On \( \omega_{\tilde{\theta}}\mu \)-Open Sets in Generalized Topological Spaces

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Abstract. In this paper analogous to [1], we introduce a new class of sets called \( \omega_{\tilde{\theta}}\mu \)-open sets in generalized topological spaces which lies strictly between the class of \( \tilde{\theta}\mu \)-open sets and the class of \( \omega\mu \)-open sets. We prove that the collection of \( \omega_{\tilde{\theta}}\mu \)-open sets forms a generalized topology. Finally, several characterizations and properties of this class have been given.

1. Introduction

One notion that has received much attention lately is the so-called \( \omega \)-open sets in a topological space \((X, \tau)\) was introduced by Hdeib [12], which forms a topology finer than \( \tau \). Recently, many topological concepts and several interesting results related to this notion have obtained by many authors such as [3], [10], [9], [2]. A collection \( \mu \) of subsets of a nonempty set \( X \) is a generalized topology (GT) if \( \emptyset \in \mu \) and \( \mu \) is closed under arbitrary unions, this notion was introduced by Császár in the sense of [5]. We call the pair \((X, \mu)\) a generalized topological space (briefly GTS) on \( X \). The elements of \( \mu \) are called \( \mu \)-open sets and their complements are called \( \mu \)-closed sets, see [7], the union of all elements of \( \mu \) will be denoted by \( \mathcal{M}_\mu \) and a GTS \((X, \mu)\) is said to be strong [7] if \( X \in \mu \). If \( A \) is a subset of a GTS \((X, \mu)\), then the \( \mu \)-closure of \( A \), \( c_\mu(A) \), is the intersection of all \( \mu \)-closed sets containing \( A \) and the \( \mu \)-interior of \( A \), \( i_\mu(A) \), is the union of all \( \mu \)-open sets contained in \( A \) (see [5, 7]). It is easy to observe that operators \( i_\mu \) and \( c_\mu \) are idempotent and monotonic A subset \( A \) of a GTS \((X, \mu)\) is \( \mu \)-open if and only if \( A = i_\mu(A) \), and and \( i_\mu(A) = X \setminus c_\mu(X \setminus A) \). Evidently, \( A \) is \( \mu \)-closed if and only if \( A = c_\mu(A) \), \( c_\mu(A) \) is the smallest \( \mu \)-closed set containing \( A \), \( i_\mu(A) \) is the largest \( \mu \)-open set contained in \( A \). Over recent years several authors have been working in formulate many topological

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concepts to establish new concepts in the structure of GTS, see [4], [8], [6] [11], [17], [15], [13] and others. Then motivated by the notion of $\omega$-$\mu$-open set in a topological space $(X, \tau)$, Al Ghour and Wafa Zareer (2016) [1] defined the notions of $\omega$-$\mu$-closed sets and $\omega$-$\mu$-open sets in the structure of GTS as follows: A subset $A$ of GTS $(X, \mu)$ is called $\omega$-$\mu$-closed if it contains all its condensation points. The complement of an $\omega$-$\mu$-closed set is called $\omega$-$\mu$-open. The family of all $\omega$-$\mu$-open subsets of $X$ forms a GT on $X$, denoted by $\omega_\mu$. Let us now recall some notions defined in [14]. A subset $A$ of GTS $(X, \tau)$ is said to be $\tilde{\theta}_\mu$-open if and only if for each $x \in A$, there exists $U \in \mu$ such that $x \in U \subseteq c_\mu(U) \cap \mathcal{M}_\mu \subseteq A$ and the collection of all $\tilde{\theta}_\mu$-open subsets of a GTS $(X, \mu)$ is denoted by $\tilde{\theta}_\mu$. Then $\tilde{\theta}_\mu$ is also a GT included in $\mu$. Analogous to [1] and by using the notion of $\tilde{\theta}_\mu$-open, we introduce the relatively new notions of $\omega_{\tilde{\theta}}$-$\mu$-open as a new class of sets. We present several characterizations, properties, and examples related to the new concepts.

In section 2, we use the the notion of $\tilde{\theta}_\mu$-open to introduce $\omega_{\tilde{\theta}}$-$\mu$-open sets in GTS as a new class of sets and we prove that this class lies strictly between the class of $\tilde{\theta}_\mu$-open sets and the class of $\omega$-$\mu$-open sets. Moreover, we give some sufficient conditions for the equivalence between the class of $\omega_{\tilde{\theta}}$-$\mu$-open sets and the class of $\omega$-$\mu$-open sets.

In section 3, several interesting properties of $\omega_{\tilde{\theta}}$-$\mu$-open subsets are discussed via the operations of $\omega_{\tilde{\theta}}$-interior and $\omega_{\tilde{\theta}}$-closure.

**Definition 1.1.** [16] A GTS $(X, \mu)$ is said to be $\mu$-locally indiscrete if every $\mu$-open set in $(X, \mu)$ is $\mu$-closed.

**Definition 1.2.** [1] A GTS $(X, \mu)$ is called $\mu$-locally countable if $\mathcal{M}_\mu$ is nonempty and for every point $x \in \mathcal{M}_\mu$, there exists a $U \in \mu$ such that $x \in U$ and $U$ is countable.

**Definition 1.3.** [14] Let $(X, \mu)$ be a GTS, $A \subseteq X$ and $\gamma_{\tilde{\theta}} : P(X) \to P(X)$ be an operation defined as the following:

$$\gamma_{\tilde{\theta}}(A) = \{x \in X : c_\mu(U) \cap \mathcal{M}_\mu \cap A \neq \emptyset \text{ for all } U \in \tilde{\theta}_\mu, x \in U\}.$$ 

**Theorem 1.1.** [1] Let $(X, \mu)$ be a GTS. Then $\mathcal{M}_\mu = \mathcal{M}_{\omega_{\mu}}$.

**Theorem 1.2.** [1] If $(X, \mu)$ is a $\mu$-locally countable GTS, then $\omega_{\mu}$ is the discrete topology on $\mathcal{M}_\mu$.

