Fuzzy Ideals and Fuzzy Filters on Topologies Generated by Fuzzy Relations

Kheir Saadaoui\textsuperscript{1,*}, Soheyb Milles\textsuperscript{2}, Lemnaouar Zedam\textsuperscript{3}

\textsuperscript{1}Laboratory LMPA, Department of Mathematics, University of M’sila, Algeria
\textsuperscript{2}Laboratory LMPA, Department of Mathematics and Informatics, University Center of Barika, Algeria
\textsuperscript{3}Laboratory LMPA, Department of Mathematics, University of M’sila, Algeria

*Corresponding author: kheir.saadaoui@univ-msila.dz

Abstract. Recently, Mishra and Srivastava have introduced and studied the notion of fuzzy topology generated by fuzzy relation and several properties were proved. In this paper, we mainly investigate the lattice structure of fuzzy open sets in this topology, and show its various properties and characteristics. Additionally, we extend to this lattice the notions of fuzzy ideal and fuzzy filter. For each of these notions, we fully characterize them in terms of this lattice meet and join operations.

1. Introduction

The notions of ideals and filters have studied in many algebraic structures (e.g., semi-groups, rings, MV-algebras, lattices, et cetera) and used as tools to investigate properties, representations and characterizations of these algebraic structures [16], [7], [13], [14]. In addition to their theoretical roles, they have used in some areas of applied mathematics, especially, in topology and its analysis approaches. Ideals and filters are appeared to provide very general contexts to unify the various notions of sequences convergence and limit in arbitrary topological spaces, and to express completeness and compactness in metric spaces [3], [17].

In the fuzzy setting and its extensions, several authors introduced and investigated the concepts of fuzzy ideals and fuzzy filters in different structures. The first approach considered fuzzy ideal and fuzzy filter as fuzzy sets on crisp structures, like on lattices or on residuated lattices [5], [9], [15]. The
second approach proposed similar notions on (intuitionistic) fuzzy structures [2], [10], [12]. The third approach considered neutrosophic ideal and neutrosophic filter as neutrosophic sets [1], [20].

The present study is motivated by the work of Mishra and Srivastava [11] that have considered the notion of fuzzy topology generated by a fuzzy relation. More specifically, we deepen the study of a lattice structure of fuzzy open sets on this topology, and providing its various characteristics and properties. We pay particular attention to the notion of fuzzy ideal (resp. fuzzy filter) on this topology generated by a fuzzy relation. Furthermore, we provide a characterization of these notion of fuzzy ideal (resp. fuzzy filter) based on the meet and the join operations of the introduced lattice.

This paper is organized as follows. In Section 2, we recall some basic concepts related to fuzzy sets, fuzzy relations and fuzzy topology. In Section 3, we provide the lattice structure of fuzzy open sets in a fuzzy topology generated by a fuzzy relation, and we show its various properties and characteristics. In Section 4, we introduce the notions of fuzzy ideal (resp. fuzzy filter) on the lattice of fuzzy open sets, and some basic properties are given. Finally, some conclusions and future research in Section 5 are presented.

2. Basic concepts

This section contains the basic definitions and properties of fuzzy sets, fuzzy topology and some related notions that will be needed throughout this paper.

2.1. Fuzzy sets. In this subsection, we recall some basic concepts of fuzzy sets.

Let $X$ be a universe, a fuzzy subset $A = \{\langle x, \mu_A(x) \rangle \mid x \in X\}$ of $X$ defined by Zadeh in 1965 [18] is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element $x$ in the fuzzy subset $A$ for each $x \in X$.

For fuzzy sets, several operations are defined. Here we present only those which are related to the present paper.

