Tri-Endomorphisms on BCH-Algebras

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Abstract. In this paper, we use the concept of endomorphisms and bi-endomorphisms as a model to create tri-endomorphisms on BCH-algebras. We introduce the concepts of left tri-endomorphisms, central tri-endomorphisms, right tri-endomorphisms, and complete tri-endomorphisms of BCH-algebras and provide some properties. In addition, we obtain the properties between those tri-endomorphisms and some subsets of BCH-algebras.

1. Introduction

The algebraic structures of BCK-algebras and BCI-algebras were studied by Iséki and his colleague [4, 5]. In 1983, Hu and Li [3] generalized a new class of algebras from BCI-algebras, namely, a BCH-algebra. Next, Bandru and Rafi [1] introduced a new algebra, called a G-algebra. BCH-algebras are also being studied extensively later, [2, 3].

In this paper, we use the concept of endomorphisms and bi-endomorphisms as a model to create tri-endomorphisms. We introduce the concepts of left tri-endomorphisms, central tri-endomorphisms, right tri-endomorphisms, and complete tri-endomorphisms of BCH-algebras and provide some properties.

Before studying, we will review the definitions and well-known results.

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**Definition 1.1.** [3] A BCH-algebra is a non-empty set \( X \) with an element 0 and a binary operation * satisfying the following conditions:

(BCH1) \((\forall x \in X)(x * x = 0)\),
(BCH2) \((\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)\),
(BCH3) \((\forall x, y, z \in X)((x * y) * z = (x * z) * y)\).

In a BCH-algebra \( X = (X, *, 0) \), the binary relation \( \leq \) on \( X \) is defined as follows:

\((\forall x, y \in X)(x \leq y \iff x * y = 0)\).

**Example 1.1.** Let \( X = \{0, a, b, c\} \) with the following Cayley table as follows:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( X = (X, *, 0) \) is a BCH-algebra.

**Proposition 1.1.** [2, 3] Let \( X = (X, *, 0) \) be a BCH-algebra. Then the following hold: for all \( x, y \in X \),

(BCH4) \((\forall x, y \in X)(x * (x * y) \leq y)\),
(BCH5) \((\forall x \in X)(x * 0 = 0 \Rightarrow x = 0)\),
(BCH6) \((\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y))\),
(BCH7) \((\forall x \in X)(x * 0 = x)\),
(BCH8) \((\forall x, y \in X)((x * y) * x = 0 * y)\),
(BCH9) \((\forall x, y \in X)(x \leq y \Rightarrow 0 * x = 0 * y)\).

For a BCH-algebra \( X = (X, *, 0) \), some interesting subsets of \( X \) play a significant rule in the investigation of its properties described below.

**Definition 1.2.** A non-empty subset \( Y \) of a BCH-algebra \( X = (X, *, 0) \) is called a subalgebra of \( X \) if \( x * y \in Y \) for all \( x, y \in Y \). A non-empty subset \( I \) of a BCH-algebra \( X = (X, *, 0) \) is called an ideal of \( X \) if

(1) \( 0 \in I \),
(2) \((\forall x, y \in X)(x * y \in I, x \in I \Rightarrow y \in I)\).

2. Main results

In this section, we introduce the concepts of left tri-endomorphisms, central tri-endomorphisms, right tri-endomorphisms, and complete tri-endomorphisms of BCH-algebras as follows.

**Definition 2.1.** Let \( X = (X, *, 0) \) be a BCH-algebra. A mapping \( f : X^3 \rightarrow X \) is called
(1) a left tri-endomorphism on $X$ if $(\forall w, x, y, z \in X)(f(x \ast w, y, z) = f(x, y, z) \ast f(w, y, z))$,
(2) a central tri-endomorphism on $X$ if $(\forall w, x, y, z \in X)(f(x, y \ast w, z) = f(x, y, z) \ast f(x, w, z))$,
(3) a right tri-endomorphism on $X$ if $(\forall w, x, y, z \in X)(f(x, y, z \ast w) = f(x, y, z) \ast f(x, y, w))$,
(4) a complete tri-endomorphism on $X$ if $(\forall a, b, c, x, y, z \in X)(f(x \ast a, y \ast b, z \ast c) = f(x, y, z) \ast f(a, b, c))$.