### 2. $\omega_{\tilde{\theta}}$-$\mu$-open sets

We begin this section by introducing the following definition.

**Definition 2.1.** Let $(X, \mu)$ be a GTS and $A \subseteq X$. Consider an operation $\Gamma_{\omega_{\tilde{\theta}}} : P(X) \to P(X)$ defined as the following:

$$\Gamma_{\omega_{\tilde{\theta}}}(A) = \{x \in X : U \cap A \text{ is uncountable for all } U \in \tilde{\theta}_\mu \text{ and } x \in U\}.$$ 

A point $x \in X$ is called a
\( \tilde{\theta}_\mu \)-condensation point of \( A \) if for all \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \cap A \) is uncountable. The set of all \( \tilde{\theta}_\mu \)-condensation points of \( A \) is denoted by \( \Gamma_{\omega_\mu}(A) \).

**Lemma 2.1.** Let \((X, \mu)\) be a GTS. The operation \( \Gamma_{\omega_\mu} : P(X) \to P(X) \) has the following properties:

1. if \( A \subset B \subset X \), then \( \Gamma_{\omega_\mu}(A) \subset \Gamma_{\omega_\mu}(B) \) (monotonic property);
2. \( \Gamma_{\omega_\mu}(\Gamma_{\omega_\mu}(A)) \subset \Gamma_{\omega_\mu}(A) \) for any \( A \subset X \) (restricting property);
3. if \( A \) is any countable subset of \( X \), then \( \Gamma_{\omega_\mu}(A) = \emptyset \).

**Proof.** (1) Let \( A \subset B \subset X \) and \( x \in \Gamma_{\omega_\mu}(A) \). Then \( U \cap A \) is uncountable for each \( U \in \tilde{\theta}_\mu \) and \( x \in U \). Since \( A \subset B \), then \( U \cap B \) is uncountable. Thus \( x \in \Gamma_{\omega_\mu}(B) \) and hence \( \Gamma_{\omega_\mu}(A) \subset \Gamma_{\omega_\mu}(B) \).

(2) Let \( x \in \Gamma_{\omega_\mu}(\Gamma_{\omega_\mu}(A)) \). Then \( U \cap \Gamma_{\omega_\mu}(A) \) is an uncountable for all \( U \in \tilde{\theta}_\mu \) and \( x \in U \). Let \( y \in U \cap \Gamma_{\omega_\mu}(A) \). Then \( y \in U \) and \( y \in \Gamma_{\omega_\mu}(A) \) which implies that \( U \cap A \) is an uncountable set. Hence \( x \in \Gamma_{\omega_\mu}(A) \) and therefore \( \Gamma_{\omega_\mu}(\Gamma_{\omega_\mu}(A)) \subset \Gamma_{\omega_\mu}(A) \).

(3) The proof is obvious by Definition 2.1.

**Definition 2.2.** Let \((X, \mu)\) be a GTS and \( A \subset X \). Then \( A \) is said to be \( \omega_\tilde{\theta}-\mu \)-closed if \( \Gamma_{\omega_\tilde{\theta}}(A) \subset A \). The complement of an \( \omega_\tilde{\theta}-\mu \)-closed set is said to be \( \omega_\tilde{\theta}-\mu \)-open.

The family of all \( \omega_\tilde{\theta}-\mu \)-open subsets of \((X, \mu)\) is denoted by \( \omega_\tilde{\theta} \), where \( \omega_\tilde{\theta} = \{ W \subset X : \Gamma_{\omega_\tilde{\theta}}(X \setminus W) \subset X \setminus W \} \). The following theorem and lemma give a necessary and sufficient condition for \( \omega_\tilde{\theta}-\mu \)-open sets.

**Theorem 2.1.** Let \((X, \mu)\) be a GTS and \( W \subset X \). Then the following statements are equivalent:

1. \( W \) is \( \omega_\tilde{\theta}-\mu \)-open;
2. if for every \( x \in W \) there exists a \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \setminus W \) is a countable set.

**Proof.** (1) \( \Rightarrow \) (2): Suppose \( W \) is \( \omega_\tilde{\theta}-\mu \)-open. Since \( X \setminus W \) is \( \omega_\tilde{\theta}-\mu \)-closed set, then \( \Gamma_{\omega_\tilde{\theta}}(X \setminus W) \subset X \setminus W \). This means that for every \( x \in W \), \( x \notin \Gamma_{\omega_\tilde{\theta}}(X \setminus W) \) and hence there exists a \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \cap (X \setminus W) = U \setminus W \) is countable.

(2) \( \Rightarrow \) (1): Let \( x \in W \). Then by assumption there exists a \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \cap (X \setminus W) \) is countable. Which implies that \( x \notin \Gamma_{\omega_\tilde{\theta}}(X \setminus W) \), \( \Gamma_{\omega_\tilde{\theta}}(X \setminus W) \subset X \setminus W \) and hence \( X \setminus W \) is \( \omega_\tilde{\theta}-\mu \)-closed.

Therefore \( W \) is \( \omega_\tilde{\theta}-\mu \)-open set.

**Lemma 2.2.** A subset \( W \) of a GTS \((X, \mu)\) is \( \omega_\tilde{\theta}-\mu \)-open if and only if for every \( x \in W \) there exists a \( U \in \tilde{\theta}_\mu \) and a countable set \( C \subset M_\mu \) such that \( x \in U \setminus C \subset W \).

**Proof.** Necessity. Let \( W \) be \( \omega_\tilde{\theta}-\mu \)-open and \( x \in W \). By Theorem 2.1, there exists \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \setminus W \) is countable. Let \( C = U \setminus W \). Then \( C \) is countable, \( C \subset M_\mu \) and \( x \in U \cap (X \setminus C) = U \cap (X \setminus (U \cap X \setminus W)) = U \cap W \subset W \) and hence \( x \in U \setminus C \subset W \).

