Pairwise Semiregular Properties on Generalized Pairwise Lindelöf Spaces

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Abstract. Let \((X, \tau_1, \tau_2)\) be a bitopological space and \((X, \tau^s_1, \tau^s_2)\) its pairwise semiregularization. Then a bitopological property \(P\) is called pairwise semiregular provided that \((X, \tau_1, \tau_2)\) has the property \(P\) if and only if \((X, \tau^s_1, \tau^s_2)\) has the same property. In this work we study pairwise semiregular property of \((i,j)\)-nearly Lindelöf, pairwise nearly Lindelöf, \((i,j)\)-almost Lindelöf, pairwise almost Lindelöf, \((i,j)\)-weakly Lindelöf and pairwise weakly Lindelöf spaces. We prove that \((i,j)\)-almost Lindelöf, pairwise almost Lindelöf, \((i,j)\)-weakly Lindelöf and pairwise weakly Lindelöf are pairwise semiregular properties, on the contrary of each type of pairwise Lindelöf space which are not pairwise semiregular properties.

1. Introduction

Semiregular properties in topological spaces have been studied by many topologist. Some of them related to this research studied by Mrsevic et al. [14, 15] and Fawakhreh and Kılıçman [3]. But in bitopological space, the study of this topic is still open for investigation. The purpose of this paper is to study pairwise semiregular properties on generalized pairwise Lindelöf spaces, that we have studied in [9, 13, 16, 17], namely, \((i,j)\)-nearly Lindelöf, pairwise nearly Lindelöf, \((i,j)\)-almost Lindelöf, pairwise almost Lindelöf, \((i,j)\)-weakly Lindelöf and pairwise weakly Lindelöf spaces.

In 2010, Salleh and Kılıçman [19] studied the pairwise semiregular properties of \((i,j)\)-almost regular-Lindelöf, pairwise almost regular-Lindelöf, \((i,j)\)-weakly regular-Lindelöf and pairwise weakly regular-Lindelöf spaces. They also show that the \((i,j)\)-nearly regular-Lindelöf and pairwise nearly-regular-Lindelöf spaces are pairwise semiregular invariant properties.

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The main results is that the Lindelöf, $B$-Lindelöf, $s$-Lindelöf and $p$-Lindelöf spaces are not pairwise semiregular properties. While $(i, j)$-almost Lindelöf, pairwise almost Lindelöf, $(i, j)$-weakly Lindelöf and pairwise weakly Lindelöf spaces are pairwise semiregular properties. We also show that $(i, j)$-nearly Lindelöf and pairwise nearly Lindelöf spaces are satisfying pairwise semiregular invariant properties.

2. Preliminaries

Throughout this paper, all spaces $(X, \tau)$ and $(X, \tau_1, \tau_2)$ (or simply $X$) are always mean topological spaces and bitopological spaces, respectively unless explicitly stated. If $P$ is a topological property, then $(\tau_i, \tau_j)$-$P$ denotes an analogue of this property for $\tau_i$ has property $P$ with respect to $\tau_j$, and $p$-$P$ denotes the conjunction $(\tau_1, \tau_2)$-$P \land (\tau_2, \tau_1)$-$P$, i.e., $p$-$P$ denotes an absolute bitopological analogue of $P$.

Also note that $(X, \tau_i)$ has a property $P \iff (X, \tau_1, \tau_2)$ has a property $\tau_i$-$P$. Sometimes the prefixes $(\tau_i, \tau_j)$- or $\tau_i$- will be replaced by $(i, j)$- or $i$- respectively, if there is no chance for confusion. By $i$-open cover of $X$, we mean that the cover of $X$ by $i$-open sets in $X$; similar for the $(i, j)$-regular open cover of $X$ etc. By $i$-int $(A)$ and $i$-cl $(A)$, we shall mean the interior and the closure of a subset $A$ of $X$ with respect to topology $\tau_i$, respectively. In this paper always $i, j \in \{1, 2\}$ and $i \neq j$. The reader may consult [2] for the detail notations.

The following are some basic concepts.

**Definition 2.1.** [6,20] A subset $S$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-regular open (resp. $(i, j)$-regular closed) if $i$-int $(j$-cl $(S)) = S$ (resp. $i$-cl $(j$-int $(S)) = S)$, where $i, j \in \{1, 2\}, i \neq j$. $S$ is called pairwise regular open (resp. pairwise regular closed) if it is both $(1, 2)$-regular open and $(2, 1)$-regular open (resp. $(1, 2)$-regular closed and $(2, 1)$-regular closed).

**Definition 2.2.** [6,21] A bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-almost regular if for each point $x \in X$ and for each $(i, j)$-regular open set $V$ containing $x$, there exists an $(i, j)$-regular open set $U$ such that $x \in U \subseteq j$-cl $(U) \subseteq V$. $X$ is called pairwise almost regular if it is both $(1, 2)$-almost regular and $(2, 1)$-almost regular.

In any bitopological space $(X, \tau_1, \tau_2)$, the family of all $(i, j)$-regular open sets is closed under finite intersections. Thus the family of $(i, j)$-regular open sets in any bitopological space $(X, \tau_1, \tau_2)$ forms a base for a coarser topology called $(i, j)$-semiregularization of $(X, \tau_1, \tau_2)$, which is defined as follows.

