SOME RESULTS BY USING CLR’s-PROPERTY IN PROBABILISTIC 2-METRIC SPACE

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ABSTRACT. The aim of this paper is to generate two fixed point theorems in probabilistic 2-metric space by applying CLR’s-property and occasionally weakly compatible mappings (OWC), these two results generalize the theorem proved by V. K. Gupta, Arihant Jain and Rajesh Kumar. Further these results are justified with suitable examples.

1. INTRODUCTION

Menger [1] pioneered the statistical metric (SM) space theory. One of the major achievements was the translation of probabilistic concepts into geometry. Menger used the notation of new distance distribution function from p to q by a Fpq. B. Schweizer, and A. Sklar [2] introduced a new notion of a probabilistic-norm. This norm naturally generates topology, convergence, continuity and completeness in SM-space. Mishra [3] used compatible mappings and generated some fixed points in Menger space. Altumn Turkoglu [4] proved some more results of SM-space by utilizing the implicit relation in multivalued mappings. Zhang, Xiaohong, Huacan He, and Yang Xu [5] employed the Schweizer-Sklar t-norm established fuzzy...
logic system to contribute in development of SM-space. Sehgal, V. M., and A. T. Bharucha-Reid [6] used classical Banach contraction to establish the first result of Menger space for coincidence points. Weakly compatible mappings were generalized by Al-Thagafi and Shahzad [7], by introducing occasionally weakly compatible mappings. Further Chauhan, Sunny, Wutiphol Sintunavarat, and Poom Kumam[9] proved some more theorems by using CLR’S-property in fuzzy metric space. Further some more results can be witnessed by using the concepts of sub sequentially continuous and semi compatible mappings in Menger space [10].

2. PRELIMINARIES

**Definition 2.1** [8] A continuously t-norm is mapping $t: [0, 1] \times [0, 1] \rightarrow [0, 1]$ and it satisfies the following properties

1. \( t\) is abelian & associative
2. \( t(y, 1) = y, \forall y \in [0,1] \)
3. \( t(y, \omega) \leq t(\alpha, \theta) \) for \( y \leq \alpha \) and \( \omega \leq \theta \) \( \forall y, \alpha, \theta, \omega \in [0,1] \).

**Definition 2.2** [8] The pair \((X,F)\) named as Probabilistic 2-metric space (2-PM space) where \( X \neq \emptyset \) and \( F: X \times X \times X \rightarrow L \) here \( L \) is the set of all distribution functions and the \( F \) value at \((u,v,w)\) is represented by \( F_{u,v,w} \) and obeys properties as under

1. \( F_{u,v,w}(0) = 0, \)
2. For all distinct \( u, v \) in \( X \) \( \exists w \in X \) with \( F_{u,v,w}(t) < 1 \) for some \( t > 0, \)
3. \( F_{u,v,w}(t) = 1 \ \forall t > 0, \)If any two of the three points have to be the same,
4. \( F_{u,v,w}(t) = F_{v,w,u}(t) = F_{w,u,v}(t), \)
5. \( F_{u,v,w}(t_a) = F_{v,w,u}(t_b) = F_{w,u,v}(t_c) = 1 \) then \( F_{u,v,w}(t_a + t_b + t_c) = 1. \)

**Definition 2.3** [8] A sequence \( \langle x_n \rangle \) in 2-Menger space \((X,F,t)\) is

1. **Converges** to \( \beta \) if for each \( \epsilon^* > 0, \) \( \exists N(\epsilon^*) \in \mathbb{N} \) implies \( F_{x_n,\beta,a}(\epsilon^*) > 1 - t, \) \( \forall a \in X \) and \( n \geq N(\epsilon^*). \)
2. **Cauchy** if for each \( \epsilon > 0, t > 0, \) \( \exists N(\epsilon) \in \mathbb{N} \) implies \( F_{x_n,x_m,a}(\epsilon) > 1 - t, \) \( \forall a \in X \) and \( n, m \geq N(\epsilon). \)
3. If the cauchy sequence converges in \( X \) then it is referred as a **complete** 2-Menger space.
Definition 2.4 [8] Self-mappings \( P, S \) in 2-Menger space \((X, F, t)\) are called

(i) **Compatible** If \( F_{PSx_n,SPx_n} \alpha(\delta) \to 1, \forall a \in X \) and \( \delta > 0 \) whenever a sequence \( \langle x_n \rangle \) in \( X \)

\[ \exists P x_n, S x_n \to z \] where \( z \) is an element of \( X \) as \( n \to \infty \).