Sufficiency. Let \( x \in W \). From assumption there exists \( U \in \tilde{\theta}_\mu \) and a countable set \( C \subset M_\mu \) such that \( x \in U \setminus C \subset W \). Therefore, \( U \setminus W \subset C \) and \( U \setminus W \) is a countable set and this completes the proof.
Example 2.1. Consider \( \omega_{\tilde{\theta}} \)-\( \mu \)-closed set \( F \subseteq \omega \) and a countable subset \( B \subseteq M \) such that \( x \in U \setminus B \subseteq X \). Thus \( C \subseteq X \setminus (U \setminus B) = X \setminus (U \cap (X \setminus B)) = (X \setminus U) \cup B \). Let \( F = X \setminus U \). Then \( F \) is \( \omega_{\tilde{\theta}} \)-\( \mu \)-closed such that \( C \subseteq F \cup B \). \( \square \)

Theorem 2.3. Let \( (X, \mu) \) be a GTS. Then the collection \( \omega_{\tilde{\theta}} \) forms a generalized topology on \( X \).

Proof. It is clear that \( \emptyset \in \omega_{\tilde{\theta}} \). Let \( \{ W_\lambda : \lambda \in \Delta \} \) be a collection of \( \omega_{\tilde{\theta}} \)-\( \mu \)-open subsets of \( (X, \mu) \) and \( x \in \bigcup \lambda \subseteq W_\lambda \). There exists an \( \lambda_0 \in \Delta \) such that \( x \in W_{\lambda_0} \). Since \( W_{\lambda_0} \) is \( \omega_{\tilde{\theta}} \)-open set, then by Lemma 2.2, there exist \( U \in \tilde{\theta}_\mu \) and a countable set \( C \subseteq M_\mu \) such that \( x \in U \setminus C \subseteq W_{\lambda_0} \subseteq \bigcup \lambda \subseteq \Delta \). By Lemma 2.2, it follows that \( \bigcup \lambda \subseteq W_\lambda \) is \( \omega_{\tilde{\theta}} \)-\( \mu \)-open. Hence the collection \( \omega_{\tilde{\theta}} \) is generalized topology on \( X \). \( \square \)

The next theorem obtains that the new class of \( \omega_{\tilde{\theta}} \)-\( \mu \)-open sets lies strictly between the class of \( \tilde{\theta}_\mu \)-\( \mu \)-open sets and the class of \( \omega_\mu \)-open sets.

Theorem 2.4. Let \( (X, \mu) \) be a GTS. Then \( \tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}} \subseteq \omega_\mu \).

Proof. To show that \( \tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}} \), let \( W \in \tilde{\theta}_\mu \) and \( x \in W \). Take \( U = W \) and \( C = \emptyset \). Then \( U \in \tilde{\theta}_\mu \), \( C \subseteq M_\mu \) such that \( x \in U \setminus C \subseteq W \). Therefore, by Lemma 2.2, it follows that \( W \in \omega_{\tilde{\theta}} \).

To show that \( \omega_{\tilde{\theta}} \subseteq \omega_\mu \), let \( W \in \omega_{\tilde{\theta}} \). By Theorem 2.1, for each \( x \in W \) there exists a \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \setminus W \) is countable. Since \( \tilde{\theta}_\mu \subseteq \mu \), then \( U \in \mu \) and hence \( W \) is \( \omega_\mu \)-open. Therefore \( \omega_{\tilde{\theta}} \subseteq \omega_\mu \). \( \square \)

The following diagram follows immediately from the definitions and Theorem 2.4.

\[
\begin{align*}
\tilde{\theta}_\mu - \text{open} & \implies \omega_{\tilde{\theta}} - \text{open} \\
\mu - \text{open} & \implies \omega - \mu - \text{open}
\end{align*}
\]

The converse of these implications need not be true in general as shown by the following examples.

Example 2.1. Consider \( X = \mathbb{R}, A = \{4n : n \in \mathbb{N}\} \) and \( \mu = \{0, [0, 2], [1, 3] \cup A, [0, 3] \cup A\} \). Then \( (X, \mu) \) is a generalized topological space and the family of all \( \tilde{\theta}_\mu \)-open sets is \( \tilde{\theta}_\mu = \{0, [0, 3] \cup A\} \). Then \( [1, 3] \subseteq \omega_\mu \setminus \omega_{\tilde{\theta}} \), i.e. \( [1, 3] \) is \( \omega \)-\( \mu \)-open but it is not \( \omega_{\tilde{\theta}} \)-\( \mu \)-open. Also, it is easy to check that \( \Gamma_{\omega_{\tilde{\theta}}}(\mathbb{R} \setminus [0, 3]) \subseteq \mathbb{R} \setminus [0, 3] \). Thus \( [0, 3] \subseteq \omega_{\tilde{\theta}} \), i.e. \( [0, 3] \) is \( \omega_{\tilde{\theta}} \)-\( \mu \)-open but it is not \( \tilde{\theta}_\mu \)-open.

Example 2.2. Let \( X = \{a, b, c, d\} \) with \( GT \mu = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\} \). Then \( \{a, c\} \subseteq \omega_{\tilde{\theta}} \), \( \emptyset \subseteq \tilde{\theta}_\mu \), i.e. the set \( \{a, c\} \) is \( \omega_{\tilde{\theta}} \)-\( \mu \)-open but it is not \( \tilde{\theta}_\mu \)-open.

Note that the previous examples show that \( \tilde{\theta}_\mu \neq \omega_{\tilde{\theta}} \neq \omega_\mu \) in general.
Remark 2.1. The notions of $\mu$-open and $\omega_\theta$-$\mu$-open sets are independent of each other. For more clarity in Example 2.1, the set $[0, 3]$ is $\omega_\theta$-$\mu$-open but it is not $\mu$-open and the set $[1, 3] \cup A$ is $\mu$-open but it is not $\omega_\theta$-$\mu$-open.

Theorem 2.5. If a GTS $(X, \mu)$ is a $\mu$-locally indiscrete, then $\mu \subseteq \omega_\theta$.