Let $A = \{\langle x, \mu_A(x) \rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x) \rangle \mid x \in X\}$ be two fuzzy subsets on $X$, then

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, for any $x \in X$;
(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$, for any $x \in X$;
(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x) \rangle \mid x \in X\}$;
(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x) \rangle \mid x \in X\}$;
(v) $\overline{A} = \{\langle x, 1 - \mu_A(x) \rangle \mid x \in X\}$;
(vi) $\text{Supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$;
(vii) $\text{Ker}(A) = \{x \in X \mid \mu_A(x) = 1\}$.

In the sequel, we need the following definition of level set (which is also often called $\alpha$-cuts) of a fuzzy set.
Definition 2.1. [6] Let \( A \) be a fuzzy set on a nonempty set \( X \). The \( \alpha \)-cut of \( A \) is the crisp subset 
\[
A_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha \},
\]
for any \( \alpha \in [0, 1] \).


Definition 2.2. [19] A fuzzy binary relation (a fuzzy relation, for short) from a nonempty set \( X \) to a nonempty set \( Y \) is a fuzzy subset in \( X \times Y \), i.e., is an expression \( R \) given by
\[
R = \{ (x, y), \mu_R(x, y) \mid (x, y) \in X \times Y \},
\]
where
\[
\mu_R : X \times Y \to [0, 1]
\]
for any \((x, y) \in X \times Y\). The value \( \mu_R(x, y) \) is called the degree of relation between \( x \) and \( y \) under the fuzzy relation \( R \).

Next, we need to recall the following definitions [19].

Let \( R \) and \( P \) are two fuzzy relations from a nonempty set \( X \) to a nonempty set \( Y \). \( R \) is said to be contained in \( P \) or we say that \( P \) contains \( R \), denoted by \( R \subseteq P \), if for all \((x, y) \in X \times Y \) it holds that \( \mu_R(x, y) \leq \mu_P(x, y) \).

The transpose (inverse) \( R^t \) of \( R \) is the fuzzy relation from the nonempty set \( Y \) to the nonempty set \( X \) defined by
\[
R^t = \{ (x, y), \mu_{R^t}(x, y) \mid (x, y) \in X \times Y \},
\]
where \( \mu_{R^t}(x, y) = \mu_R(y, x) \) for any \((x, y) \in X \times Y\).

The intersection of two fuzzy relations \( R \) and \( P \) from a nonempty set \( X \) to a nonempty set \( Y \) is defined as
\[
R \cap P = \{ (x, y), \mu_{R \cap P}(x, y) \},
\]
where \( \mu_{R \cap P}(x, y) = \min(\mu_R(x, y), \mu_P(x, y)) \) for any \((x, y) \in X \times Y\).

The union of two fuzzy relations \( R \) and \( P \) from a nonempty set \( X \) to a nonempty set \( Y \) is defined as
\[
R \cup P = \{ (x, y), \mu_{R \cup P}(x, y) \},
\]
where \( \mu_{R \cup P}(x, y) = \max(\mu_R(x, y), \mu_P(x, y)) \) for any \((x, y) \in X \times Y\).

In general, if \( A \) is a set of fuzzy relations from a nonempty set \( X \) to a nonempty set \( Y \), then
\[
\bigcap_{R \in A} R = \{ (x, y), \mu_{\bigcap_{R \in A} R}(x, y) \},
\]
where \( \mu_{\cap R}(x, y) = \inf_{R \in A} \mu_R(x, y) \) for any \((x, y) \in X \times Y; \)

\[
\bigcup_{R \in A} R = \{(x, y), \mu_{\cup R}(x, y)\},
\]

where \( \mu_{\cup R}(x, y) = \sup_{R \in A} \mu_R(x, y) \) for any \((x, y) \in X \times Y.\)

2.3. Fuzzy topology.

**Definition 2.3.** [4] [Fuzzy topology] A fuzzy topology (FT, for short) on a nonempty set \( X \) is a family \( \tau \) of fuzzy sets on \( X \) which satisfies the following axioms:

(i) \( \emptyset, X \in \tau \);

(ii) \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \);

(iii) \( \bigcup G_i \in \tau \) for any \( \{G_i : i \in J\} \subseteq \tau \).