(ii) **Weakly compatible** if the mappings commute at their coincidence points.

(iii) **Occasionally weakly compatible** (OWC) if \( \exists x \in X \) \( \exists P x = S x \Rightarrow PS x = SP x \).

Remark 2.5 Two weakly compatible mappings are obviously OWC mappings, but the converse does not have to be the case.

Example 2.6. By treating \( X = [0, 1] \) and \( d \) be the usual metric on \( X \) and for all \( t_1 \in [0, 1] \),

define

\[ F_{u,v,a}(t_1) = \begin{cases} \frac{t_1}{t_1 + |\alpha - \beta|}, & t_1 > 0 \\ 0, & t_1 = 0 \end{cases} \quad \forall \alpha, \beta \in X \text{ and fixed } a, t_1 > 0. \]

Define mappings \( P, S : X \to X \) as \( P(x) = \frac{x^2}{2}, x \in [0, 1] \) and \( S(x) = \frac{x}{3}, x \in [0, 1] \).

We notice that the pair \( (P, S) \) has two coincidence points \( 0, \frac{2}{3} \).

If \( x = \frac{2}{3} \) then \( P(\frac{2}{3}) = S(\frac{2}{3}) = \frac{2}{9} \) \hspace{1cm} (2.6.1)

\[ PS(\frac{2}{3}) = P(\frac{2}{9}) = \frac{2}{81}, \] \hspace{1cm} (2.6.2)

\[ SP(\frac{2}{3}) = S(\frac{2}{9}) = \frac{2}{27}. \] \hspace{1cm} (2.6.3)

From (2.6.2) and (2.6.3) \( PS(\frac{2}{3}) \neq SP(\frac{2}{3}) \).

At \( x = 0 \), \( P(0) = S(0) \) and \( PS(0) = SP(0) \).

This shows the mappings \( P, S \) are OWC but not weakly compatible.

Definition 2.7 [9] “Self maps \( P \) and \( S \) of a 2-Menger space \((X, F, t)\) are said to satisfy

CLR\(_S^\prime\) property (common limit range property) if there exists a sequence

\[ \langle x_n \rangle \in X \ \exists P x_n, S x_n \to S z, \text{ for some element } z \in X \text{ as } n \to \infty. \]

This example shows that mappings \( P, S \) satisfy CLR\(_S^\prime\) property but they do not have closed ranges.

Example 2.8. Take \( X = (0, 1] \) and \( t \in [0, 1] \), define
\[ F_{u,v}(t) = \begin{cases} \frac{t}{t+|\alpha-\beta|}, & t > 0 \\ 0, & t = 0 \end{cases}, \quad \forall \, \alpha, \beta \text{ in } X \text{ and } t > 0. \]

Define \( P, S : X \to X \) as \( P(x) = \begin{cases} 1 - x, & x \in (0, \frac{2}{3}) \\ x, & x \in [\frac{2}{3}, 1] \end{cases} \) \quad (2.8.1)

and

\[ S(x) = \begin{cases} 2x, & x \in (0, \frac{2}{3}] \\ 1, & x \in [\frac{2}{3}, 1] \end{cases}. \] \quad (2.8.2)

Consider a sequence \( x_n = \frac{1}{3} - \frac{1}{3n} \) for \( n = 1, 2, 3 \ldots \) then

\[ P x_n = 1 - \left( \frac{1}{3} - \frac{1}{3n} \right) = \frac{2}{3} + \frac{1}{3n} \to \frac{2}{3} \quad (2.8.4) \]

\[ S x_n = 2\left( \frac{1}{3} - \frac{1}{3n} \right) = \frac{2}{3} - \frac{2}{9n} \to \frac{2}{3} \text{ as } n \to \infty. \] \quad (2.8.5)

Thus \( P x_n S x_n \to S\left( \frac{1}{3} \right) = \frac{2}{3} \) as \( n \to \infty. \) \quad (2.8.6)

Where \( P(X) = \left( \frac{1}{3}, 1 \right], S(X) = (0, \frac{4}{3}] \cup \{1\} \) this shows that \( P, S \) are satisfy CLR’- property but they do not have closed ranges.

Now we give the statement of Theorem (A). It is proved by V. K. Gupta et al.

