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Generalized Result on Global Existence of Weak Solutions for Parabolic Reaction-Diffusion Systems

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Abstract. In this paper, we study global existence of weak solutions for 2×2 parabolic reaction-diffusion systems with a full matrix of diffusion coefficients on a bounded domain, such as, we treat the main properties related: the positivity of the solutions and the total mass of the components are preserved with time. Moreover, we suppose that the non-linearities have critical growth with respect to the gradient. The technique used is based on compact semigroup methods and some estimates. Our objective is to show, under appropriate hypotheses, that the proposed model has a global solution with a large choice of non-linearities.

1. Introduction

The study of reaction-diffusion systems (or systems of parabolic partial differential equations) was extensively developed in the literature, see for example in [10, 13, 26, 29, 30].

The question on the existence of solution for reaction-diffusion systems have long been a subject of active research like their global existence, their positivity, and some other qualitative properties.

In the present paper, we study a mathematical model of reaction-diffusion system

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f\left(t, x, u, v, \nabla u, \nabla v\right), & \text{in } Q_T, \\ \frac{\partial v}{\partial t} - c\Delta u - d\Delta v = -f\left(t, x, u, v, \nabla u, \nabla v\right), & \text{in } Q_T, \\ u = v = 0, \text{ or } \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & \text{in } \Sigma_T, \\ u\left(0, x\right) = u_0\left(x\right), \; v\left(0, x\right) = v_0\left(x\right), & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where Ω is an open bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, $T>0$, and Δ denotes the Laplacian operator on $L^1(\Omega)$ with Dirichlet or Neumann boundary conditions, the constants a, b, c and d are supposed to be positives, $a\le d$, and $(b+c)^2 \le 4ad$ which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is positive definite, that is the eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$) of its transposed are positives.

We consider the problem (1.1) where we suppose the following hypotheses

$$\begin{cases} \left(1+\frac{a-\lambda_1}{c}\right)\left(u-\frac{a-\lambda_1}{c}v\right) f(t,x,u,v,\nabla u,\nabla v) \le 0, \\ \forall \frac{a-\lambda_2}{c}v \le u \le \frac{a-\lambda_1}{c}v, \text{ a.e. } (t,x)\in Q_T, \end{cases} \tag{1.2}$$

$$\begin{cases} \left(1+\frac{a-\lambda_1}{c}\right) f\left(t,x,\frac{a-\lambda_1}{c}v,v,\frac{a-\lambda_1}{c}s,s\right) \le 0, \\ \left(1+\frac{a-\lambda_2}{c}\right) f\left(t,x,\frac{a-\lambda_2}{c}v,v,\frac{a-\lambda_2}{c}s,s\right) \ge 0, \\ \text{for all } \forall v\ge 0,\ \forall s\in\mathbb{R}^N, \text{ a.e. } (t,x)\in Q_T, \end{cases} \tag{1.3}$$

$$\begin{cases} \frac{\lambda_1-\lambda_2}{c} f(t,x,u,v,\nabla u,\nabla v) \le L_1\left(\frac{\lambda_2-\lambda_1}{c}v+1\right) \\ \forall \frac{a-\lambda_2}{c}v \le u \le \frac{a-\lambda_1}{c}v, \text{ a.e. } (t,x)\in Q_T, \end{cases} \tag{1.4}$$

where L_1 is a positive constant.

Now, we condider the following hypotheses:

$$\begin{cases} f:]0,T[\times\Omega\times\mathbb{R}^2\times\mathbb{R}^{2N}\to\mathbb{R} \text{ is measurable}, \\ f:\mathbb{R}^2\times\mathbb{R}^{2N}\to\mathbb{R} \text{ is function locally Lipschitz continuous}. \end{cases} \tag{1.5}$$

$$\begin{aligned} -\left(1+\frac{a-\lambda_1}{c}\right) f(t,x,u,v,\nabla u,\nabla v) &\le C_1\left(\left|-u(t,x)+\frac{a-\lambda_1}{c}v(t,x)\right|\right) \\ \times\left(F_1(t,x)+\left|-\nabla u(t,x)+\frac{a-\lambda_1}{c}\nabla v(t,x)\right|^2\right. &\left.+\left|\nabla u(t,x)-\frac{a-\lambda_2}{c}\nabla v(t,x)\right|^\alpha\right), \end{aligned} \tag{1.6}$$

$$\begin{aligned} \left(1+\frac{a-\lambda_2}{c}\right) f(t,x,u,v,\nabla u,\nabla v) &\le C_2\left(\left|-u(t,x)+\frac{a-\lambda_1}{c}v(t,x)\right|,\left|u(t,x)-\frac{a-\lambda_2}{c}v(t,x)\right|\right) \\ \times\left(G_1(t,x)+\left|-\nabla u(t,x)+\frac{a-\lambda_1}{c}\nabla v(t,x)\right|^2\right. &\left.+\left|\nabla u(t,x)-\frac{a-\lambda_2}{c}\nabla v(t,x)\right|^\alpha\right), \end{aligned} \tag{1.7}$$

where $C_1 : [0,\infty) \to [0,\infty)$, $C_2 : [0,\infty)^2 \to [0,\infty)$ are non-decreasing, $F_1, G_1 \in L^1(Q_T)$ and $1\le\alpha<2$.