In this case, the pair \((X, \tau)\) is called a fuzzy topological space (FTS, for short) and any FS in \( \tau \) is known as a fuzzy open set (FOS, for short) in \( X \). The complement of a fuzzy open set is called a fuzzy closed set (FCS, for short) in \( X \).

**Example 2.1.** Let \( X = \{x_1, x_2, x_3\} \) and \( A_1, A_2, A_3 \in FS(X) \) such that

\[
A_1 = \{(x_1, 0.4), (x_2, 0.7), (x_3, 0.1)\},
\]

\[
A_2 = \{(x_1, 0.3), (x_2, 0.6), (x_3, 0.2)\},
\]

\[
A_3 = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.2)\}.
\]

Then, \( \tau = \{\emptyset, X, A_1, A_2, A_3\} \) is a fuzzy topology on \( X \).

2.4. Fuzzy topology generated by a fuzzy relation. The notion of fuzzy topology generated by a fuzzy relation was previously proposed by Mishra and Srivastava [11].

**Definition 2.4.** [11] Let \( X \) be a nonempty crisp set and \( R = \{(x, y), \mu_R(x, y)\} \) \( x, y \in X \) a fuzzy relation on \( X \). Then for any \( x \in X \), the fuzzy sets \( L_x \) and \( R_x \) defined by:

\[
\mu_{L_x}(y) = \mu_R(y, x), \text{ for any } y \in X,
\]

\[
\mu_{R_x}(y) = \mu_R(x, y), \text{ for any } y \in X,
\]

are called respectively the lower and the upper contour of \( x \).

We denote by \( \tau_1 \), the fuzzy topology generated by the set of all lower contours and \( \tau_2 \), the fuzzy topology generated by the set of all upper contours. Consequently, we denote by \( \tau_R \), the fuzzy topology generated by \( S \) the set of all lower and upper contours and it’s called the fuzzy topology generated by \( R \).

**Definition 2.5.** Let \( R \) be a fuzzy relation on the set \( X \) and \( \tau_R \) is the fuzzy topology generated by \( R \) and let \( U_1, U_2 \) are two fuzzy open sets on \( \tau_R \). The \( U_1 \) is said to be contained in \( U_2 \) (in symbols, \( U_1 \subseteq U_2 \)) if \( \mu_{U_1}(x) \leq \mu_{U_2}(x) \) for any \( x \in X \). In this case, we also say that \( U_1 \) is smaller than \( U_2 \).
Example 2.2. Let $X = \{x, y\}$ and $R$ be a fuzzy relation on $X$ given by:

<table>
<thead>
<tr>
<th>$\mu_R(\cdot)$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>$y$</td>
<td>0.4</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Then $L_x$, $L_y$, $R_x$ and $R_y$ are the fuzzy sets on $X$ given by:

$L_x = \{(x, 0.5); (y, 0.4)\}$,
$L_y = \{(x, 0.7); (y, 0.6)\}$,
$R_x = \{(x, 0.5); (y, 0.7)\}$,
$R_y = \{(x, 0.4); (y, 0.6)\}$.

Notice that $R_y \subseteq R_x$ and $R_y \subseteq L_y$. The fuzzy topology $\tau_R$ generated by the fuzzy topology generated by $R$ is the fuzzy topology generated by $S = \{L_x, L_y\} \cup \{R_x, R_y\}$, i.e., $\tau_R = \{\emptyset, X, L_x, L_y, R_x, R_y, L_x \cap R_x, L_y \cap R_y, L_x \cup R_x, L_y \cup R_y\}$, where

$L_x \cap R_y = \{(x, 0.4); (y, 0.4)\}$,
$L_y \cap R_x = \{(x, 0.5); (y, 0.6)\}$,
$L_x \cup R_y = \{(x, 0.5); (y, 0.6)\}$, and $L_y \cup R_x = \{(x, 0.7); (y, 0.7)\}$.

