Quasi-Ideals and Bi-Ideals of Near Left Almost Rings

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Abstract. In this paper, we define quasi-ideal, bi-ideal, and weak bi-ideal of nLA-ring, and investigate it properties.

1. Introduction

M.A. Kazim and MD. Naseeruddin defined LA-semigroup as the following; a groupoid $S$ is called a left almost semigroup, abbreviated as LA-semigroup if

$$(ab)c = (cb)a, \quad \forall a, b, c \in S$$

M.A. Kazim and MD. Naseeruddin [2] asserted that, in every LA-semigroups $G$ a medial law hold

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall a, b, c, d \in G.$$

Q. Mushtaq and M. Khan [4] asserted that, in every LA-semigroups $G$ with left identity

$$(a \cdot b) \cdot (c \cdot d) = (d \cdot b) \cdot (c \cdot a), \quad \forall a, b, c, d \in G.$$

Further M. Khan, Faisal, and V. Amjid [3], asserted that, if an LA-semigroup $G$ with left identity the following law holds

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall a, b, c \in G.$$ 

M. Sarwar (Kamran) [6] defined LA-group as the following; a groupoid $G$ is called a left almost group, abbreviated as LA-group, if (i) there exists $e \in G$ such that $ea = a$ for all $a \in G$, (ii) for every $a \in G$ there exists $a' \in G$ such that, $a' a = e$, (iii) $(ab)c = (cb)a$ for every $a, b, c \in G$. 

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A non-empty subset $A$ of an LA-group $G$ is called an LA-subgroup of $G$ if $A$ is itself an LA-group under the same operation as defined in $G$.

S.M. Yusuf in [8] introduced the concept of a left almost ring (LA-ring). That is, a non-empty set $R$ with two binary operations “$+$” and “$\cdot$” is called a left almost ring, if $\langle R, + \rangle$ is an LA-group, $\langle R, \cdot \rangle$ is an LA-semigroup and distributive laws of “$\cdot$” over “$+$” holds. T. Shah and I. Rehman [8] asserted that a commutative ring $\langle R, +, \cdot \rangle$, we can always obtain an LA-ring $\langle R, \oplus, \cdot \rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $\langle R, +, \cdot \rangle$ is said to be an LA-integral domain if for all $a, b \in R$, with $a \cdot b = 0$, then $a = 0$ or $b = 0$. Let $\langle R, +, \cdot \rangle$ be an LA-ring and $S$ be a non-empty subset of $R$ and $S$ is itself and LA-ring under the binary operation induced by $R$, then $S$ is called an LA-subring of $\langle R, +, \cdot \rangle$. If $S$ is an LA-subring of an LA-ring $\langle R, +, \cdot \rangle$, then $S$ is called a left ideal of $R$ if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.

By [5] a near-ring is a non-empty set $N$ together with two binary operations “$+$” and “$\cdot$” such that $\langle N, + \rangle$ is a group (not necessarily abelian), $\langle N, \cdot \rangle$ is a semigroup and one sided distributive (left or right) of “$\cdot$” over “$+$” holds.

By [1] If a subgroup $Q$ of $\langle N, + \rangle$ has the property $QN \cap NQ \subseteq Q$, then it is called a quasi-ideal of $N$.

By [9] If a subgroup $B$ of $\langle N, + \rangle$ is said to be a bi-ideal of $N$ if $BNB \cap (BN) \ast B \subseteq B$. If $N$ has a zero symmetric near-ring a subgroup $B$ of $\langle N, + \rangle$ is a bi-ideal if and only if $BNB \subseteq B$.

By [10] a subgroup $B$ of $\langle N, + \rangle$ is said to be a weak bi-ideal of $N$ if $B^3 \subseteq B$. In this paper we will define bi-ideal of near-ring has a zero symmetric.

2. Near Left Almost Rings


Definition 2.1. [7]. A non-empty set $N$ with two binary operation “$+$” and “$\cdot$” is called a near left almost ring (or simply an nLA-ring) if and only if

1. $\langle N, + \rangle$ is an LA-group.
2. $\langle N, \cdot \rangle$ is an LA-semigroup.
3. Left distributive property of $\cdot$ over $+$ holds, that is $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$.

Definition 2.2. [7]. An nLA-ring $\langle N, + \rangle$ with left identity $1$, such that $1 \cdot a = a$ for all $a \in N$, is called an nLA-ring with left identity.

Definition 2.3. [7]. A non-empty subset $S$ of an nLA-ring $N$ is said to be an nLA-subring if and only if $S$ is itself an nLA-ring under the same binary operations as in $N$. 

Definition 2.4. [7]. An $n$LA-subring $I$ of an nLA-ring $N$ is called a left ideal of $N$ if $NI \subseteq I$, and $I$ is called a right ideal if for all $n, m \in N$ and $i \in I$ such that $(i + n)m - nm \in I$, and is called two sided ideal or simply ideal if it is both left and right ideal.