\textbf{Theorem (A) [8]} “ Let \( A, B, S \) and \( T \) be self -mappings on a complete probabilistic 2-metric space \( (\bar{X}, F, t) \) satisfying:

- \( (A_1) \) \( A(\bar{X}) \subseteq T(\bar{X}), B(\bar{X}) \subseteq S(\bar{X}) \)
- \( (A_2) \) one of \( A(\bar{X}), B(\bar{X}), T(\bar{X}) \) or \( S(\bar{X}) \) is complete,
- \( (A_3) \) pairs \( (A, S) \) and \( (B, T) \) are weakly compatible,
- \( (A_4) \) \( F_{A(x, y), a(t)} \geq r F_{S(x, y), a(t)} \) for all \( x, y \) and \( t > 0, \)

where \( r : [0, 1] \to [0, 1] \) is some continuous function such that \( r(t) > t \) for each \( 0 < t < 1, \)

then \( A,B,S \) and \( T \) have unique common fixed point in \( \bar{X} \).”

We now generalize Theorem(A) as under.

\section{3. MAIN RESULT}

\textbf{Theorem 3.1} Let \( A, B, S \) and \( T \) be self -mappings on a complete probabilistic 2-metric space \( (\bar{X}, F, t^*) \) satisfying:

- \( (3.1.1) \) \( A(\bar{X}) \subseteq T(\bar{X}), B(\bar{X}) \subseteq S(\bar{X}), \)
- \( (3.1.2) \) the pairs \( (A, S), (B, T) \) share the CLR’ property with OWC,
\((3.1.3)\) \(F_{Ax, By, a}(t^*) \geq r F_{Sa, Ty, a}(t^*)\) for all \(x, y\) and \(t^* > 0\),

where \(r: [0, 1] \rightarrow [0, 1]\) is some continuous function such that \(r(t^*) > t^*\) for each \(0 < t^* < 1\).

then \(A, B, S\) and \(T\) have unique common fixed point in \(\bar{X}\).

**Proof:**

Iteratively the sequences \(\langle y_n \rangle\) and \(\langle x_n \rangle\) can be constructed as

\(x_0 \in \bar{X} = Ax_0 \in A(\bar{X}) \subseteq T(\bar{X}), \exists x_1 \in \bar{X}\) in such a way that \(Ax_0 = Tx_1\),

\(Bx_1 \in B(\bar{X}) \subseteq S(\bar{X})\) then we have \(x_2 \in \bar{X}\) with \(Bx_1 = Sx_2\)

\(\langle y_{2n} \rangle = Ax_{2n} = Tx_{2n+1}\) and \(\langle y_{2n+1} \rangle = Bx_{2n+1} = Sx_{2n+2}\). \hspace{1cm} (3.1.4)

Now our claim is to show \(\langle y_n \rangle\) is cauchy sequence.

For this take \(x = x_{2n}, y = x_{2n+1}\) in (3.1.3) we get

\(F_{Ax_{2n}, Bx_{2n+1}, a}(t^*) \geq rF_{Sx_{2n}, Tx_{2n+1}, a}(t^*)\). \hspace{1cm} (3.1.5)

\(\Rightarrow F_{y_{2n}, y_{2n+1}, a}(t^*) \geq rF_{y_{2n-1}, y_{2n}, a}(t^*) > F_{y_{2n-1}, y_{2n}, a}(t^*)\). \hspace{1cm} (3.1.6)

Similarly

\(F_{y_{2n+1}, y_{2n+2}, a}(t^*) > F_{y_{2n}, y_{2n+1}, a}(t^*)\). \hspace{1cm} (3.1.7)

In general we have \(F_{y_{n+1}, y_n, a}(t^*) > F_{y_{n-1}, y_{n-1}, a}(t^*)\) for all values of \(n\).

Then \(F_{y_{n+1}, y_n, a}(t^*) > 1\) is an increasing sequence bounded above by 1 therefore it must converge to \(L\), where \(L \leq 1\).

If \(L < 1\) then \(F_{y_{n+1}, y_n, a}(t^*) = L > r(1) > 1\) as a result of the contradiction, \(L = 1\).

Hence \(F_{y_{n+1}, y_n, a}(t) = 1\) for all \(n\) and \(p\).

As a result, because Cauchy sequence exists in complete space \(X\), it has a limit \(z\) in \(X\) and consequently each sub sequence has the same limit \(z\).