Proof. To show that $\mu \subseteq \omega_\theta$, let $A \in \mu$ and $x \in A$. Take $U = A$. Since $(X, \mu)$ is $\mu$-locally indiscrete, then $c_\mu(U) = U$ and we have $x \in U \subseteq c_\mu(U) \cap M_\mu \subseteq A$. Thus $A \in \tilde{\theta}_\mu$ and by Theorem 2.4, $\tilde{\theta}_\mu \subseteq \omega_\theta$. Therefore $A \in \omega_\theta$. □

Lemma 2.3. Let $(X, \mu)$ be a GTS. Then $M_\mu \subseteq \tilde{\theta}_\mu$.

Proof. Let $A = M_\mu$ and $x \in A$. Then there exists $U_x \in \mu$ such that $x \in U_x$. Since $U_x \subseteq c_\mu(U_x) \cap M_\mu \subseteq A$, then $A = M_\mu \subseteq \tilde{\theta}_\mu$. □

For a GT $\mu$ on a nonempty set $X$, let $M_{\omega_\theta} = \bigcup\{U \subseteq X : U \in \omega_\theta\}$. Thus we have the following theorem.

Theorem 2.6. Let $(X, \mu)$ be a GTS. Then $M_\mu = M_{\omega_\theta}$.

Proof. By Lemma 2.3, $M_\mu \subseteq \tilde{\theta}_\mu$ and form Theorem 2.4, $\tilde{\theta}_\mu \subseteq \omega_\theta$ and hence $M_\mu \subseteq M_{\omega_\theta}$. On the other hand, let $x \in M_{\omega_\theta}$. Since, $M_{\omega_\theta} \subseteq \omega_\theta$, then by Lemma 2.2, there exists a $U \in \tilde{\theta}_\mu$ and a countable set $C \subseteq M_\mu$ such that $x \in U \setminus C \subseteq M_{\omega_\theta}$. Since $U \subseteq M_\mu$ and $U$ is $\mu$-open, it follows that $x \in M_\mu$ and hence $M_{\omega_\theta} \subseteq M_\mu$. Therefore $M_\mu = M_{\omega_\theta}$. □

By Theorem 1.1 and Theorem 2.6, we obtain the following corollary

Corollary 2.1. Let $(X, \mu)$ be a GTS. Then $M_\mu = M_{\omega_\theta} = M_{\omega_\mu}$.

We will denote by $(\tau_{coc})_X$, the cocountable topology on a nonempty set $X$.

Theorem 2.7. Let $(X, \mu)$ be a GTS. Then $(\tau_{coc})_U \subseteq \omega_\theta$ for all $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$.

Proof. Let $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$, $W \in (\tau_{coc})_U$ and $x \in W$. Since $W \subseteq U$, we have $x \in U$ and $U \setminus W = U \setminus (U \cap V)$ for some $V \in \tau_{coc}$. Now, $U \setminus W = U \setminus (U \cap V) = U \setminus V$. Thus $U \setminus W$ is countable set and by Theorem 2.1, it follows that $W \in \omega_\theta$. This shows that $(\tau_{coc})_U \subseteq \omega_\theta$. □

Theorem 2.8. For any GTS $(X, \mu)$, the following statements are equivalent.

1) $\tilde{\theta}_\mu = \omega_\theta$.
2) $(\tau_{coc})_U \subseteq \tilde{\theta}_\mu$ for all $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$.

Proof. (1) $\implies$ (2): Assume that $\tilde{\theta}_\mu = \omega_\theta$ and $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$. Then by Theorem 2.7, $(\tau_{coc})_U \subseteq \omega_\theta = \tilde{\theta}_\mu$.

(2) $\implies$ (1): Suppose that $(\tau_{coc})_U \subseteq \tilde{\theta}_\mu$ for all $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$. It is enough to show that $\omega_\theta \subseteq \tilde{\theta}_\mu$. Let
Let \( W \in \omega_{\tilde{\theta}} \) and \( x \in W \). By Lemma 2.2, there exists \( U_x \in \tilde{\theta}_\mu \) and a countable set \( C_x \subseteq M_\mu \) such that \( x \in U_x \setminus C_x \subseteq W \). Thus \( U_x \cap X \setminus C_x \in (\tau_{\text{coc}})_{\omega_\mu} \), where \( X \setminus C_x \in \tau_{\text{coc}} \). From assumption \( U_x \setminus C_x \in (\tau_{\text{coc}})_{\omega_\mu} \) for all \( x \in W \), and so \( U_x \setminus C_x \in \tilde{\theta}_\mu \). It follows that \( W = \bigcup \{ U_x \setminus C_x : x \in W \} \in \tilde{\theta}_\mu \), and hence \( \tilde{\theta}_\mu = \omega_{\tilde{\theta}} \).
\[ \square \]

**Proposition 2.1.** Let \((X, \mu)\) be a GTS. If \( \tilde{\theta}_\mu \) is a topology on \( X \), then \( \omega_{\tilde{\theta}} \) is a topology.

**Proof.** Suppose that \( \tilde{\theta}_\mu \) is a topology. By Theorem 2.3, \( \omega_{\tilde{\theta}} \) is generalized topology. It is enough to show that the collection \( \omega_{\tilde{\theta}} \) is closed under finite intersection. Let \( W, G \) be \( \omega_{\tilde{\theta}}-\mu \)-open sets and \( x \in W \cap G \). Then by Theorem 2.1, there exist \( U, V \in \tilde{\theta}_\mu \) containing \( x \) such that \( U \setminus W \) and \( V \setminus G \) are countable sets. Since \( \tilde{\theta}_\mu \) is a topology, we have \( x \in U \cap V \in \tilde{\theta}_\mu \). Furthermore, \((U \cap V) \setminus (W \cap G) = (U \cap V) \setminus [X \setminus (W \cup G)] = [(U \cap V) \setminus W] \cup [(U \cap V) \setminus G] \subseteq (U \setminus W) \cup (V \setminus G)\). Therefore, \((U \cap V) \setminus (W \cap G)\) is a countable set and hence \( W \cap G \) is \( \omega_{\tilde{\theta}}-\mu \)-open.
\[ \square \]

**Definition 2.3.** Let \((X, \mu)\) be a GTS. Then \((X, \mu)\) is said to be \( \tilde{\theta}_\mu \)-locally countable if \( M_\mu \) is nonempty and for every point \( x \in M_\mu \), there exists a \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \) is countable.