Example 2.1. In Example 1.1, we define $f_l : X^3 \to X$ by

$$f_l(x, y, z) = \begin{cases} x & \text{if } y = z = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then $f_l$ is a left tri-endomorphism on $X$.

Proposition 2.1. Let $X = (X, \ast, 0)$ be a BCH-algebra and $f_l$ be a left tri-endomorphism on $X$. Then

(1) $(\forall y, z \in X)(f_l(0, y, z) = 0)$,
(2) $(\forall w, x, y, z \in X)(x \leq w \Rightarrow f_l(x, y, z) \leq f_l(w, y, z))$.

Proof. (1) Let $y, z \in X$. Then, by BCH1, we have $f_l(0, y, z) = f_l(0 \ast 0, y, z) = f_l(0, y, z) \ast f_l(0, y, z) = 0$.

(2) Let $w, x, y, z \in X$ be such that $x \leq w$. Then, by (1), we have $0 = f_l(0, y, z) = f_l(x \ast w, y, z) = f_l(x, y, z) \ast f_l(w, y, z)$. Hence, $f_l(x, y, z) \leq f_l(w, y, z)$.

Similarly, the properties of central and right tri-endomorphisms are easily obtained.

Proposition 2.2. Let $X = (X, \ast, 0)$ be a BCH-algebra and $f_c$ be a central tri-endomorphism on $X$. Then

(1) $(\forall x, z \in X)(f_c(x, 0, z) = 0)$,
(2) $(\forall w, x, y, z \in X)(y \leq w \Rightarrow f_c(x, y, z) \leq f_c(x, w, z))$.

Proposition 2.3. Let $X = (X, \ast, 0)$ be a BCH-algebra and $f_r$ be a right tri-endomorphism on $X$. Then

(1) $(\forall x, y \in X)(f_r(x, y, 0) = 0)$,
(2) $(\forall w, x, y, z \in X)(z \leq w \Rightarrow f_r(x, y, z) \leq f_r(x, y, w))$.

Theorem 2.1. Let $X = (X, \ast, 0)$ be a BCH-algebra and $f$ be a complete tri-endomorphism on $X$. Then

(1) $f(0, 0, 0) = 0$,
(2) if $S$ is a subalgebra of $X$, then $f(S^3)$ is also a subalgebra of $X$,
(3) if $S$ is an ideal of $X$ and $f$ is bijective, then $f(S^3)$ is also an ideal of $X$,
(4) if $f$ is a left tri-endomorphism on $X$, then $f(x, y, z) \ast f(x, 0, 0) = 0$ for any $x, y, z \in X$,
(5) if $f$ is a central tri-endomorphism on $X$, then $f(x, y, z) \ast f(0, y, 0) = 0$ for any $x, y, z \in X$,
(6) if $f$ is a right tri-endomorphism on $X$, then $f(x, y, z) \ast f(0, 0, z) = 0$ for any $x, y, z \in X$. 

(7) if \( f \) is a left and right (central and right, left and central) tri-endomorphism on \( X \), then \( f(x, y, z) = 0 \) for any \( x, y, z \in X \), i.e., \( f \) is the zero map.

**Proof.** (1) By BCH1, we have \( f(0, 0, 0) = f(0 * 0 * 0 * 0 * 0) = f(0, 0, 0) * f(0, 0, 0) = 0 \).

(2) Suppose that \( S \) is a subalgebra of \( X \). Let \( a, b \in f(S^3) \). Then there exist \((x_1, y_1, z_1), (x_2, y_2, z_2) \in S^3\) such that \( a = f(x_1, y_1, z_1) \) and \( b = f(x_2, y_2, z_2) \). Thus \( a * b = f(x_1, y_1, z_1) * f(x_2, y_2, z_2) = f(x_1 * x_2, y_1 * y_2, z_1 * z_2) \in f(S^3) \). Hence, \( f(S^3) \) is a subalgebra of \( X \).