That is \(Ax_{2n}, Sx_{2n} \rightarrow z\) and \(Bx_{2n+1}, Tx_{2n+1} \rightarrow z\) as \(n \rightarrow \infty\). \hspace{1cm} (3.1.8)

On using CLR0-Property of \((A, S)\), \((B, T)\) implies there are sequences \((a_n)\) as well as \((b_n)\) in order for

\(Aa_n, Sa_n, Bb_n, Tb_n \rightarrow S\mu\) as \(n \rightarrow \infty\) for some \(\mu\) in \(\bar{X}\). \hspace{1cm} (3.1.9)

To prove \(z = S\mu\) put \(x = a_{2n}, y = x_{2n+5}\) in (3.1.3) we get

\(F_{Aa_{2n}, Bx_{2n+5}, a}(t^*) \geq r(F_{Sa_{2n}, Tx_{2n+5}, a}(t^*)\) as \(n \rightarrow \infty\) \hspace{1cm} (3.1.10)

\(\Rightarrow F_{S\mu, z, a}(t^*) \geq r(F_{S\mu, z, a}(t^*) > F_{S\mu, z, a}(t^*)\). \hspace{1cm} (3.1.11)

Resulting \(F_{S\mu, z, a}(t^*) > F_{S\mu, z, a}(t^*)\) \hspace{1cm} (3.1.12)

which is a contradiction. Hence \(S\mu = z\). \hspace{1cm} (3.1.13)
Claim $A \mu = S \mu.$

Put $x = \mu, y = x^{2n+3}$ in (3.1.3) we get

$$F_{A \mu, Bx^{2n+3}}(t^*) \geq r(F_{S \mu, Tx^{2n+3}}(t^*)) \text{ as } n \to \infty$$  \hspace{1cm} (3.1.14)

$$\Rightarrow F_{A \mu, z}(t^*) \geq r(F_{S \mu, z}(t^*)) \text{ using (3.1.13)}$$  \hspace{1cm} (3.1.15)

$$\Rightarrow F_{A \mu, z}(t^*) \geq r(F_{z, z}(t^*)) = r(1) = 1.$$  \hspace{1cm} (3.1.16)

This results $A \mu = S \mu = z.$  \hspace{1cm} (3.1.17)

Since the pair $(A, S)$ obeys OWC resulting

$$A \mu = S \mu = z = z.$$  \hspace{1cm} (3.1.18)

Claim $Az = z.$

Substitute $y = x^{2n+3}, x = z$ in (3.1.3) we have

$$F_{Az, Bx^{2n+3}}(t^*) \geq r(F_{Sz, Tx^{2n+3}}(t^*)) \text{ letting } n \to \infty.$$  \hspace{1cm} (3.1.19)

$$\Rightarrow F_{Az, z}(t^*) \geq r(F_{Sz, z}(t^*)) \text{ using (3.1.18)}$$  \hspace{1cm} (3.1.20)

$$\Rightarrow F_{Az, z}(t^*) > r(F_{Az, z}(t^*)) \text{ for } t^* \to \infty.$$  \hspace{1cm} (3.1.21)

This is a contradiction. Thus $z = Az.$

Resulting $Az = Sz = z.$  \hspace{1cm} (3.1.22)

Since $Az \in A(X) \subseteq T(X)$ then $\exists \rho \in X$ such that $Az = T\rho.$  \hspace{1cm} (3.1.23)

Claim $z = Br.$

By employing $x = x_{4n}, y = \rho$ of (3.1.3) we obtain

$$F_{Ax^{2n}, Br}(t^*) \geq r(F_{Sx^{2n}, T\rho}(t^*)) \text{ as } n \to \infty.$$  \hspace{1cm} (3.1.24)

From (3.1.22) & (3.1.23)

$$\Rightarrow F_{z, Br}(t^*) \geq r(F_{z, T\rho}(t^*)) = r(1) = 1.$$  \hspace{1cm} (3.1.25)

Thus $z = Br = T\rho.$

Since the pair of mappings $(B, T)$ obeys OWC, this results

$Br = T\rho \Rightarrow BT\rho = TB\rho.$ That is $Bz = Tz.$  \hspace{1cm} (3.1.26)

Claim $z = Bz.$

By substituting $y = z, x = z$ in (3.1.3) results

$$F_{Az, Bz}(t^*) \geq r(F_{Sz, Tz}(t^*)) \text{ using (3.1.22) & (3.1.26)}$$  \hspace{1cm} (3.1.27)

$$F_{z, Bz}(t^*) \geq r(F_{z, Bz}(t^*)) \text{ for } t^* \to \infty.$$  \hspace{1cm} (3.1.28)

Resulting $F_{z, Bz}(t^*) > F_{z, Bz}(t^*).$ It is impossible. Therefore $Bz = z.$

Combining all we get $Az = Bz = z = Sz = Tz.$
Thus \( z \) is the required common fixed point for these mappings \( A, B, S \) and \( T \).