Definition 2.5. [7]. Let $\langle N, +, \cdot \rangle$ be an nLA-ring. A non-zero element $a$ of $N$ is called a left zero divisor if there exists $0 \neq b \in N$ such that $a \cdot b = 0$. Similarly $a$ is a right zero divisor if $b \cdot a = 0$. If $a$ is both a left and a right zero divisor, then $a$ is called a zero divisor.

Definition 2.6. [7]. An nLA-ring $\langle D, +, \cdot \rangle$ with left identity 1, is called an nLA-ring integral domain if it has no left zero divisor.

Definition 2.7. [7]. An nLA-ring $\langle F, +, \cdot \rangle$ with left identity 1, is called a near almost field (n-almost field) if and only if each non-zero element of $F$ has inverse under “$\cdot$”.

3. Quasi-ideals of Near Left Almost rings

Definition 3.1. If an LA-subgroup $Q$ of $\langle N, + \rangle$ has the property $QN \cap NQ \subseteq Q$, then it is called a quasi-ideal of $N$.

Lemma 3.1. Let $N$ be a nLA-ring and $Q_1, Q_2$ are quasi-ideals of $N$. Then $Q_1 \cap Q_2$ is a quasi-ideal of $N$.

Proof. Since $Q_1, Q_2$ are LA-subgroups of $\langle N, + \rangle$ we have $Q_1 \cap Q_2$ is a LA-subgroup of $\langle N, + \rangle$. We must show that $(Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) \subseteq Q_1 \cap Q_2$. Then

\[
(Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) \subseteq Q_1 N \cap Q_2 N \cap NQ_1 \cap NQ_2 \\
= (Q_1 N \cap NQ_1) \cap (Q_2 N \cap NQ_2) \\
\subseteq Q_1 \cap Q_2.
\]

Thus $Q_1 \cap Q_2$ is a quasi-ideal of $N$. \hfill \Box

Theorem 3.1. Each quasi-ideal of an nLA-ring $N$ is an nLA-subring.

Proof. Let $Q$ be a quasi-ideal an nLA-ring $N$. Then $Q$ is a nLA-subring of $\langle N, + \rangle$. Let $a, b \in Q \subseteq N$. Then $ab \in NQ \subseteq NQ$ and $ab \in QN \subseteq QN$. Thus $ab \in NQ \cap QN \subseteq Q$, since $Q$ is a quasi-ideal of $N$. Hence $ab \in Q$. Therefore $Q$ is a nLA-subring of $N$. \hfill \Box

Theorem 3.2. The set of all quasi-ideal of nLA-ring.

Proof. Let $\{Q_i\}_{i \in I}$ be a set of quasi-ideal in $N$ and $Q = \cap_{i \in I} Q_i$. Then

\[
QN \cap NQ \subseteq \bigcap_{i \in I} Q_i N \cap N \bigcap_{i \in I} Q_i \subseteq Q_i
\]

for every $i \in I$. Thus $Q$ is a quasi-ideal of $N$. \hfill \Box
4. Bi-ideals and Weak Bi-ideals of Near Left Almost Rings

Next we defined of a bi-ideal and weak bi-ideal in nLA-ring is defines the same as a bi-ideal and weak bi-ideal in near-ring in [9] and [10].

**Definition 4.1.** Let $N$ be an nLA-ring. An LA-subgroup $B$ of $\langle N,+ \rangle$ is a bi-ideal if $(BN)B \subseteq B$.

**Theorem 4.1.** If $B$ be a bi-ideal of a nLA-ring $N$ and $S$ is an nLA-subring of $N$. Then $B \cap S$ is a bi-ideal of $S$.

**Proof.** Since $B$ is a bi-ideal of $N$ we have $(BN)B \subseteq B$. Assume that $C := B \cap S$. Then $(CS)C \subseteq (SS)S \subseteq S$, since $S$ is a nLA-subring of $N$ and $C \subseteq S$.

On the other hand $(CS)C \subseteq (BS)B \subseteq (BN)B \subseteq B$. Hence $(CS)C \subseteq B \cap S = C$. Therefore $C$ is a bi-ideal of $S$.

**Theorem 4.2.** Let $N$ be an nLA-ring and $A, B$ be bi-ideals of an nLA-ring $N$. Then $A \cap B$ is a bi-ideal of $N$.

**Proof.** Since $A, B$ is bi-ideals of an nLA-ring $N$, we have $A \cap B$ is an LA-subgroup of $\langle N,+ \rangle$. Thus $[(A \cap B)N](A \cap B) \subseteq (AN)(A \cap B) = [(A \cap B)N]A \subseteq (AN)A \subseteq A$ and $[(A \cap B)N](A \cap B) \subseteq (BN)(A \cap B) = [(A \cap B)N]B \subseteq (BN)B \subseteq B$. It following that $A \cap B$ is a bi-ideal of $N$.