**Definition 2.3.** [16] The topology generated by the $(i, j)$-regular open subsets of $(X, \tau_1, \tau_2)$ is denoted by $\tau^s_{(i,j)}$ and it is called $(i, j)$-semiregularization of $X$. The topologies is pairwise semiregularization of $X$ if the first topology is $(1, 2)$-semiregularization of $X$ and the second topology is $(2, 1)$-semiregularization of $X$. If $\tau_i \equiv \tau^s_{(i,j)}$, then $X$ is said to be $(i, j)$-semiregular. $(X, \tau_1, \tau_2)$ is called pairwise semiregular if it is both $(1, 2)$-semiregular and $(2, 1)$-semiregular, that is, whenever
\( \tau_i \equiv \tau_{(i,j)}^s \) for each \( i, j \in \{1, 2\} \) and \( i \neq j \). In other words, \( (X, \tau_1, \tau_2) \) is \((i, j)\)-semiregular if the family of \((i, j)\)-regular open sets form a base for the topology \( \tau_i \).

It is very clear that \( \tau_{(i,j)}^s \subseteq \tau_i \), but it is not necessary \( \tau_i \subseteq \tau_{(i,j)}^s \). Thus with every given bitopological space \((X, \tau_1, \tau_2)\) there is associated another bitopological space \((X, \tau_{(1,2)}, \tau_{(2,1)}^s)\) in the manner described above (see [20]). We provide the following example in order to understand the concept of pairwise semiregular spaces clearly.

**Example 2.1.** For the set of all real numbers \( \mathbb{R} \), let \( \tau_u \) denotes the usual topology and \( \tau_s \) denote the Sorgenfrey topology, i.e., topology generated by right half-open intervals (see [22]). Then \((\mathbb{R}, \tau_u, \tau_s)\) is \((\tau_u, \tau_s)\)-semiregular since \( \tau_u = \tau_{(\tau_u, \tau_s)} \), i.e., \( \tau_u \) generated by \((\tau_u, \tau_s)\)-regular open subsets of \( \mathbb{R} \). \((\mathbb{R}, \tau_u, \tau_s)\) is also \((\tau_s, \tau_u)\)-semiregular since \( \tau_s = \tau_{(\tau_s, \tau_u)}^s \) because any set \( E \in \tau_s \) is the union of a collection of \((\tau_s, \tau_u)\)-regular open sets in \( \mathbb{R} \). Thus \((\mathbb{R}, \tau_u, \tau_s)\) is pairwise semiregular.

Khedr and Alshibani [6] defined the equivalent definition of \((i, j)\)-semiregular spaces as follows.

**Definition 2.4.** A bitopological space \( X \) is said to be \((i, j)\)-semiregular if for each \( x \in X \) and for each \( i\)-open subset \( V \) of \( X \) containing \( x \), there is an \( i\)-open set \( U \) such that \( x \in U \subseteq i\text{-int} (j\text{-cl} (U)) \subseteq V \). \( X \) is called pairwise semiregular if it is both \((1, 2)\) and \((2, 1)\)-semiregular.

**Definition 2.5.** [1] A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((i, j)\)-extremally disconnected if the \( i\)-closure of every \( j\)-open set is \( j\)-open. \( X \) is called pairwise extremally disconnected if it is both \((1, 2)\) and \((2, 1)\)-extremally disconnected.

Recall that a property \( \mathcal{P} \) will be called bitopological property (resp. \( p\)-topological property) if whenever \((X, \tau_1, \tau_2)\) has property \( \mathcal{P} \), then every space homeomorphic (resp. \( p\)-homeomorphic) to \((X, \tau_1, \tau_2)\) also has property \( \mathcal{P} \) (see [8]). If a bitopological space \( X \) has bitopological (or \( p\)-topological) property \( \mathcal{P} \), one may ask, does the pairwise semiregularization of \( X \) satisfies the property \( \mathcal{P} \) also? Now we arrive to the concept of pairwise semiregular property.

**Definition 2.6.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and let \((X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)\) its pairwise semiregularization. A bitopological property \( \mathcal{P} \) is called pairwise semiregular provided that \((X, \tau_1, \tau_2)\) has the property \( \mathcal{P} \) if and only if \((X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)\) has the property \( \mathcal{P} \).