Example 2.3. Let $X = \{x, y, z\}$ and $R$ be a fuzzy relation on $X$ given by:

<table>
<thead>
<tr>
<th>$\mu_R(\cdot)$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>$z$</td>
<td>0.7</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$L_x$, $L_y$, $L_z$, $R_x$, $R_y$ and $R_z$ are the fuzzy sets on $X$ given by:

$L_x = \{(x, 1); (y, 0); (z, 0.7)\}$,
$L_y = \{(x, 0.5); (y, 1); (z, 0)\}$,
$L_z = \{(x, 0); (y, 0.8); (z, 1)\}$,
$R_x = \{(x, 1); (y, 0.5); (z, 0)\}$,
$R_y = \{(x, 0); (y, 1); (z, 0.8)\}$,
$R_z = \{(x, 0.7); (y, 0); (z, 1)\}$.

The fuzzy topology $\tau_R$ is generated by $S = \{L_x, L_y, L_z\} \cup \{R_x, R_y, R_z\}$. Thus,

$\tau_R = \{\emptyset, X, L_x, L_y, L_z, R_x, R_y, R_z, \{(x, 0.5); (y, 0); (z, 0)\}, \{(x, 0); (y, 0); (z, 0.7)\}, \{(x, 1); (y, 0); (z, 0)\}, \{(x, 0.7); (y, 0); (z, 0.7)\}, \{(x, 0); (y, 0.8); (z, 0)\}, \{(x, 0.5); (y, 0.5); (z, 0)\}.$
Remark 3.1.

Theorem 3.1. by a fuzzy relation.

Proof. The fact that \( R \) is a lattice.

hold:

Definitions. 4.1. A fuzzy set \( A \) on \( X \) is called a fuzzy ideal if for all \( A, B \in \mathcal{L} \) the following conditions hold:

\( \mu_I(A \cup B) \geq \mu_I(A) \land \mu_I(B) \),

\( \mu_I(A \cap B) \geq \mu_I(A) \lor \mu_I(B) \).

Definition 4.2. A fuzzy set \( F \) on \( \mathcal{L} \) is called a fuzzy filter if for all \( A, B \in \mathcal{L} \) the following conditions hold:
(i) $\mu_F(A \cup B) \geq \mu_F(A) \lor \mu_F(B)$.
(ii) $\mu_F(A \cap B) \geq \mu_F(A) \land \mu_F(B)$.

The following proposition expresses the relationship between a fuzzy ideal and a fuzzy filter on a lattice of fuzzy open sets. Its proof is straightforward.

**Proposition 4.1.** Let $\mathcal{L}^d$ be the order-dual lattice of $\mathcal{L}$ and $A$ be fuzzy set on $(\mathcal{L})$. Then it holds that $A$ is a fuzzy ideal on $\mathcal{L}$ if and only if it is a fuzzy filter on $\mathcal{L}^d$, and conversely.

We need also the following result.

**Proposition 4.2.** Let $A$ and $B$ be two fuzzy sets on $\mathcal{L}$, then it holds that

(i) If $A$ and $B$ are two fuzzy ideals on $\mathcal{L}$, then $A \cap B$ is a fuzzy ideal on $\mathcal{L}$;
(ii) If $A$ and $B$ are two fuzzy filters on $\mathcal{L}$, then $A \cap B$ is a fuzzy filter on $\mathcal{L}$.

### 4.2. Basic characterization of fuzzy ideals and filters on a lattice of fuzzy open sets

In this subsection, we provide interesting characterization of fuzzy ideals (resp. filters) on the lattice of fuzzy open sets in terms of its meet and its join operations.