In the diagonal case (i.e. when $b=c=0$), Alikakos [5] established global existence and $L^\infty-$bounds of solutions for positive initial data for

$$f(u,v) = -uv^\sigma, \text{ and } 1<\sigma<\frac{n+2}{n}.$$

Masuda [23] showed that the solutions to this system exist globally for every $\sigma>1$ and converge to a constant vector as $t\to+\infty$.

Haraux and Youkana [14] have generalized the method of Masuda to non-linearities $f(u,v) = -u\Psi(v)$ satisfying

$$\lim_{v\to+\infty} \frac{[\log(1+\Psi(v))]}{v} = 0.$$

In [24, 25], Moumeni and Barrouk have obtained a global existence result of solutions for reaction-diffusion systems with a diagonal and triangular matrix of diffusion coefficents. By combining the compact semigroup methods and some L^1 estimates, we show that global solutions exist for a large class of the function f.

Recently, Kouachi and Youkana [18] have generalized the method of Haraux and Youkana to the triangular case, i.e. when $b = 0$.

In the same direction, Kouachi [17] has proved the global existence of solutions for two-component reaction-diffusion systems with a general full matrix of diffusion coefficients, non-homogeneous boundary conditions and polynomial growth conditions on non-linear terms and he obtained in [18] the global existence of solutions for the same system with homogeneous Neumann boundary conditions.

Rebiai and Benachour [28] have treated the case of a general full matrix of diffusion coefficients with homogeneous boundary conditions and non-linearities of exponential growth.

This article is a continuation of [3] where $c, b \neq 0$. In that article the calculations were relatively simple since the system can be regarded as a perturbation of the simple and trivial case where $b = c = 0$; for which non-negative solutions exist globally in time.

In the present paper, to show global existence result for reaction-diffusion system with critical growth with respect to the gradient ($m = 2$), we truncate the system (1.1) then we give suitable estimates. To that end, we show the convergence of the approximating problem by using a technique introduced by Boccardo et al. [7] and Dall'aglio and Orsina [11].

2. Existence

Multiplying second equation of (1.1) one time through by $\frac{a-\lambda_1}{c}$ and subtracting first equation of (1.1) and another time by $-\frac{a-\lambda_2}{c}$ and adding first equation of (1.1) we get,

$$\begin{cases} \frac{\partial w}{\partial t} - \lambda_1 \Delta w = F(t,x,w,z,\nabla w,\nabla z), & \text{in } Q_T, \\ \frac{\partial z}{\partial t} - \lambda_2 \Delta z = G(t,x,w,z,\nabla w,\nabla z), & \text{in } Q_T, \\ w = z = 0 \text{ or } \frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0, & \text{in } \Sigma_T, \\ w(0,x) = w_0(x) \geq 0,\ z(0,x) = z_0(x) \geq 0, & \text{in } \Omega, \end{cases} \tag{2.1}$$

where

$$w(t,x) = -u(t,x) + \frac{a-\lambda_1}{c} v(t,x),\ z(t,x) = u(t,x) - \frac{a-\lambda_2}{c} v(t,x), \tag{2.2}$$

and

$$\begin{cases} F(t,x,w,z,\nabla w,\nabla z) = -\left(1 + \frac{a-\lambda_1}{c}\right) f(t,x,u,v,\nabla u,\nabla v), \\ G(t,x,w,z,\nabla w,\nabla z) = \left(1 + \frac{a-\lambda_2}{c}\right) f(t,x,u,v,\nabla u,\nabla v). \end{cases} \tag{2.3}$$

2.1. **Assumptions.** Suppose that the hypotheses (1.2)-(1.7) are satisfied, then the problem (2.1) true the following hypotheses:

- The non-linearities F, G have critical growth with respect to $|\nabla w|, |\nabla z|$. With respect to w, z.

We assume that the hypotheses (1.2) are satisfied and we obtain,

$$\begin{cases} \left(1+\frac{a-\lambda_1}{c}\right)\left(u-\frac{a-\lambda_1}{c}v\right) f(t,x,u,v,\nabla u,\nabla v) \leq 0, \\ \forall \frac{a-\lambda_2}{c}v \leq u \leq \frac{a-\lambda_1}{c}v, \text{ a.e. } (t,x) \in Q_T, \end{cases}$$

i.e.

$$\begin{cases} -\left(1+\frac{a-\lambda_1}{c}\right)\left(-u+\frac{a-\lambda_1}{c}v\right) f(t,x,u,v,\nabla u,\nabla v) \leq 0, \\ \forall -u+\frac{a-\lambda_1}{c}v \geq 0, \ \forall u-\frac{a-\lambda_2}{c}v \geq 0, \text{ a.e. } (t,x) \in Q_T, \end{cases}$$

by (2.2)-(2.3), then F satisfies the sign condition

$$wF(t,x,w,z,\nabla w,\nabla z) \leq 0, \quad \forall w, z \geq 0, \text{ a.e. } (t,x) \in Q. \tag{2.4}$$