The following corollary is a direct result from Definition 2.3 and Definition 1.2.

**Corollary 2.2.** Let \((X, \mu)\) be a GTS. If \((X, \mu)\) is \( \tilde{\theta}_\mu \)-locally countable, then \((X, \mu)\) is \( \mu \)-locally countable.

**Theorem 2.9.** If \((X, \mu)\) is a \( \tilde{\theta}_\mu \)-locally countable GTS, then \( \omega_{\tilde{\theta}} \) is the discrete topology on \( M_\mu \).

**Proof.** It is enough to show that every singleton subset of \( M_\mu \) is \( \omega_{\tilde{\theta}}-\mu \)-open. Since \((X, \mu)\) is \( \tilde{\theta}_\mu \)-locally countable, then for each \( x \in M_\mu \), there exists a \( U \in \tilde{\theta}_\mu \) such that \( x \in U \) and \( U \) is countable. By Theorem 2.7, we have \((\tau_{\text{coc}})_U \subseteq \omega_{\tilde{\theta}} \). Therefore \( U \setminus (U \setminus \{x\}) = \{x\} \in \omega_{\tilde{\theta}} \).
\[ \square \]

The following corollary is a direct result of Theorem 2.9.

**Corollary 2.3.** Let \((X, \mu)\) be a strong GTS. If \((X, \mu)\) is a \( \tilde{\theta}_\mu \)-locally countable, then \( \omega_{\tilde{\theta}} \) is the discrete topology on \( M_\mu \).

**Proposition 2.2.** If \((X, \mu)\) is a \( \tilde{\theta}_\mu \)-locally countable GTS, then \( \omega_{\tilde{\theta}} = \omega_\mu \).

**Proof.** Since \((X, \mu)\) is \( \tilde{\theta}_\mu \)-locally countable, then by Theorem 2.9, \( \omega_{\tilde{\theta}} \) is the the discrete topology on \( M_\mu \). From Corollary 2.2 and Theorem 1.2, we get \( \omega_{\tilde{\theta}} = \omega_\mu \).
\[ \square \]

**Corollary 2.4.** Let \((X, \mu)\) be a GTS. If \( M_\mu \) is a countable nonempty set, then \( \omega_{\tilde{\theta}} \) is the discrete topology on \( M_\mu \).

**Proof.** Since \( M_\mu \) is countable nonempty set, then for \( x \in M_\mu \), there exists \( U \in \tilde{\theta}_\mu \) such that \( U \) is countable set. Thus \((X, \mu)\) is \( \tilde{\theta}_\mu \)-locally countable. From Theorem 2.9, we get \( \omega_{\tilde{\theta}} \) is the discrete topology on \( M_\mu \).
\[ \square \]
Definition 3.1. Let \((X, \mu)\) be a GTS and \(A \subseteq X\). A point \(x \in X\) is called an \(\omega_{\theta}\)-closure point of \(A\) if and only if \(U \cap A \neq \emptyset\) for all \(U \in \omega_{\theta}\) and \(x \in U\). Consider the following operations are defined as follows:

1. \(\gamma_{\omega_{\theta}}(A) = \{x \in X : U \cap A \neq \emptyset, \text{ for all } U \in \omega_{\theta} \text{ and } x \in U\}\);
2. \(c_{\omega_{\theta}}(A) = \cap\{F : A \subseteq F, F \text{ is } \omega_{\theta}\text{-}\mu\text{-closed in } X\}\).

Lemma 3.1. Let \((X, \mu)\) be a GTS. Then \(c_{\omega_{\theta}}(A) = \gamma_{\omega_{\theta}}(A)\) for any \(A \subseteq X\).

Proof. It is enough to show that \(\gamma_{\omega_{\theta}}(A)\) is the smallest \(\omega_{\theta}\text{-}\mu\text{-closed set containing } A\). Clearly \(A \subseteq \gamma_{\omega_{\theta}}(A)\). Further \(\gamma_{\omega_{\theta}}(A)\) is \(\omega_{\theta}\text{-}\mu\text{-closed}, that is \(X \setminus \gamma_{\omega_{\theta}}(A)\) is \(\omega_{\theta}\text{-}\mu\text{-open because for each } x \in X \setminus \gamma_{\omega_{\theta}}(A)\) there is \(U_x \in \omega_{\theta}\) such that \(x \in U_x\) and \(U_x \cap A = \emptyset\). Now, for any \(y \in U_x\) implies \(y \in X \setminus \gamma_{\omega_{\theta}}(A)\) so that \(X \setminus \gamma_{\omega_{\theta}}(A) = \bigcup_{x \in X \setminus \gamma_{\omega_{\theta}}(A)} U_x \in \omega_{\theta}\).

Finally if \(A \subseteq F\) and \(F\) is any \(\omega_{\theta}\text{-}\mu\text{-closed}, then \(X \setminus F\) is \(\omega_{\theta}\text{-}\mu\text{-open and } (X \setminus F) \cap A = \emptyset\) so that \(X \setminus F \subseteq X \setminus \gamma_{\omega_{\theta}}(A)\) and hence \(\gamma_{\omega_{\theta}}(A) \subseteq F\). Therefore \(\gamma_{\omega_{\theta}}(A)\) is the smallest \(\omega_{\theta}\text{-}\mu\text{-closed set containing } A\), and by Definition 3.1(2), \(\gamma_{\omega_{\theta}}(A) = c_{\omega_{\theta}}(A)\).

The proof of the following theorem is straightforward and thus omitted.

