$$
Using (H1), we obtain
\[(\alpha (t) \mathcal{L}(t) + cE(t))' \leq -k_1 \alpha (t) E(t) .\]

By exploiting (3.14), it can easily be shown that
\[S_0(t) := \alpha (t) \mathcal{L}(t) + cE(t) \sim E(t). \tag{4.3}\]

So, for some positive constant $\lambda_1$, we obtain
\[S_0'(t) + \lambda_1 \alpha (t) S_0(t) \leq 0, \quad \forall t \geq 0 \tag{4.4}\]

The combination of Eq. (4.3) and (4.4), gives
\[E(t) \leq E(0) e^{-\lambda_1 \int_0^t \alpha(s)ds} = E(0) G_1^{-1} \left( \lambda_1 \int_0^t (s) ds \right). \tag{4.5}\]

**Case II:** $G$ is nonlinear on $[0, \eta]$. In this case, we first choose $0 < \eta_1 < \eta$ such that
\[sg(s) \leq \min \{\eta, G(\eta)\}, \quad \forall |s| \leq \eta_1. \tag{4.6}\]

Using (H2) along with fact that $g$ is continuous and $|g(s)| > 0$, for $s \neq 0$, it follows that
\[ \begin{cases} 
  s^2 + g^2(s) \leq G^{-1}(sg(s)), & \forall |s| \leq \eta_1 \\
  c_1 |s| \leq |sg(s)| \leq c_2 |s|, & \forall |s| \geq \eta_1 
\end{cases} \tag{4.7}\]

To estimate the last integral in Eq. (4.2), we consider the following partition of $(0, 1)$:
\[I_1 = \{x \in (0, 1) : |\phi_t| \leq \eta_1\}, \quad I_2 = \{x \in (0, 1) : |\phi_t| > \eta_1\}. \]

Now, with $I(t)$ defined by
\[I(t) = \int_{I_1} \phi_t g(\phi_t) \, dx, \]
we have, using Jensen inequality (note that $G^{-1}$ is concave and recall (4.6))
\[G^{-1}(I(t)) \geq c \int_{I_1} G^{-1}(\phi_t g(\phi_t)) \, dx. \tag{4.8}\]

The combination of Eq. (4.7) and (4.8) yields
\[\alpha(t) \int_0^1 (\phi_t^2 + g^2(\phi_t)) \, dx = \alpha(t) \int_{I_1} (\phi_t^2 + g^2(\phi_t)) \, dx + \alpha(t) \int_{I_2} (\phi_t^2 + g^2(\phi_t)) \, dx \leq \alpha(t) \int_{I_1} G^{-1}(\phi_t g(\phi_t)) \, dx + c\alpha(t) \int_{I_2} \phi_t g(\phi_t) \, dx \leq c\alpha(t) G^{-1}(I(t)) - cE'(t). \tag{4.9}\]

So, by substituting (4.9) into (4.2) and using (4.3) and (H1), we have
\[S_0'(t) \leq -k_1 \alpha(t) E(t) + c\alpha(t) G^{-1}(I(t)) \tag{4.10}\]
Now, for \( \eta_1 < \eta \) and \( \delta_0 > 0 \), using (4.10) and the fact that \( E' \leq 0 \), \( G' > 0 \), \( G'' > 0 \) on \((0, \eta]\), we find that the functional \( S_1 \), defined by

\[
S_1 (t) := G' \left( \eta_0 \frac{E (t)}{E (0)} \right) S_0 (t) + \delta_0 E (t),
\]

satisfies, for some \( b_1, b_2 > 0 \),

\[
b_1 S_1 (t) \leq E (t) \leq b_2 S_1 (t)
\]

and

\[
S'_0 (t) := \eta_0 E' (t) G'' \left( \eta_0 \frac{E (t)}{E (0)} \right) S_0 (t) + G' \left( \eta_0 \frac{E (t)}{E (0)} \right) S'_0 (t) + \delta_0 E' (t)
\]

\[
\leq -k_1 \alpha (t) E (t) G' \left( \eta_0 \frac{E (t)}{E (0)} \right) + c\alpha (t) G' \left( \eta_0 \frac{E (t)}{E (0)} \right) G^{-1} (I (t)) + \delta_0 E' (t)
\]

Let \( G^\star \) be the convex conjugate of \( G \) defined by

\[
G^\star (s) = s \left( G' \right)^{-1} (s) - G \left[ \left( G' \right)^{-1} (s) \right], \text{ if } s \in (0, G' (\eta)],
\]

satisfying the following general Young's inequality

\[
AB \leq G^\star (A) + G (B), \text{ if } A \in (0, G' (\eta)], \ B \in (0, \eta].
\]

With

\[
A = G' \left( \eta_0 \frac{E (t)}{E (0)} \right) \text{ and } B = G^{-1} (I (t)),
\]

using (4.6), we obtain

\[
\alpha (t) G' \left( \eta_0 \frac{E (t)}{E (0)} \right) G^{-1} (I (t)) \leq \alpha (t) G^\star \left( G' \left( \eta_0 \frac{E (t)}{E (0)} \right) \right) + \alpha (t) I (t).
\]

By exploiting (3.1) and the fact that

\[
G^\star (s) \leq s \left( G' \right)^{-1} (s), \text{ we get}
\]

\[
\alpha (t) G' \left( \eta_0 \frac{E (t)}{E (0)} \right) G^{-1} (I (t)) \leq \alpha (t) \eta_0 E (t) G' \left( \eta_0 \frac{E (t)}{E (0)} \right) - cE' (t)
\]

(4.13)

By substituting (4.12) into Eq. (4.13), we obtain

\[
S'_1 (t) \leq -k \alpha (t) E (t) G' \left( \eta_0 \frac{E (t)}{E (0)} \right) = -k_1 \alpha (t) G_0 \left( \frac{E (t)}{E (0)} \right)
\]

(4.14)

where \( k > 0 \) and \( G_0 (t) = t G' (\eta_0 t) \).

Note that

\[
G'_0 (t) = G' (\eta_0 t) + \eta_0 t G'' (\eta_0 t).
\]

So, using the strict convexity of \( G \) on \((0, \eta]\), we find that \( G_0 (t), G'_0 (t) > 0 \) on \((0, 1]\). With \( S (t) := \frac{b_1 S_1 (t)}{E (0)} \) it is obvious that \( S (t) \leq \frac{E (t)}{E (0)} \leq 1 \). Now, using (4.11) and (4.14), we have

\[
S (t) \sim E (t)
\]

(4.15)
and, for some $a_2 > 0$

$$S'(t) \leq -a_2 \alpha(t) G_0(S(t)). \tag{4.16}$$

Inequality (4.16) implies that

$$\frac{d}{dt} G_1(S(t)) \geq a_2 \alpha(t),$$

where

$$G_1(t) = \int_1^t \frac{1}{G_0(s)} ds.$$

Thus, by integrating over $[0, t]$, we obtain, for some $a_3 > 0$,

$$S(t) \leq G_1^{-1} \left( a_2 \int_0^t \alpha(s) ds + a_3 \right). \tag{4.17}$$

Here, we used, based on the properties of $G_0$, the fact that $G_1$ is strictly decreasing on $(0, 1]$. Finally, using (4.17) and (4.15), we obtain (4.1).

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**REFERENCES**