**Uniqueness:**

Assume \( z_1 \) is second common fixed point.

Now assume \( z \neq z_1 \).

By considering \( y = z_1, x = z \) in (3.1.4) we obtain

\[
F_{Az, Bz_1} a (t^*) \geq r F_{Sz, Tz_1} a (t^*)
\]

\[
F_{z, z_1} a (t^*) \geq r F_{z, z_1} a (t^*) > F_{z, z_1} a (t^*)
\]

\[
F_{z, z_1} a (t^*) > F_{z, z_1} a (t^*) \text{ which is absurd. Hence } z = z_1.
\]

As a result, for self- mappings \( A, B, S, \) and \( T \) have the only one common fixed point.

Now we justify our theorem as under.

3.2 Example

Let us take \( X = [0, \pi] \) and each \( t \in [0, 1] \), define

\[
F_{u,v}(t) = \begin{cases} t & t > 0 \\ \frac{t}{t + \alpha - \beta} & , t = 0 \\ 0 & for all \alpha, \beta \text{ in } X, t > 0. 
\end{cases}
\]

Define mappings \( P, S, T \) & \( Q : X \to X \) as

\[
A(x) = B(x) = \begin{cases} 2e^{-\pi x}, & x \in [0, \frac{\pi}{2}] \\ \pi - x, & x \in [\frac{\pi}{2}, \pi] \end{cases} \quad (3.2.1)
\]

and \( S(x) = T(x) = \begin{cases} 2e^{-\pi x^2}, & x \in [0, \frac{\pi}{2}] \\ x, & x \in [\frac{\pi}{2}, \pi] \end{cases} \quad (3.2.2)
\]

Now \( A(X) = B(X) = [0, 2] \) and \( S(X) = T(X) = [0, \pi] \) implies \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \).

Clearly \( \frac{\pi}{2} \) and 1 are coincidence points for the mappings \( B, T \).

At \( x = \frac{\pi}{2}, B(\frac{\pi}{2}) = T(\frac{\pi}{2}) \) and \( BT(\frac{\pi}{2}) = B(\frac{\pi}{2}) = \frac{\pi}{2} \).

\( TB(\frac{\pi}{2}) = T(\frac{\pi}{2}) = \frac{\pi}{2} \) implies \( BT(\frac{\pi}{2}) = TB(\frac{\pi}{2}). \)

At \( x = 1, B(1) = T(1) \) and \( BT(1) \neq TB(1) \).

Thus the pairs \( (A, S), (B, T) \) satisfy OWC but are not weakly compatible.

If \( x_n = \frac{\pi}{2} - \frac{1}{n} \) for all \( n \geq 1 \). Then

\[
Sx_n = Tx_n = S\left( \frac{\pi}{2} - \frac{1}{n} \right) = \frac{\pi}{2} - \frac{1}{n} \to \frac{\pi}{2} \quad (3.2.3)
\]

\[
Ax_n = Bx_n = A\left( \frac{\pi}{2} - \frac{1}{n} \right) = \pi - \left( \frac{\pi}{2} - \frac{1}{n} \right) = \pi + \frac{1}{n} \to \frac{\pi}{2} \text{ as } n \to \infty. \quad (3.2.5)
\]
\[ \Rightarrow \text{Ax}_n, \text{Sx}_n, \text{Tx}_n, \text{Bx}_n \to S\left(\frac{\pi}{2}\right) \text{ as } n \to \infty. \]  

(3.2.6)

This gives the pairs of maps (A, S), (B, T) sharing the CLR’S property with OWC.

Thus A, B, S and T satisfy all the norms of Theorem and having the unique commonly fixed point at \[ \frac{\pi}{2} \] as 

\[ \text{A}\left(\frac{\pi}{2}\right) = \text{S}\left(\frac{\pi}{2}\right) = \text{B}\left(\frac{\pi}{2}\right) = \text{T}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}. \]

Now we present another generalization of Theorem (A) as under.