**Lemma 2.1.** [16] Let \((X, \tau_1, \tau_2)\) be a bitopological space and let \((X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)\) its pairwise semiregularization. Then

(a) \( \tau_j\text{-int} (C) = \tau_{(i,j)}^s\text{-int} (C) \) for every \( \tau_j\)-closed set \( C \);

(b) \( \tau_j\text{-cl} (A) = \tau_{(i,j)}^s\text{-cl} (A) \) for every \( A \in \tau_j \);

(c) the family of \((\tau_i, \tau_j)\)-regular open sets of \((X, \tau_1, \tau_2)\) are the same as the family of \((\tau_{(i,j)}^s, \tau_{(i,j)}^s)\)-regular open sets of \((X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)\);
(d) the family of \((\tau_i, \tau_j)\)-regular closed sets of \((X, \tau_1, \tau_2)\) are the same as the family of \(\left(\tau_{i,j}^s, \tau_{j,i}^s\right)\)-regular closed sets of \(\left(X, \tau_{1,2}^s, \tau_{2,1}^s\right)\):
\[
(e) \left(\tau_{i,j}^s\right)^s_{(i,j)} = \tau_{j,i}^s_{(i,j)}.
\]

3. Pairwise Semiregularization of Pairwise Lindelöf Spaces

**Definition 3.1.** \([5, 7]\). A bitopological space \((X, \tau_1, \tau_2)\) is said to be i-Lindelöf if the topological space \((X, \tau_i)\) is Lindelöf. \(X\) is called Lindelöf if it is i-Lindelöf for each \(i = 1, 2\). In other words, \((X, \tau_1, \tau_2)\) is called Lindelöf if the topological space \((X, \tau_1)\) and \((X, \tau_2)\) are both Lindelöf.

Note that i-Lindelöf property as well as Lindelöf property is not a pairwise semiregular property by the following example.

**Example 3.1.** Let \(X\) be a set with cardinality \(2^c\), where \(c = \text{card}(\mathbb{R})\). Let \(\tau_1\) be a co-c topology on \(X\) consisting of \(\emptyset\) and all subsets of \(X\) whose complements have cardinality at most \(c\) and let \(\tau_2\) be a cofinite topology on \(X\). Then \((X, \tau_1, \tau_2)\) is \(\tau_2\)-Lindelöf but not \(\tau_1\)-Lindelöf and hence not Lindelöf. Observe that \(\left(X, \tau_{1,2}^s, \tau_{2,1}^s\right)\) is \(\tau_{1,2}^s\)-Lindelöf and \(\tau_{2,1}^s\)-Lindelöf since \(\tau_{1,2}^s\) and \(\tau_{2,1}^s\) are indiscrete topologies. Hence \(\left(X, \tau_{1,2}^s, \tau_{2,1}^s\right)\) is Lindelöf.

**Definition 3.2.** A bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-Lindelöf \([5, 7]\) if for every \(i\)-open cover of \(X\) there is a countable \(j\)-open subcover. \(X\) is called B-Lindelöf \([5]\) or \(p_1\)-Lindelöf \([7]\) if it is both \((1, 2)\)-Lindelöf and \((2, 1)\)-Lindelöf.

An \((i, j)\)-Lindelöf property as well as B-Lindelöf property is not pairwise semiregular property by the following example.

**Example 3.2.** Let \((X, \tau_1, \tau_2)\) be a bitopological space as in Example 3.1. Then \((X, \tau_1, \tau_2)\) is not \((\tau_1, \tau_2)\)-Lindelöf but it is \((\tau_2, \tau_1)\)-Lindelöf and hence not B-Lindelöf. Observe that \(\left(X, \tau_{1,2}^s, \tau_{2,1}^s\right)\) \(\left(\tau_{1,2}^s, \tau_{2,1}^s\right)\)-Lindelöf and \(\left(\tau_{2,1}^s, \tau_{1,2}^s\right)\)-Lindelöf since \(\tau_{1,2}^s\) and \(\tau_{2,1}^s\) are indiscrete topologies. Hence \(\left(X, \tau_{1,2}^s, \tau_{2,1}^s\right)\) is B-Lindelöf.

**Definition 3.3.** A cover \(U\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \(\tau_1\tau_2\)-open \([23]\) if \(U \subseteq \tau_1 \cup \tau_2\). If, in addition, \(U\) contains at least one nonempty member of \(\tau_1\) and at least one nonempty member of \(\tau_2\), it is called \(p\)-open \([4]\).

**Definition 3.4.** \([5]\) A bitopological space \((X, \tau_1, \tau_2)\) is called s-Lindelöf (resp. \(p\)-Lindelöf) if every \(\tau_1\tau_2\)-open (resp. \(p\)-open) cover of \(X\) has a countable subcover.

A \(p\)-Lindelöf property is not pairwise semiregular property by the following example. Thus the s-Lindelöf property is also not pairwise semiregular property.
Example 3.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space as in Example 3.1. Then \((X, \tau_1, \tau_2)\) is not \(p\)-Lindelöf and hence not \(s\)-Lindelöf. Observe that \((X, \tau_{(1,2)}, \tau^s_{(2,1)})\) is \(p\)-Lindelöf and \(s\)-Lindelöf since \(\tau^s_{(1,2)}\) and \(\tau^s_{(2,1)}\) are indiscrete topologies.