Theorem 3.1. For subsets \(A, B\) of GTS \((X, \mu)\), the following properties hold:

1. If \(A \subseteq B \subseteq X\), then \(c_{\omega_{\theta}}(A) \subseteq c_{\omega_{\theta}}(B)\);
2. \(A \subseteq c_{\omega_{\theta}}(A)\) for \(A \subseteq X\);
3. \(c_{\omega_{\theta}}(c_{\omega_{\theta}}(A)) = c_{\omega_{\theta}}(A)\) for \(A \subseteq X\);
4. \(A\) is \(\omega_{\theta}\text{-}\mu\text{-closed if and only if } c_{\omega_{\theta}}(A) = A\).

Definition 3.2. Let \((X, \mu)\) be a GTS and \(A \subseteq X\). Then we define the following notions:

1. \(c_{\tilde{\theta}_{\mu}}(A) = \cap\{F : A \subseteq F, F \text{ is } \tilde{\theta}_{\mu}\text{-}\mu\text{-closed in } X\}\);
2. \(c_{\omega_{\theta}}(A) = \cap\{F : A \subseteq F, F \text{ is } \omega_{\theta}\text{-}\mu\text{-closed in } X\}\).

The proof of the following corollary is straightforward and thus omitted.

Corollary 3.1. For a subset \(A\) of a GTS \((X, \mu)\), the following properties hold:

1. \(A\) is \(\tilde{\theta}_{\mu}\text{-}\mu\text{-closed if and only if } c_{\tilde{\theta}_{\mu}}(A) = A\);
2. \(A\) is \(\omega_{\theta}\text{-}\mu\text{-closed if and only if } c_{\omega_{\theta}}(A) = A\).

Lemma 3.2. Let \((X, \mu)\) be a GTS. Then \(\gamma_{\tilde{\theta}_{\mu}}(A) \subseteq c_{\tilde{\theta}_{\mu}}(A)\) for any \(A \subseteq X\).

Proof. Let \(x \notin c_{\tilde{\theta}_{\mu}}(A)\). Then \(x \in X \setminus c_{\tilde{\theta}_{\mu}}(A)\) so that there is \(U \in \tilde{\theta}_{\mu}\) satisfying \(x \in U\) and \(U \cap A = \emptyset\). Since \(U \in \tilde{\theta}_{\mu}\), then there is \(V \in \mu\) such that \(x \in V \subseteq c_{\mu}(V) \cap M_{\mu} \subseteq U\) and \(c_{\mu}(V) \cap M_{\mu} \cap A = \emptyset\), consequently \(x \notin \gamma_{\tilde{\theta}}(A)\). Thus we have \(\gamma_{\tilde{\theta}}(A) \subseteq c_{\tilde{\theta}}(A)\).
Theorem 3.2. Let \((X, \mu)\) be a GTS and \(A \subseteq X\). Then the following properties hold:

1. \(c_{\omega_{\mu}}(A) \subseteq c_{\omega_{\tilde{\mu}}}(A)\);
2. If \(A\) is \(\tilde{\theta}_{\mu}\)-closed, then \(A\) is \(\omega_{\tilde{\mu}}\)-\(\mu\)-closed;
3. If \(A\) is \(\omega_{\tilde{\mu}}\)-\(\mu\)-closed, then \(A\) is \(\omega\)-\(\mu\)-closed.

Proof. (1) To show that \(c_{\omega_{\mu}}(A) \subseteq c_{\omega_{\tilde{\mu}}}(A)\), let \(x \notin c_{\omega_{\tilde{\mu}}}(A)\) and so there is a \(U \in \omega_{\tilde{\mu}}\) containing \(x\) such that \(U \cap A = \emptyset\). From Theorem 2.4, we have \(\omega_{\tilde{\mu}} \subseteq \omega_{\mu}\), \(U \subseteq \omega_{\mu}\), and hence \(x \notin c_{\omega_{\mu}}(A)\). To show that \(c_{\omega_{\tilde{\mu}}}(A) \subseteq c_{\tilde{\theta}_{\mu}}(A)\), let \(x \notin c_{\tilde{\theta}_{\mu}}(A)\) and so there is a \(U \in \tilde{\theta}_{\mu}\) containing \(x\) such that \(U \cap A = \emptyset\). From Theorem 2.4, we have \(\tilde{\theta}_{\mu} \subseteq \omega_{\tilde{\mu}}\), \(U \in \omega_{\tilde{\mu}}\), and hence \(x \notin c_{\omega_{\tilde{\mu}}}(A)\).

(2) Suppose that \(A\) is \(\tilde{\theta}_{\mu}\)-closed. Then by Corollary 3.1(1), \(c_{\tilde{\theta}_{\mu}}(A) = A\). Thus by (1), \(c_{\omega_{\tilde{\mu}}}(A) = A\) and hence \(A\) is \(\omega_{\tilde{\mu}}\)-\(\mu\)-closed.

(2) Suppose that \(A\) is \(\omega_{\tilde{\mu}}\)-\(\mu\)-closed. Then by Theorem 3.1(4), \(c_{\omega_{\mu}}(A) = A\). Thus by (1), \(c_{\omega_{\mu}}(A) = A\) and hence \(A\) is \(\omega\)-\(\mu\)-closed.

Proposition 3.1. Let \((X, \mu)\) be a \(\tilde{\theta}_{\mu}\)-locally countable GTS and \(A \subseteq X\). Then \(c_{\omega_{\mu}}(A) = c_{\omega_{\tilde{\mu}}}(A)\)

Proof. By Theorem 3.2(1), \(c_{\omega_{\mu}}(A) \subseteq c_{\omega_{\tilde{\mu}}}(A)\). Let \(x \in c_{\omega_{\tilde{\mu}}}(A)\). Then \(U \cap A \neq \emptyset\) for all \(U \in \omega_{\tilde{\mu}}\) and \(x \in U\). Since \((X, \mu)\) is a \(\tilde{\theta}_{\mu}\)-locally countable, then by Theorem 2.9, \(\omega_{\tilde{\mu}}\) is the discrete topology on \(\mathcal{M}_{\mu}\) and hence \(\omega_{\mu} = \omega_{\tilde{\mu}}\). Which implies that \(x \in c_{\omega_{\mu}}(A)\) and \(c_{\omega_{\mu}}(A) \subseteq c_{\omega_{\mu}}(A)\). Hence \(c_{\omega_{\mu}}(A) = c_{\omega_{\tilde{\mu}}}(A)\).