**Theorem 3.3** Let A, B, S and T be self - mappings on a complete probabilistic 2-metric space \( (\tilde{X}, F, t^*) \) satisfying :

(3.3.1) \( A(\tilde{X}) \subseteq T(\tilde{X}), B(\tilde{X}) \subseteq S(\tilde{X}) \)

(3.3.2) the pair (A, S) satisfies CLR’S property with OWC and (B, T) satisfies OWC.

(3.3.3) Further \( F_{Ax, By, a} \ (t^*) \geq F_{Sx, Ty, a} \ (t^*) \) for all elements \( x, y \) in \( \tilde{X} \) and \( t^* > 0 \)

r is continuous self-map on \([0, 1]\) such that \( r(t^*) > t^* \) for each \( o < t^* < 1 \).

Then A, B, S and T have unique common fixed point in \( \tilde{X} \).

**Proof:**

Take the constructed sequences \( < x_n>, < y_n> \) in Theorem (3.1) as

\[ \langle y_{2n} \rangle = Ax_{2n} = Tx_{2n+1} \text{ and } \langle y_{2n+1} \rangle = Bx_{2n+1} = Sx_{2n+2}. \]  

(3.3.4)

It is already shown that \( \langle y_n \rangle \) as cauchy sequence.

As a result each sub sequence has the same limit point z in complete space \( \tilde{X} \).

That is \( Ax_{2n}, Sx_{2n} \to z \) and \( Bx_{2n+1}, Tx_{2n+1} \to z \).

The pair (A, S) obeys CLR’S-property this implies there is a sequence \( \langle z_n \rangle \) such that

\[ A z_n, S z_n \to Sv \text{ for some } v \text{ in } \tilde{X}. \]

Claim \( z = Sv \).

By putting \( y = x_{2n+1}, x = z_n \) in (3.3.3), that results

\[ F_{Az_n, Bx_{2n+1}, a} \ (t^*) \geq r F_{Sx_n, Tx_{2n+1}, a} \ (t^*) \text{ as } n \to \infty \]  

(3.3.5)

\[ \Rightarrow F_{Sv, z, a} \ (t^*) \geq r F_{Sv, z, a} \ (t^*) \]  

(3.3.6)

\[ \Rightarrow F_{Sv, z, a} \ (t^*) > F_{Sv, z, a} \ (t^*). \]  

(3.3.7)

This is absurd. As a result \( Sv = z \).

Claim \( Av = Sv \).

(3.3.8)

By inserting \( x = v, y = x_{2n+3} \) in (3.3.3), that results

\[ F_{Av, Bx_{2n+3}, a} \ (t^*) \geq r F_{Sv, Tx_{2n+3}, a} \ (t^*) \text{ letting as } n \to \infty \]  

(3.3.10)

\[ \Rightarrow F_{Av, z, a} \ (t^*) \geq r F_{Sv, z, a} \ (t^*) \]  

(3.3.11)

\[ \Rightarrow F_{Av, Sv, a} \ (t^*) \geq r F_{Sv, Sv, a} \ (t^*) = r(1) = 1. \]  

(3.3.12)
\[ \Rightarrow Av = Sv = z. \quad (3.3.13) \]

Since the pair (A, S) satisfies OWC property, that results
\[ Av = Sv \Rightarrow SAv = ASv. \text{ This gives } Az = Sz. \quad (3.3.14) \]

Claim \( Az = z. \)

By replacing \( y = x_{2n+1}, x = z \) in (3.3.3), as a result
\[ F_{Az, Bx_{2n+1}, a}(t^*) \geq rF_{Sx_{2n+1}, Tω}. \quad (3.3.15) \]

\[ \Rightarrow F_{Az, z, a}(t^*) \geq rF_{Sz, z, a}(t^*), \text{ using (3.3.14)} \quad (3.3.16) \]

\[ \Rightarrow F_{Az, z, a}(t^*) \geq rF_{Az, z, a}(t^*) \quad (3.3.17) \]

This is a contradiction. Consequently \( Az = z. \quad (3.3.18) \)

By combining (3.3.14) and (3.3.18) gives \( z = Sz = Az. \quad (3.3.19) \)

Since \( Az \in A(X) \subseteq T(X) \) then \( \exists \omega \in X \text{ such that } Az = T\omega. \quad (3.3.20) \)