(3) Suppose that \( S \) is an ideal of \( X \) and \( f \) is bijective. Since \( 0 \in S \) and by (1), we have \( 0 = f(0, 0, 0) \in f(S^3) \). Assume that \( x * y \in f(S^3) \) and \( x \in f(S^3) \). There exist \((x_1, y_1, z_1), (x_2, y_2, z_2) \in S^3\) such that \( x * y = f(x_1, y_1, z_1) \) and \( x = f(x_2, y_2, z_2) \). Since \( f \) is surjective, there exists \((a, b, c) \in X^3\) such that \( y = f(a, b, c) \). Thus \( f(S^3) \ni f(x_1, y_1, z_1) = x * y = f(x_2, y_2, z_2) * f(a, b, c) = f(x_2 * a, y_2 * b, z_2 * c) \). Since \( f \) is injective, we have \( x_2 * a, y_2 * b, z_2 * c \in S \). Since \( S \) is an ideal of \( X \), we get \( a, b, c \in S \). Thus \( y = f(a, b, c) \in f(S^3) \). Hence, \( f(S^3) \) is an ideal of \( X \).

(4)-(6) It is obvious from Propositions 2.1-2.3.

(7) Suppose that \( f \) is a left and right tri-endomorphism on \( X \). Let \( x, y, z \in X \). Then, by Propositions 2.1 and 2.3, BCH1, BCH7 \( 0 = f(0, y, z) = f(x * x, y * 0, z * 0) = f(x, y, z) * f(x, 0, 0) = f(x, y, z) * 0 = f(x, y, z) \). Hence, \( f \) is the zero map on \( X \). □

Let \( T_1(X) \) (resp., \( T_c(X) \), \( T_r(X) \) and \( T(X) \)) be the set of all left tri-endomorphisms (resp., right, central and complete tri-endomorphisms) on a BCH-algebra \( X = (X, *, 0) \). We define an operation \( \ast \) on \( T_1(X) \) by \((\forall x, y, z \in X)((f \ast g)(x, y, z) = f(x, y, z) * g(x, y, z))\). Let \( f \in T_1(X) \) and \( x, y, z \in X \). Then \((f \ast f)(x, y, z) = f(x, y, z) * f(x, y, z) = 0 \). This means that \( f \ast f = 0_X \), where \( 0_X : X^3 \to X \) is a function that maps all members to 0. Let \( f, g \in T_1(X) \) be such that \( f \ast g = 0_X \) and \( g \ast f = 0_X \). Then for all \( x, y, z \in X \), \( 0 = (f \ast g)(x, y, z) = f(x, y, z) * g(x, y, z) \) and \( 0 = (g \ast f)(x, y, z) = g(x, y, z) * f(x, y, z) \). Since \( g(x, y, z), f(x, y, z) \in X \), we have \( f(x, y, z) = g(x, y, z) \) for all \( x, y, z \in X \). Hence, \( f = g \). Let \( f, g, h \in T_1(X) \) and \( x, y, z \in X \). Then \(((f \ast g) \ast h)(x, y, z) = (f \ast g)(x, y, z) * h(x, y, z) = (f(x, y, z) * g(x, y, z)) * h(x, y, z) = (f(x, y, z) * h(x, y, z)) * g(x, y, z) \). Hence, \((f \ast g) \ast h = (f \ast h) \ast g \).

**Theorem 2.2.** \((T_1(X), \ast, 0_X), (T_c(X), \ast, 0_X), (T_r(X), \ast, 0_X), \) and \((T(X), \ast, 0_X) \) are BCH-algebras.

Let \( X = (X, *, 0) \) be a BCH-algebra. We define the binary operation \( \circ \) on \( X^3 \) as follows: \((\forall (a, b, c), (x, y, z) \in X^3)((a, b, c) \circ (x, y, z) = (a * x, b * y, c * z))\). Then \( X^3 = (X, \circ, (0, 0, 0)) \) is a BCH-algebra.