4. Pairwise Semiregularization of Generalized Pairwise Lindelöf Spaces

Definition 4.1. [9, 13, 16] A bitopological space \(X\) is said to be \((i,j)\)-nearly Lindelöf (resp. \((i,j)\)-almost Lindelöf, \((i,j)\)-weakly Lindelöf) if for every \(i\)-open cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(X\), there exists a countable subset \(\{\alpha_n : n \in \mathbb{N}\}\) of \(\Delta\) such that \(X = \bigcup_{n \in \mathbb{N}} i\text{-int}\,(j\text{-cl}(U_{\alpha_n}))\) (resp. \(X = \bigcup_{n \in \mathbb{N}} j\text{-cl}\,(U_{\alpha_n})\), \(X = j\text{-cl}\,\left(\bigcup_{n \in \mathbb{N}} (U_{\alpha_n})\right)\)). \(X\) is called pairwise nearly Lindelöf (resp. pairwise almost Lindelöf, pairwise weakly Lindelöf) if it is both (1,2)-nearly Lindelöf (resp. (1,2)-almost Lindelöf, (1,2)-weakly Lindelöf) and (2,1)-nearly Lindelöf (resp. (2,1)-almost Lindelöf, (2,1)-weakly Lindelöf).

Our first result is analogue with the result of Mršević et al. [15, Theorem 1].

Theorem 4.1. A bitopological space \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-nearly Lindelöf if and only if \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is \(\tau^s_{(i,j)}\)-Lindelöf.

Proof. Let \((X, \tau_1, \tau_2)\) be a \((\tau_i, \tau_j)\)-nearly Lindelöf and let \(\{U_\alpha : \alpha \in \Delta\}\) be a \(\tau^s_{(i,j)}\)-open cover of \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\). For each \(x \in X\), there exists \(\alpha_x \in \Delta\) such that \(x \in U_{\alpha_x}\) and since for each \(\alpha_x \in \Delta\), \(U_{\alpha_x} \in \tau^s_{(i,j)}\), there exists a \((\tau_i, \tau_j)\)-regular open set \(V_{\alpha_x}\) in \((X, \tau_1, \tau_2)\) such that \(x \in V_{\alpha_x} \subseteq U_{\alpha_x}\). So \(X = \bigcup_{x \in X} V_{\alpha_x}\) and hence \(\{V_{\alpha_x} : x \in X\}\) is a \((\tau_i, \tau_j)\)-regular open cover of \(X\). Since \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-nearly Lindelöf, there exists a countable subset of points \(x_1, \ldots, x_n, \ldots\) of \(X\) such that \(X = \bigcup_{n \in \mathbb{N}} V_{\alpha_{x_n}} \subseteq \bigcup_{n \in \mathbb{N}} U_{\alpha_{x_n}}\). This shows that \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is \(\tau^s_{(i,j)}\)-Lindelöf.

Conversely, suppose that \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is \(\tau^s_{(i,j)}\)-Lindelöf and let \(\{V_\alpha : \alpha \in \Delta\}\) be a \((\tau_i, \tau_j)\)-regular open cover of \((X, \tau_1, \tau_2)\). Since \(V_\alpha \in \tau^s_{(i,j)}\) for each \(\alpha \in \Delta\), \(\{V_\alpha : \alpha \in \Delta\}\) is a \(\tau^s_{(i,j)}\)-open cover of \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\). Since \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is \(\tau^s_{(i,j)}\)-Lindelöf, there exists a countable subcover such that \(X = \bigcup_{n \in \mathbb{N}} V_{\alpha_n}\). This implies that \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-nearly Lindelöf. \(\Box\)

Corollary 4.1. A bitopological space \((X, \tau_1, \tau_2)\) is pairwise nearly Lindelöf if and only if \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is Lindelöf.

Proposition 4.1. A bitopological space \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is \((\tau^s_{(i,j)}, \tau^s_{(i,j)})\)-nearly Lindelöf if and only if \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is \(\tau^s_{(i,j)}\)-Lindelöf.

Proof. The sufficient condition is obvious by the definitions. So we need only to prove necessary condition. Suppose that \(\{U_\alpha : \alpha \in \Delta\}\) is a \(\tau^s_{(i,j)}\)-open cover of \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\). For each \(x \in X\), there exists \(\alpha_x \in \Delta\) such that \(x \in U_{\alpha_x}\). Since \((X, \tau^s_{(1,2)}, \tau^s_{(2,1)})\) is \(\tau^s_{(i,j)}\)-semiregular, there exists a...
Proof. By Theorem 4.3, let \( \text{Proposition 4.2.} \)

Theorem 4.3. If \((X, \tau_{(1,2)}, \tau_{(2,1)})\) prove in the following theorems.

Corollary 4.2. A bitopological space \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is pairwise nearly Lindelöf if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is Lindelöf.

From the Definition 2.6, if the property \(P\) is not bitopological property but it satisfies the condition \((X, \tau_1, \tau_2)\) has the property \(P\) if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) has the property \(P\), then the property \(P\) will be called pairwise semiregular invariant property. The following theorem prove that \((i,j)\)-nearly Lindelöf as well as pairwise nearly Lindelöf property satisfying the pairwise semiregular invariant property since \((i,j)\)-nearly Lindelöf and pairwise nearly Lindelöf are not \(i\)-topological property [8] and bitopological property, respectively. This is because the \(i\)-continuity and \((i,j)\)-\(\delta\)-continuity (resp. continuity and \(p\)-\(\delta\)-continuity) are independent notions (see [12]).

Theorem 4.2. A bitopological space \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \((\tau_i, \tau_j)\)-nearly Lindelöf if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \((\tau_{(i,j)}, \tau_{(i,j)})\)-nearly Lindelöf.