**Theorem 4.3.** The set of all bi-ideal of nLA-ring.

**Proof.** Let $\{B_i\}_{i \in I}$ be a set of bi-ideal in $N$ and $B := \cap_{i \in I} B_i$. Then $(BN)B \subseteq \cap_{i \in I} B_i \subseteq B_i$ for every $i \in I$. Thus $B$ is a bi-ideal of $N$.

**Definition 4.2.** Let $N$ be an nLA-ring. An element $d$ of $N$ is called distributive if $(n + n')d = nd + n'd$ for all $n, n' \in N$.

**Theorem 4.4.** Let $N$ be an nLA-ring. If $B$ is a bi-ideal of $N$ then $Bn$ and $n'B$ are bi-ideals of $N$ where $n, n' \in N$ and $n'$ is a distributive element in $N$.

**Proof.** Since $B$ is a bi-ideal we have $Bn$ and $n'B$ are LA-subgroup $\langle N,+ \rangle$. Thus $((Bn)N)(Bn) \subseteq (BN)(Bn) = ((BN)B)n \subseteq Bn$.

Hence $Bn$ is a bi-ideal of $N$.

Again $((n'B)N)(n'B) \subseteq ((n'B)N)B = (n'BN)B \subseteq n'B$.

Thus $n'B$ are bi-ideal of $N$.

**Corollary 4.1.** Let $B$ be a bi-ideal of nLA-ring. For $b, c \in B$, if $b$ is a distributive element in $N$, then $bBc$ is a bi-ideal of $N$. 

Proof. Let $B$ be a bi-ideal of nLA-ring and $b$ is a distributive element in $N$. Then $(n + n')b = nb + n'b$ for all $n, n' \in N$. Since $B$ is a bi-ideal we have $bBc$ is an LA-subgroup $\langle N, + \rangle$ then $((bBc)N)(bBc) \subseteq (BN)B \subseteq B$.

**Definition 4.3.** An nLA-ring $N$ is said to be $B$-simple if it has no proper bi-ideals.

**Theorem 4.5.** Let $N$ be an nLA-ring with more than one element. If $N$ is a near almost field. Then $N$ is a $B$-simple.

Proof. Let $N$ be a near almost field then $\{0\}$ and $N$ are the only bi-ideals of $N$. For if $0 \neq B$ is a bi-ideal of $N$, then for $0 \neq b \in B$ we get $N = Nb$ and $N = bN$.

Now $N = N^2 = (bN)(Nb) \subseteq bNb \subseteq B$, since $B$ is a bi-ideal of $N$. Then $N = B$. Thus $N$ is a $B$-simple. □

The following we defined weak bi-ideal and study properties it.

**Definition 4.4.** An LA-subgroup $B$ of $\langle N, + \rangle$ is said to be a weak bi-ideal of $N$ if $B^3 \subseteq B$.

**Theorem 4.6.** Every bi-ideal of an nLA-ring is a weak bi-ideal.

Proof. Since $B^3 = (BB)B \subseteq (BN)B \subseteq B$ we have every bi-ideal is a weak bi-ideal. □

**Theorem 4.7.** If $B$ is a weak bi-ideal of a nLA-ring $N$ and $S$ is a nLA-subring of $N$. Then $B \cap S$ is a weak bi-ideal of $N$.

Proof. Assume that $C := B \cap S$. Then

$$
C^3 = ((B \cap S)(B \cap S))(B \cap S) \\
= ((B \cap S)(B \cap S))B \cap ((B \cap S)(B \cap S))S \\
\subseteq (BB)B \cap SSS \\
= B^3 \cap SSS \\
\subseteq B^3 \cap SS \\
\subseteq B^3 \cap S \\
\subseteq B \cap S \\
= C.
$$

Thus $C^3 \subseteq C$. Hence $C$ is a weak bi-ideal of $N$. □

**Theorem 4.8.** Let $N$ be an nLA-ring. If $B$ is a weak bi-ideal of $N$ then $Bn$ and $n'B$ are weak bi-ideal of $N$ where $n, n' \in N$ and $n'$ is a distributive element in $N$.

Proof. Since $B$ is a weak bi-ideal we have $Bn$ and $n'B$ an LA-subgroup of $\langle N, + \rangle$. Thus

$$(Bn)^3 = (BnBn)Bn \subseteq (BB)Bn \subseteq B^3n \subseteq Bn.$$ 

Hence $Bn$ is a weak bi-ideal of $N$. 

Again
\[(n'B)^3 = (n'Bn'B)n'B \subseteq (n'Bn'Bn'B)B = n'B^3 \subseteq n'B.\]

Thus \(n'B\) is a weak bi-ideal of \(N\). \(\Box\)

**Corollary 4.2.** Let \(B\) be a weak bi-ideal of \(n\text{LA-ring}\). For \(b, c \in B\), if \(b\) is a distributive element in \(N\), then \(bBc\) is a weak bi-ideal of \(N\).

5. Conclusion

In this article, we give the concept of a quasi-ideals and bi-ideals in \(n\text{LA-rings}\). We study properties of quasi-ideals and bi-ideals. In the future we study primary and quasi-primary in \(n\text{LA-ring}\).

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