Theorem 3.3. Let \((X, \mu)\) be a \(\mu\)-locally indiscrete GTS and let \(A \subseteq X\). Then the following properties hold.

1. \(c_{\mu}(A) = c_{\tilde{\theta}_{\mu}}(A)\);
2. \(c_{\omega_{\tilde{\mu}}}(A) \subseteq c_{\mu}(A)\);
3. If \(A\) is \(\mu\)-closed in \((X, \mu)\), then \(A\) is \(\tilde{\theta}_{\mu}\)-\(\mu\)-closed in \((X, \mu)\).
4. If \(A\) is \(\tilde{\mu}\)-\(\mu\)-closed in \((X, \mu)\), then \(A\) is \(\omega_{\tilde{\mu}}\)-\(\mu\)-closed in \((X, \mu)\).

Proof. (1) Clearly \(c_{\mu}(A) \subseteq c_{\tilde{\theta}_{\mu}}(A)\). To show that \(c_{\tilde{\theta}_{\mu}}(A) \subseteq c_{\mu}(A)\), let \(x \notin c_{\mu}(A)\). Then there exists \(U \in \mu\) such that \(x \in U\) and \(U \cap A = \emptyset\). Since \((X, \mu)\) is a \(\mu\)-locally indiscrete, \(c_{\mu}(U) = U\). It follows that \(U \subseteq c_{\mu}(U) \cap \mathcal{M}_{\mu} \subseteq U\) and hence \(U \in \tilde{\theta}_{\mu}\). Thus \(x \notin c_{\tilde{\theta}_{\mu}}(A)\).

(2) Since \((X, \mu)\) is \(\mu\)-locally indiscrete, then by Theorem 2.5, \(\mu \subseteq \omega_{\tilde{\mu}}\) and hence \(c_{\omega_{\tilde{\mu}}}(A) \subseteq c_{\mu}(A)\).

(3) Suppose that \(A\) is \(\mu\)-closed in \((X, \mu)\), then \(c_{\mu}(A) = A\). Thus by (1), \(A = c_{\tilde{\theta}_{\mu}}(A)\) and hence \(A\) is \(\tilde{\theta}_{\mu}\)-\(\mu\)-closed in \((X, \mu)\).

(4) Suppose that \(A\) is \(\mu\)-closed in \((X, \mu)\), then \(c_{\mu}(A) = A\). Thus by (2), \(A = c_{\omega_{\tilde{\mu}}}(A)\) and hence \(A\) is \(\omega_{\tilde{\mu}}\)-\(\mu\)-closed in \((X, \mu)\).

Definition 3.3. A GTS \((X, \mu)\) is said to be \(\omega_{\tilde{\mu}}\)-anti-locally countable if the intersection of any two \(\omega_{\tilde{\mu}}\)-\(\mu\)-open sets is either empty or uncountable.

The following lemma is used to prove the theorem which is stated below.
Lemma 3.3. Let \( (X, \mu) \) be \( \omega_\tilde{g} \)-anti-locally countable and \( A \subseteq X \). If \( A \in \omega_\tilde{g} \), then \( c_{\tilde{g}_\mu}(A) = c_{\omega_\tilde{g}}(A) \).

Proof. Suppose that \( \emptyset \neq A \subseteq X \) and \( A \in \omega_\tilde{g} \). By Theorem 3.2(1), \( c_{\omega_\tilde{g}}(A) \subseteq c_{\tilde{g}_\mu}(A) \). To Show that \( c_{\tilde{g}_\mu}(A) \subseteq c_{\omega_\tilde{g}}(A) \), let \( x \in c_{\tilde{g}_\mu}(A) \) and \( W \in \omega_\tilde{g} \) such that \( x \in W \). Then by Lemma 2.2, there exists \( U \in \tilde{g}_\mu \) and a countable set \( C \subseteq M_\mu \) such that \( x \in U \setminus C \subseteq W \). Since \( x \in U \cap c_{\tilde{g}_\mu}(A) \), \( U \cap A \neq \emptyset \). Choose \( y \in U \cap A \). Since \( A \in \omega_\tilde{g} \), there exists \( V \in \tilde{g}_\mu \) and a countable set \( D \subseteq M_\mu \) such that \( y \in V \setminus D \subseteq A \). Since \( y \in U \cap V \) and \( (X, \mu) \) is \( \omega_\tilde{g} \)-anti-locally countable, then \( U \cap V \) is uncountable. Thus, \( (U \setminus C) \cap (V \setminus D) \neq \emptyset \) and hence \( A \cap W \neq \emptyset \). Therefore, \( x \in c_{\omega_\tilde{g}}(A) \). \( \square \)

A subset \( A \) of GTS \( (X, \mu) \) is said to be \( \tilde{\theta}_\mu \)-clopen (resp. \( \omega_\tilde{g} \)-clopen) if it is both \( \tilde{\theta}_\mu \)-open and \( \tilde{\theta}_\mu \)-closed (resp. \( \omega_\tilde{g} \)-open and \( \omega_\tilde{g} \)-closed).

In the following, by using Lemma 3.3, we prove the main result in this section.

Theorem 3.5. Let \( (X, \mu) \) be \( \omega_\tilde{g} \)-anti-locally countable and \( A \subseteq X \). Then, \( A \) is \( \tilde{\theta}_\mu \)-clopen if and only if \( A \) is \( \omega_\tilde{g} \)-clopen.

Proof. \( \Rightarrow \) Suppose that \( A \) is \( \tilde{\theta}_\mu \)-clopen, then \( A \) and \( X \setminus A \) are \( \tilde{\theta}_\mu \)-open. Since \( \tilde{\theta}_\mu \subseteq \omega_\tilde{g} \), then \( A \) and \( X \setminus A \) are \( \omega_\tilde{g} \)-open, and hence \( A \) is \( \omega_\tilde{g} \)-clopen.