Proof. It is obvious by Theorem 4.1 and Proposition 4.1. \(\square\)

Corollary 4.3. A bitopological space \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is pairwise nearly Lindelöf if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is pairwise nearly Lindelöf.

Theorem 4.3. [20] If \((X, \tau_1, \tau_2)\) is pairwise semiregular, then \((X, \tau_1, \tau_2) = (X, \tau_{(1,2)}, \tau_{(2,1)})\).

The converse of Theorem 4.3 is also true by the definitions.

Proposition 4.2. Let \((X, \tau_1, \tau_2)\) be a pairwise semiregular space. Then \((X, \tau_1, \tau_2)\) is \((i,j)\)-nearly Lindelöf if and only if it is \(i\)-Lindelöf.

Proof. By Theorem 4.3, \((X, \tau_1, \tau_2) = (X, \tau_{(1,2)}, \tau_{(2,1)})\). The result follows immediately by Proposition 4.1. \(\square\)

Corollary 4.4. Let \((X, \tau_1, \tau_2)\) be a pairwise semiregular space. Then \((X, \tau_1, \tau_2)\) is pairwise nearly Lindelöf if and only if it is Lindelöf.

Unlike all types of pairwise Lindelöf properties, the \((i,j)\)-almost Lindelöf, pairwise almost Lindelöf, \((i,j)\)-weakly Lindelöf and pairwise weakly Lindelöf properties are pairwise semiregular properties as we prove in the following theorems.
Theorem 4.4. A bitopological space \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-almost Lindelöf if and only if \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\) is \((\tau_{(i,j)}, \tau_{(i,j)})\)-almost Lindelöf.

Proof. Let \((X, \tau_1, \tau_2)\) be a \((\tau_i, \tau_j)\)-almost Lindelöf and let \(\{U_\alpha : \alpha \in \Delta\}\) be a \(\tau_{(i,j)}\)-open cover of \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\). Since \(\tau_{(i,j)} \subseteq \tau_i\), \(\{U_\alpha : \alpha \in \Delta\}\) is a \(\tau_i\)-open cover of the \((\tau_i, \tau_j)\)-almost Lindelöf space \((X, \tau_1, \tau_2)\). Then there is a countable subset \(\{\alpha_n : n \in \mathbb{N}\}\) of \(\Delta\) such that \(X = \bigcup_{n \in \mathbb{N}} \tau_j\)-cl \((U_{\alpha_n})\). By Lemma 2.1, we have \(X = \bigcup_{n \in \mathbb{N}} \tau_{(i,j)}\)-cl \((U_{\alpha_n})\), which implies \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\) is \((\tau_{(i,j)}, \tau_{(i,j)})\)-almost Lindelöf.

Conversely suppose that \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\) is \((\tau_{(i,j)}, \tau_{(i,j)})\)-almost Lindelöf and let \(\{V_\alpha : \alpha \in \Delta\}\) be a \(\tau_i\)-open cover of \((X, \tau_1, \tau_2)\). Since \(V_\alpha \subseteq \tau_i\)-int \((\tau_j\text{-cl} \,(V_\alpha))\) and \(\tau_i\)-int \((\tau_j\text{-cl} \,(V_\alpha))\) \(\in \tau_{(i,j)}\), we have \(\{\tau_i\text{-int} \,(\tau_j\text{-cl} \,(V_\alpha)) : \alpha \in \Delta\}\) is a \(\tau_{(i,j)}\)-open cover of the \((\tau_{(i,j)}, \tau_{(i,j)})\)-almost Lindelöf space \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\). So there is a countable subset \(\{\alpha_n : n \in \mathbb{N}\}\) of \(\Delta\) such that \(X = \bigcup_{n \in \mathbb{N}} \tau_{(i,j)}\text{-cl} \,(\tau_i\text{-int} \,(\tau_j\text{-cl} \,(V_{\alpha_n})))\). By Lemma 2.1, we have \(X = \bigcup_{n \in \mathbb{N}} \tau_j\text{-cl} \,(\tau_i\text{-int} \,(\tau_j\text{-cl} \,(V_{\alpha_n})))\) \(\subseteq \bigcup_{n \in \mathbb{N}} \tau_{(i,j)}\text{-cl} \,(\tau_j\text{-cl} \,(V_{\alpha_n}))\). This implies that \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-almost Lindelöf.

Corollary 4.5. A bitopological space \((X, \tau_1, \tau_2)\) is pairwise almost Lindelöf if and only if \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\) is pairwise almost Lindelöf.

Note that, the \((i,j)\)-almost Lindelöf property and the pairwise almost Lindelöf property are both bitopological properties (see [18]). Utilizing this fact, Theorem 4.4 and Corollary 4.5, we easily obtain the following corollary.

Corollary 4.6. The \((i,j)\)-almost Lindelöf property and the pairwise almost Lindelöf property are both pairwise semiregular properties.

Proposition 4.3. Let \((X, \tau_1, \tau_2)\) be a \((\tau_i, \tau_j)\)-almost regular space. Then \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-almost Lindelöf if and only if \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\) is \(\tau_{(i,j)}\)-Lindelöf.