**Theorem 4.1.** $I$ is a fuzzy ideal on $\mathcal{L}$ if and only if the following condition is satisfied:

$$\mu_I(A \cup B) = \mu_I(A) \land \mu_I(B), \text{ for any } A, B \in \mathcal{L}.$$  

**Proof.** Suppose that $I$ is a fuzzy ideal on $\mathcal{L}$, then for any $A, B \in \mathcal{L}$ it holds that $\mu_I(A \cup B) \geq \mu_I(A) \land \mu_I(B)$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows from Definition 4.1 (ii) that $\mu_I(A) = \mu_I(A \cap (A \cup B)) \geq \mu_I(A) \lor \mu_I(A \cup B) \geq \mu_I(A \cup B)$. In the same manner, $\mu_I(B) \geq \mu_I(A \cup B)$. Hence, $\mu_I(A) \land \mu_I(B) \geq \mu_I(A \cup B)$. Thus, $\mu_I(A \cup B) = \mu_I(A) \land \mu_I(B)$.

Conversely, suppose that $\mu_I(A \cup B) = \mu_I(A) \land \mu_I(B)$ for any $A, B \in \mathcal{L}$. Then it is easy to see that $\mu_I(A \cup B) \geq \mu_I(A) \land \mu_I(B)$ for any $A, B \in \mathcal{L}$. Next, we will show that $\mu_I(A \cap B) \geq \mu_I(A) \lor \mu_I(B)$ for any $A, B \in \mathcal{L}$. Let $A, B \in \mathcal{L}$, since $A \cup (A \cap B) = A$ and $B \cup (A \cap B) = B$ then it holds that $\mu_I(A \cup (A \cap B)) = \mu_I(A)$ and $\mu_I(B \cup (A \cap B)) = \mu_I(B)$. From hypothesis it follows that $\mu_I(A) \land \mu_I(A \cap B) = \mu_I(A)$ and $\mu_I(B) \land \mu_I(A \cap B) = \mu_I(B)$. Hence, $\mu_I(A \cap B) \geq \mu_I(A)$ and $\mu_I(A \cap B) \geq \mu_I(B)$. Thus, $\mu_I(A \cap B) \geq \mu_I(A) \lor \mu_I(B)$, for any $A, B \in \mathcal{L}$. Therefore, $I$ is a fuzzy ideal on $\mathcal{L}$. \hfill \Box

In the same line, the following theorem provides a characterization of fuzzy filters on the lattice of fuzzy open sets in terms of its meet operation.

**Theorem 4.2.** $F$ is a fuzzy filter on $\mathcal{L}$ if and only if the following condition is satisfied:

$$\mu_F(A \cap B) = \mu_F(A) \land \mu_F(B), \text{ for any } A, B \in \mathcal{L}.$$
Proof. The proof is a direct application of Proposition 4.1 and Theorem 4.1.

As corollaries of the above theorems, we obtain the following interesting properties of fuzzy ideals and fuzzy filters on a lattice of fuzzy open sets.

**Corollary 4.1.** Let \( I \) be a fuzzy ideal on \( \mathcal{L} \) and \( A, B \in \mathcal{L} \). If \( A \sqsubseteq B \), then \( \mu_I(A) \geq \mu_I(B) \), (i.e., the mapping \( \mu_I \) is antitone).

**Corollary 4.2.** Let \( F \) be a fuzzy filter on \( \mathcal{L} \) and \( A, B \in \mathcal{L} \). If \( A \sqsubseteq B \), then \( \mu_F(A) \leq \mu_F(B) \), (i.e., the mapping \( \mu_F \) is monotone).

5. Conclusion and Future Work

In this article, we have studied properties of lattices on fuzzy topology generated by fuzzy relation, and provided their various characteristics. We have introduced and studied the notions of fuzzy ideal (resp. fuzzy filter) on the lattice of fuzzy open sets and we have discussed some its basic properties. We anticipate that these notions of fuzzy ideals (resp. fuzzy filters) will facilitate the study and the representations of the different kinds of fuzzy lattices. Due to the usefulness of these notions, we think it makes sense to study some kinds of fuzzy ideals (resp. fuzzy filters) on fuzzy topology generated by fuzzy relation.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