**Theorem 2.3.** Let \( X = (X, *, 0) \) be a BCH-algebra and \( S_1, S_2, S_3 \) be subsets of \( X \). Then

(1) \( S_1 \times S_2 \times S_3 \) is a subalgebra of \( X^3 \) if and only if \( S_1, S_2 \) and \( S_3 \) are subsets of \( X \),

(2) \( S_1 \times S_2 \times S_3 \) is an ideal of \( X^3 \) if and only if \( S_1, S_2 \) and \( S_3 \) are ideals of \( X \).
Proof. (1) Suppose that \( S_1 \times S_2 \times S_3 \) is a subalgebra of \( X^3 \). Firstly, we will show that \( S_1 \) is a subalgebra of \( X \). Let \( a, b \in S_1 \). Let \( x \in S_2 \) and \( u \in S_3 \). Then \( (a, x, u), (b, x, u) \in S_1 \times S_2 \times S_3 \). Thus \( (a \ast b, 0, 0) = (a \ast b, x \ast x, u \ast u) = (a, x, u) \odot (b, x, u) \in S_1 \times S_2 \times S_3 \), that is, \( a \ast b \in S_1 \). Hence, \( S_1 \) is a subalgebra of \( X \). On the other hand, we can show that \( S_2 \) and \( S_3 \) are subalgebras of \( X \).

Conversely, let \( (x, y, z), (a, b, c) \in S_1 \times S_2 \times S_3 \). Then \( x \ast a \in S_1, y \ast b \in S_2, \) and \( z \ast c \in S_3 \), so \( (x, y, z) \odot (a, b, c) = (x \ast a, y \ast b, z \ast c) \in S_1 \times S_2 \times S_3 \). Hence, \( S_1 \times S_2 \times S_3 \) is a subalgebra of \( X^3 \).

(2) Suppose that \( S_1 \times S_2 \times S_3 \) is an ideal of \( X^3 \). Since \( (0, 0, 0) \in S_1 \times S_2 \times S_3 \), we have \( 0 \in S_i \) for all \( i = 1, 2, 3 \). Assume that \( a \ast x \in S_1 \) and \( a \in S_1 \). Let \( b \in S_2 \) and \( c \in S_3 \). Then \( (a, b, c) \in S_1 \times S_2 \times S_3 \) and \( (x, b, c) \in X^3 \). Thus \( (a, b, c) \odot (x, b, c) = (a \ast x, b \ast b, c \ast c) = (a \ast x, 0, 0) \in S_1 \times S_2 \times S_3 \). Since \( S_1 \times S_2 \times S_3 \) is an ideal of \( X^3 \), we have \( (x, b, c) \in S_1 \times S_2 \times S_3 \), that is, \( x \in S_1 \). Hence, \( S_1 \) is an ideal of \( X \). Similarly, we can show that \( S_2 \) and \( S_3 \) are ideals of \( X \).

Conversely, suppose that \( S_1, S_2 \) and \( S_3 \) are ideals of \( X \). Since \( 0 \in S_i \) for all \( i = 1, 2, 3 \), we have \( (0, 0, 0) \in S_1 \times S_2 \times S_3 \). Assume that \( (a, b, c) \ast (x, y, z) \in S_1 \times S_2 \times S_3 \) and \( (a, b, c) \in S_1 \times S_2 \times S_3 \). We get \( (a \ast x, b \ast y, c \ast z) \in S_1 \times S_2 \times S_3 \). Since \( a \ast x, a \in S_1 \), we have \( x \in S_1 \). Moreover, we can obtain that \( y \in S_2 \) and \( z \in S_3 \). This implies that \( (x, y, z) \in S_1 \times S_2 \times S_3 \). Hence, \( S_1 \times S_2 \times S_3 \) is an ideal of \( X^3 \). \( \square \)

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