Claim \( z = B\omega. \)

By using \( x = x_{4n}, y = \omega \) of (3.3.4), we obtain
\[ F_{Ax_{2n}, B\omega}(t^*) \geq rF_{Sx_{2n}, T\omega, a}(t^*). \quad (3.3.21) \]

Taking limit as \( n \to \infty \) and from (3.3.19) and (3.3.20) we get
\[ \Rightarrow F_{z, B\omega}(t^*) \geq rF_{z, z, a}(t^*) = r(1) = 1. \quad (3.3.22) \]

Thus \( z = B\omega = T\omega. \quad (3.3.23) \)

Since the pair (B, T) obeys OWC property gives
\[ B\omega = T\omega \Rightarrow BT\omega = TB\omega \text{ implying } Bz = Tz. \quad (3.3.24) \]

Claim \( z = Bz. \)

Applying \( x = y = z \) in (3.3.3), this resulting
\[ F_{Az, Bz, a}(t^*) \geq rF_{Sz, Tz, a}(t^*), \quad (3.3.25) \]

using (3.3.19) and (3.3.24)
\[ \Rightarrow F_{z, Bz, a}(t^*) \geq rF_{z, Bz, a}(t^*) > F_{z, Bz, a}(t^*), \quad (3.3.26) \]

\[ \Rightarrow F_{z, Bz, a}(t^*) > F_{z, Bz, a}(t^*). \quad (3.3.27) \]

Contradicting the fact implies \( Bz = z. \)

As a result \( Az = Bz = Sz = Tz = z. \)

As a consequence four self-mappings A, B, S, and T, there is a fixed point commonly.

Uniqueness can be easily proved as in the Theorem (3.1).
Now the Theorem (3.3) can be supported by discussing with suitable example.

### 3.4 Example

We choose $X = [0, 1]$, $d$ be usual metric on $X$ and each $t \in [0, 1]$, define

$$F_{u,v}(t) = \begin{cases} t & t > 0 \\ 0 & t = 0 \end{cases}$$

for all $a, \beta$ in $X$, $t > 0$.

Choose mappings $P, S, T & Q : X \rightarrow X$ as

$$P(x) = Q(x) = \begin{cases} 1 - 2x, & x \in [0, 0.2] \\ x^2, & x \in (0.2, 1) \end{cases}$$

and

$$S(x) = T(x) = \begin{cases} 3x, & x \in [0, 0.2] \\ x^3, & x \in (0.2, 1) \end{cases}$$

Now $P(X) = Q(X) = (0.04, 1]$ and $S(X) = T(X) = [0, 1]$ so that $P(X) \subseteq T(X)$ and $Q(X) \subseteq S(X)$.

Clearly 0.2 and 1 are coincidence points of the graphs $Q$, $T$.

At $x = 0.2$, $Q(0.2) = T(0.2) = 0.6$ but $QT(0.2) = Q(0.6) = 0.36$, $TQ(0.2) = T(0.6) = 0.216$.

At $x = 1$, $Q(1) = T(1)$ and $QT(1) = Q(1) = 1 = T(1) = TQ(1)$.

This demonstrates that the pairs $(P, S)$, $(Q, T)$ are OWC mappings, although they are not weakly compatible.

If we choose $x_n = 1 - \frac{4}{3^n}$ for all $n \geq 1$. Then

$$P(x_n) = 1 - \frac{4}{3^n}$$

$$S(x_n) = 1 - \frac{4}{3^n}$$

This implies $P(x_n), S(x_n) \rightarrow S(1)$ as $n \rightarrow \infty$.

This gives the pair $(P, S)$ satisfies CLR’$s$-property with OWC and the pair $(Q, T)$ is OWC.

Thus the mappings $P$, $Q$, $S$ and $T$ satisfy all the norms of the Theorem (3.3), containing unique common fixed point at 1 as $1=P(1) = Q(1) = S(1) = T(1)$.

### 4. CONCLUSION

In this paper Theorem (A) is generalized in two ways.

(a) Theorem (3.1) is formulated by employing CLR’$s$-property and applying OWC for both the pairs instead of assuming weakly compatible mappings.
(b) Theorem (3.2) is formulated by employing CLR',-property and OWC for one pair and OWC for the other pair instead of assuming weakly compatible mappings. Further these two results are justified with suitable examples.

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