Proof. Let \((X, \tau_1, \tau_2)\) be a \((\tau_i, \tau_j)\)-almost Lindelöf and let \(\{U_\alpha : \alpha \in \Delta\}\) be a \(\tau_{(i,j)}\)-open cover of \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\). For each \(x \in X\), there exists \(\alpha_x \in \Delta\) such that \(x \in U_{\alpha_x}\) and since \(U_{\alpha_x} \subseteq \tau_{(i,j)}\), there exists a \((\tau_i, \tau_j)\)-regular open set \(V_{\alpha_x}\) in \((X, \tau_1, \tau_2)\) such that \(x \in V_{\alpha_x} \subseteq U_{\alpha_x}\). Since \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-almost regular, there is a \((\tau_i, \tau_j)\)-regular open set \(C_{\alpha_x}\) in \((X, \tau_1, \tau_2)\) such that \(x \in C_{\alpha_x} \subseteq \tau_j\text{-cl} \,(C_{\alpha_x}) \subseteq V_{\alpha_x}\). Hence \(X = \bigcup_{x \in X} C_{\alpha_x}\) and thus the family \(\{C_{\alpha_x} : x \in X\}\) forms a \((\tau_i, \tau_j)\)-regular open cover of \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-almost Lindelöf, there exists a countable subset of points \(x_1, \ldots, x_n, \ldots\) of \(X\) such that \(X = \bigcup_{n \in \mathbb{N}} \tau_j\text{-cl} \,(C_{\alpha_{x_n}}) \subseteq \bigcup_{n \in \mathbb{N}} V_{\alpha_{x_n}} \subseteq \bigcup_{n \in \mathbb{N}} U_{\alpha_{x_n}}\). This shows that \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\) is \(\tau_{(i,j)}\)-Lindelöf. Conversely, let \(\left(\tau_{(1,2)}, \tau_{(2,1)}\right)\) be a \(\tau_{(i,j)}\)-Lindelöf and let \(\{U_\alpha : \alpha \in \Delta\}\) be a \(\tau_i\)-open cover of \((X, \tau_1, \tau_2)\). Since \(U_\alpha \subseteq \tau_i\text{-int} \,(\tau_j\text{-cl} \,(U_\alpha))\) and \(\tau_i\text{-int} \,(\tau_j\text{-cl} \,(U_\alpha)) \in \tau_{(i,j)}\), \(\tau_i\text{-int} \,(\tau_j\text{-cl} \,(U_\alpha)) : \alpha \in \Delta\) is \(\tau_{(i,j)}\)-open cover of the \(\tau_{(i,j)}\)-Lindelöf space.
Corollary 4.7. Let \((X, \tau_1, \tau_2)\) be a pairwise almost regular space. Then \((X, \tau_1, \tau_2)\) is pairwise almost Lindelöf if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is Lindelöf.

Proposition 4.4. Let \((X, \tau_{(1,2)}, \tau_{(2,1)})\) be a \((\tau_{(i,j)}, \tau_{(j,i)})\)-extremally disconnected space. Then \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \(\tau_{(i,j)}\)-almost Lindelöf if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \(\tau_{(i,j)}\)-Lindelöf.

Proof. The sufficient condition is obvious by the definitions. So we need only to prove necessary condition. Suppose that \(\{U_\alpha : \alpha \in \Delta\}\) is a \(\tau_{(i,j)}\)-open cover of \((X, \tau_{(1,2)}, \tau_{(2,1)})\). For each \(x \in X\), there exists \(\alpha_x \in \Delta\) such that \(x \in U_{\alpha_x}\). Since \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \(\tau_{(i,j)}\)-semiregular, there exists a \(\tau_{(i,j)}\)-open set \(V_{\alpha_x}\) in \((X, \tau_{(1,2)}, \tau_{(2,1)})\) such that \(x \in V_{\alpha_x} \subseteq \tau_{(i,j)}\)-int \((\tau_{(i,j)}\)-cl \((V_{\alpha_x})\)) \subseteq \(U_{\alpha_x}\). Hence \(X = \bigcup_{x \in X} V_{\alpha_x}\) and thus the family \(\{V_{\alpha_x} : x \in X\}\) forms a \(\tau_{(i,j)}\)-open cover of \((X, \tau_{(1,2)}, \tau_{(2,1)})\). Since \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \(\tau_{(i,j)}\)-almost Lindelöf and \(\tau_{(i,j)}, \tau_{(j,i)}\)-extremally disconnected, there exists a countable subset of points \(x_1, \ldots, x_n, \ldots\) of \(X\) such that \(X = \bigcup_{n \in \mathbb{N}} \tau_{(i,j)}\)-cl \((V_{\alpha_m})\) = \(\bigcup_{n \in \mathbb{N}} \tau_{(i,j)}\)-int \((\tau_{(i,j)}\)-cl \((V_{\alpha_m})\)) \subseteq \bigcup_{n \in \mathbb{N}} U_{\alpha_m}\). This shows that \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \(\tau_{(i,j)}\)-Lindelöf. 

Corollary 4.8. Let \((X, \tau_{(1,2)}, \tau_{(2,1)})\) be a pairwise extremally disconnected space. Then \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is pairwise almost Lindelöf if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is Lindelöf.