\( \Leftarrow \) Suppose that \( A \) is \( \omega_\tilde{g} \)-clopen. Since \( A \) and \( X \setminus A \) are \( \omega_\tilde{g} \)-open, the by Lemma 3.3,

\[ c_{\tilde{g}_\mu}(A) = c_{\omega_\tilde{g}}(A) \] and \( c_{\tilde{g}_\mu}(X \setminus A) = c_{\omega_\tilde{g}}(X \setminus A) \).

Since \( A \) is \( \omega_\tilde{g} \)-clopen., then

\[ c_{\tilde{g}_\mu}(A) = c_{\omega_\tilde{g}}(A) = A \] and \( c_{\omega_\tilde{g}}(X \setminus A) = X \setminus A \).

Therefore,

\[ c_{\tilde{g}_\mu}(A) = A \] and \( c_{\tilde{g}_\mu}(X \setminus A) = X \setminus A \)

and hence \( A \) and \( X \setminus A \) are \( \tilde{\theta}_\mu \)-closed sets. This means that \( A \) is \( \tilde{\theta}_\mu \)-clopen. \( \square \)

Definition 3.4. Let \( (X, \mu) \) be a GTS and \( A \subseteq X \). Then, we define the following notions:

(1) \( i_{\omega_\tilde{g}}(A) = \{ U \subseteq X : U \subseteq A, U \) is \( \omega_\tilde{g} \)-open \};

(2) \( i_{\tilde{g}_\mu}(A) = \{ U \subseteq X : U \subseteq A, U \) is \( \tilde{g}_\mu \)-open \};

(3) \( i_{\omega_\tilde{g}}(A) = \{ U \subseteq X : U \subseteq A, U \) is \( \omega_\mu \)-open \}.

Theorem 3.5. For subsets \( A, B \) of GTS \( (X, \mu) \), the following properties hold:

(1) if \( A \subseteq B \subseteq X \), then \( i_{\omega_\tilde{g}}(A) \subseteq i_{\omega_\tilde{g}}(B) \);

(2) for \( A \subseteq X \), then \( i_{\omega_\tilde{g}}(A) \subseteq A \);

(3) \( i_{\omega_\tilde{g}}(i_{\omega_\tilde{g}}(A)) = i_{\omega_\tilde{g}}(A) \) for \( A \subseteq X \);

(4) \( A \) is \( \omega_\tilde{g} \)-open if and only if \( i_{\omega_\tilde{g}}(A) = A \).

Proof. The proof is obvious. \( \square \)
Corollary 3.2. Let $(X, \mu)$ be a GTS and $A \subseteq X$. Then $i_{\tilde{\theta}_\mu}(A) \subseteq i_{\omega_\mu}(A) \subseteq i_{\omega_\tilde{\theta}_\mu}(A)$.

Proof. To show that $i_{\tilde{\theta}_\mu}(A) \subseteq i_{\omega_\mu}(A)$, let $x \in i_{\tilde{\theta}_\mu}(A)$. Then there is $U \in \tilde{\theta}_\mu$ such that $x \in U \subseteq A$. By Theorem 2.4, $U$ is $\omega_\mu$-$\mu$-open. Thus $x \in i_{\omega_\mu}(A)$. To show that $i_{\omega_\mu}(A) \subseteq i_{\omega_\tilde{\theta}_\mu}(A)$, let $x \in i_{\omega_\mu}(A)$. Then there is $U \in \omega_\tilde{\theta}_\mu$ such that $x \in U \subseteq A$. Then by Theorem 2.4, $U$ is $\omega_\mu$-$\mu$-open and hence $x \in i_{\omega_\mu}(A)$.

Theorem 3.6. Let $(X, \mu)$ be a GTS and $A \subseteq X$. Then the following properties hold:

1. $c_{\omega_\mu}(X \setminus A) = X \setminus i_{\omega_\mu}(A)$;
2. $i_{\omega_\mu}(X \setminus A) = X \setminus c_{\omega_\mu}(A)$.

Proof. (1) Let $x \in c_{\omega_\mu}(X \setminus A)$ and $U \in \omega_\mu$ with $x \in U$. Since $x \in c_{\omega_\mu}(X \setminus A)$, $U \cap (X \setminus A) \neq \emptyset$. This implies that $x \notin i_{\omega_\mu}(A)$ and hence $x \in X \setminus i_{\omega_\mu}(A)$.

Conversely, for $x \in X \setminus i_{\omega_\mu}(A)$, $x \notin i_{\omega_\mu}(A)$, and then $U \cap (X \setminus A) \neq \emptyset$ for all $U \in \omega_\mu$ and $x \in U$ which implies $x \in c_{\omega_\mu}(X \setminus A)$.

(2) Let $x \in X \setminus c_{\omega_\mu}(A)$ if and only if $x \notin c_{\omega_\mu}(A)$ if and only if there is $U \in \omega_\mu$ with $x \in U$ such that $U \cap A = \emptyset$ if and only if $x \in i_{\omega_\mu}(X \setminus A)$.

4. Conclusion

In this paper, we introduced the notion of $\omega_\tilde{\theta}_\mu$-$\mu$-open sets in the sense of generalized topology given in [5]. We have proved that the collection of $\omega_\tilde{\theta}_\mu$-$\mu$-open sets forms a generalized topology on $X$ that lies between the class of $\tilde{\theta}_\mu$-$\mu$-open sets and the class of $\omega_\mu$-$\mu$-open sets. The relationships of $\omega_\tilde{\theta}_\mu$-$\mu$-open and other well-known generalized open sets are given. Several properties of $\omega_\tilde{\theta}_\mu$-$\mu$-open sets which enable us to prove certain of our results are studied and verified. In the upcoming work, we plan to:

1. introduce some concepts in GTS using $\omega_\tilde{\theta}_\mu$-$\mu$-open sets such as connectedness, compactness and Lindelöfness;
2. introduce continuity and decomposition of continuity via $\omega_\tilde{\theta}_\mu$-$\mu$-open sets.

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