Proposition 4.5. Let \((X, \tau_1, \tau_2)\) be a pairwise semiregular and \((j, i)\)-extremally disconnected space. Then \((X, \tau_1, \tau_2)\) is \((i, j)\)-almost Lindelöf if and only if it is \(i\)-Lindelöf.

Proof. By Theorem 4.3, \((X, \tau_1, \tau_2) = (X, \tau_{(1,2)}, \tau_{(2,1)})\). The result follows immediately by Proposition 4.4. 

Corollary 4.9. Let \((X, \tau_1, \tau_2)\) be a pairwise semiregular and pairwise extremally disconnected space. Then \((X, \tau_1, \tau_2)\) is pairwise almost Lindelöf if and only if it is Lindelöf.

Theorem 4.5. A bitopological space \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)\)-weakly Lindelöf if and only if \((X, \tau_{(1,2)}, \tau_{(2,1)})\) is \(\tau_{(i,j)}\)-weakly Lindelöf.

Proof. The proof is similar to the proof of Theorem 4.4 by using the fact that

\[
\tau_{(j,i)}\text{-cl} \left(\bigcup_{n \in \mathbb{N}} \tau_{(j)}\text{-int} \left(\tau_{(j)}\text{-cl} \left(V_{\alpha_n}\right)\right)\right) = \tau_{j}\text{-cl} \left(\bigcup_{n \in \mathbb{N}} \tau_{(j)}\text{-int} \left(\tau_{(j)}\text{-cl} \left(V_{\alpha_n}\right)\right)\right) \subseteq \tau_{j}\text{-cl} \left(\bigcup_{n \in \mathbb{N}} V_{\alpha_n}\right).
\]
Corollary 4.10. A bitopological space $(X, \tau_1, \tau_2)$ is pairwise weakly Lindelöf if and only if $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$ is pairwise weakly Lindelöf.

Note that, the $(i, j)$-weakly Lindelöf property and the pairwise weakly Lindelöf property are both bitopological properties (see [18]). Utilizing this fact, Theorem 4.5 and Corollary 4.10, we easily obtain the following corollary.

Corollary 4.11. The $(i, j)$-weakly Lindelöf property and the pairwise weakly Lindelöf property are both pairwise semiregular properties.

Recall that, a bitopological space $X$ is called $(i, j)$-weak $P$-space [13] if for each countable family $\{U_n : n \in \mathbb{N}\}$ of $i$-open sets in $X$, we have $j \text{-cl} \left( \bigcup_{n \in \mathbb{N}} U_n \right) = \bigcup_{n \in \mathbb{N}} j \text{-cl} (U_n)$. $X$ is called pairwise weak $P$-space if it is both $(1, 2)$-weak $P$-space and $(2, 1)$-weak $P$-space.

Proposition 4.6. Let $(X, \tau_1, \tau_2)$ be a $(\tau_i, \tau_j)$-almost regular and $(\tau_i, \tau_j)$-weak $P$-space. Then $(X, \tau_1, \tau_2)$ is $(\tau_i, \tau_j)$-weakly Lindelöf if and only if $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$ is $\tau_{(i,j)}^s$-Lindelöf.

Proof. Necessity: Let $\{U_\alpha : \alpha \in \Delta\}$ be a $\tau_{(i,j)}^s$-open cover of $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x}$ and since $U_{\alpha_x} \in \tau_{(i,j)}^s$, there exists a $(\tau_i, \tau_j)$-regular open set $V_{\alpha_x}$ in $(X, \tau_1, \tau_2)$ such that $x \in V_{\alpha_x} \subseteq U_{\alpha_x}$. Since $(X, \tau_1, \tau_2)$ is $(\tau_i, \tau_j)$-almost regular, there is a $(\tau_i, \tau_j)$-regular open set $C_{\alpha_x}$ in $(X, \tau_1, \tau_2)$ such that $x \in C_{\alpha_x} \subseteq \tau_j \text{-cl}(C_{\alpha_x}) \subseteq V_{\alpha_x}$. Hence $X = \bigcup_{x \in X} C_{\alpha_x}$ and thus the family $\{C_{\alpha_x} : x \in X\}$ forms a $(\tau_i, \tau_j)$-regular open cover of $(X, \tau_1, \tau_2)$. Since $(X, \tau_1, \tau_2)$ is $(\tau_i, \tau_j)$-weakly Lindelöf and $(\tau_i, \tau_j)$-weak $P$-space, there exists a countable subset of points $x_1, \ldots, x_n, \ldots$ of $X$ such that $X = \tau_j \text{-cl}\left( \bigcup_{n \in \mathbb{N}} C_{\alpha_{x_n}} \right) = \bigcup_{n \in \mathbb{N}} \tau_j \text{-cl}(C_{\alpha_{x_n}}) \subseteq \bigcup_{n \in \mathbb{N}} U_{\alpha_{x_n}}$. This shows that $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$ is $\tau_{(i,j)}^s$-Lindelöf.

Sufficiency: Let $\{U_\alpha : \alpha \in \Delta\}$ be a $\tau_j$-open cover of $(X, \tau_1, \tau_2)$. Since $U_\alpha \subseteq \tau_i \text{-int}(\tau_j \text{-cl}(U_\alpha))$ and $\tau_i \text{-int}(\tau_j \text{-cl}(U_\alpha)) \in \tau_{(i,j)}^s$, $\{\tau_i \text{-int}(\tau_j \text{-cl}(U_\alpha)) : \alpha \in \Delta\}$ is $\tau_{(i,j)}^s$-open cover of the $\tau_{(i,j)}^s$-Lindelöf space $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$. Then there exists a countable subset $\{C_{\alpha_n} : n \in \mathbb{N}\}$ of $\Delta$ such that $X = \bigcup_{n \in \mathbb{N}} \tau_i \text{-int}(\tau_j \text{-cl}(U_{\alpha_n})) \subseteq \bigcup_{n \in \mathbb{N}} \tau_j \text{-cl}(U_{\alpha_n}) = \tau_j \text{-cl}\left( \bigcup_{n \in \mathbb{N}} U_{\alpha_n} \right)$. This implies that $(X, \tau_1, \tau_2)$ is $(\tau_i, \tau_j)$-weakly Lindelöf.

Corollary 4.12. Let $(X, \tau_1, \tau_2)$ be a pairwise almost regular and pairwise weak $P$-space. Then $(X, \tau_1, \tau_2)$ is pairwise weakly Lindelöf if and only if $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$ is Lindelöf.

Proposition 4.7. Let $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$ be a $(\tau_{(i,j)}, \tau_{(i,j)}^s)$-extremally disconnected and $(\tau_{(i,j)}^s, \tau_{(i,j)}^s)$-weak $P$-space. Then $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$ is $(\tau_{(i,j)}^s, \tau_{(i,j)}^s)$-weakly Lindelöf if and only if $(X, \tau_{(1,2)}^s, \tau_{(2,1)}^s)$ is $\tau_{(i,j)}^s$-Lindelöf.
Proof. The sufficient condition is obvious by the definitions. So we need only to prove necessary condition. Suppose that \( \{U_\alpha : \alpha \in \Delta \} \) is a \( \tau^s_{(i,j)} \)-open cover of \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \). For each \( x \in X \), there exists \( \alpha_x \in \Delta \) such that \( x \in U_{\alpha_x} \). Since \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \) is \( (\tau^s_{(i,j)}, \tau^s_{(j,i)}) \)-semiregular, there exists a \( \tau^s_{(i,j)} \)-open set \( V_{\alpha_x} \) in \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \) such that \( x \in V_{\alpha_x} \subseteq \tau^s_{(i,j)}\)-int \( (\tau^s_{(j,i)}\-cl (V_{\alpha_x})) \subseteq U_{\alpha_x} \). Hence \( X = \bigcup_{x \in X} V_{\alpha_x} \) and thus the family \( \{V_{\alpha_x} : x \in X \} \) forms a \( \tau^s_{(i,j)} \)-open cover of \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \).

Since \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \) is \( (\tau^s_{(i,j)}, \tau^s_{(j,i)}) \)-weakly Lindelöf, \( (\tau^s_{(i,j)}, \tau^s_{(j,i)}) \)-extremely disconnected and \( (\tau^s_{(i,j)}, \tau^s_{(j,i)}) \)-weak \( P \)-space, there exists a countable subset of points \( x_1, \ldots, x_n, \ldots \) of \( X \) such that \( X = \tau^s_{(i,j)}\-cl \left( \bigcup_{n \in \mathbb{N}} V_{\alpha_{x_n}} \right) = \bigcup_{n \in \mathbb{N}} \tau^s_{(j,i)}\-cl (V_{\alpha_{x_n}}) = \bigcup_{n \in \mathbb{N}} \tau^s_{(j,i)}\-int \left( \tau^s_{(j,i)}\-cl (V_{\alpha_{x_n}}) \right) \subseteq \bigcup_{n \in \mathbb{N}} U_{\alpha_{x_n}} \). This shows that \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \) is \( \tau^s_{(i,j)} \)-Lindelöf.

Corollary 4.13. Let \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \) be a pairwise extremally disconnected and pairwise weak \( P \)-space. Then \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \) is pairwise weakly Lindelöf if and only if \( (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \) is Lindelöf.

Proposition 4.8. Let \( (X, \tau_1, \tau_2) \) be a pairwise semiregular, \( (j, i) \)-extremally disconnected and \( (i, j) \)-weak \( P \)-space. Then \( (X, \tau_1, \tau_2) \) is \( (i, j) \)-weak Lindelöf if and only if it is \( i \)-Lindelöf.

Proof. By Theorem 4.3, \( (X, \tau_1, \tau_2) = (X, \tau^s_{(1,2)}, \tau^s_{(2,1)}) \). The result follows immediately by Proposition 4.7.

Corollary 4.14. Let \( (X, \tau_1, \tau_2) \) be a pairwise semiregular, pairwise extremally disconnected and pairwise weak \( P \)-space. Then \( (X, \tau_1, \tau_2) \) is pairwise weakly Lindelöf if and only if it is Lindelöf.

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