Moreover, the following properties hold:

$$w(t,x) = -u(t,x) + \frac{a-\lambda_1}{c}v(t,x), \ z(t,x) = u(t,x) - \frac{a-\lambda_2}{c}v(t,x), \tag{2.5}$$

- By (2.2)

$$\begin{cases} w(t,x) = 0, \quad \text{if } u(t,x) = \frac{a-\lambda_1}{c}v(t,x), \quad \text{and in the case } z(t,x) = \frac{\lambda_2-\lambda_1}{c}v(t,x), \\ z(t,x) = 0, \quad \text{if } u(t,x) = \frac{a-\lambda_2}{c}v(t,x), \quad \text{and in the case } w(t,x) = \frac{\lambda_2-\lambda_1}{c}v(t,x), \end{cases}$$

from (1.3), we get that

$$\begin{cases} -\left(1+\frac{a-\lambda_1}{c}\right) f\left(t,x,\frac{a-\lambda_1}{c}v,v,\frac{a-\lambda_1}{c}s,s\right) \geq 0, \\ \left(1+\frac{a-\lambda_2}{c}\right) f\left(t,x,\frac{a-\lambda_2}{c}v,v,\frac{a-\lambda_2}{c}s,s\right) \geq 0. \\ \text{for all } \forall v \geq 0, \ \forall s \in \mathbb{R}^N, \text{ a.e. } (t,x) \in Q_T, \end{cases}$$

$$\begin{cases} F(t,x,0,z,0,s) \geq 0, \qquad G(t,x,w,0,r,0) \geq 0, \\ \forall w,z \geq 0, \ \forall r,s \in \mathbb{R}^N, \text{ a.e. } (t,x) \in Q_T. \end{cases} \tag{2.6}$$

- From (2.3), we obtain that

$$\begin{aligned} F+G &= \left(-1-\frac{a-\lambda_1}{c}+1+\frac{a-\lambda_2}{c}\right) f(t,x,u,v,\nabla u,\nabla v) \\ &= \frac{\lambda_1-\lambda_2}{c} f(t,x,u,v,\nabla u,\nabla v) \end{aligned}$$

and (2.2) is given

$$\begin{aligned} w(t,x)+z(t,x) &= -u(t,x)+\frac{a-\lambda_1}{c}v(t,x)+u(t,x)-\frac{a-\lambda_2}{c}v(t,x) \\ &= \frac{\lambda_2-\lambda_1}{c}v, \end{aligned}$$

then by the hypotheses (1.4), we obtain that

$$F + G \leq L_1 (w + z + 1), \quad \forall w, z \geq 0, \text{ a.e. } (t, x) \in Q_T, \tag{2.7}$$

where L_1 is a positive constant.

- Let us, now by (2.3) and (1.5) introduce for F and G the hypotheses

$$F, G :]0, T[\times \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R} \text{ are measurable.} \tag{2.8}$$

$$F, G : \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R} \text{ are locally Lipschitz continuous,} \tag{2.9}$$

namely

$$|F(t, x, w, z, p, q) - F(t, x, \hat{w}, \hat{z}, \hat{p}, \hat{q})| + |G(t, x, w, z, p, q) - G(t, x, \hat{w}, \hat{z}, \hat{p}, \hat{q})|$$
$$\leq K(r) (|w - \hat{w}| + |z - \hat{z}| + \|p - \hat{p}\| + \|q - \hat{q}\|)$$

for a.e. $(t, x) \in Q_T$ and for all $0 \leq |w|, |\hat{w}|, |z|, |\hat{z}|, \|p\|, \|\hat{p}\|, \|q\|, \|\hat{q}\| \leq r$.

- By (2.2)-(2.3) and the hypotheses (1.6)-(1.7), we obtain that

$$F(t, x, w, z, \nabla w, \nabla z) \leq C_1 (|w|) \left(F_1 (t, x) + |\nabla w|^2 + |\nabla z|^{\alpha} \right), \tag{2.10}$$

$$G(t, x, w, z, \nabla w, \nabla z) \leq C_2 (|w|, |z|) \left(G_1 (t, x) + |\nabla w|^2 + |\nabla z|^{\alpha} \right), \tag{2.11}$$

where $C_1 : [0, \infty) \to [0, \infty)$, $C_2 : [0, \infty)^2 \to [0, \infty)$ are non-decreasing function, $F_1, G_1 \in L^1(Q_T)$ and $1 \leq \alpha < 2$.

Let us now point out that if the non-linearities F and G do not depend on the gradient (system (2.1) is semi-linear), the existence of global positive solutions has been obtained by Haraux and Youkana [14], Hollis *et al.* [15], Hollis and Morgan [16], and Martin and Pierre [22]. One can see that in all of these works, the triangular structure, namely hypotheses (2.7) and

$$F \leq L_2 (w + z + 1), \quad \forall w, z \geq 0, \text{ a.e. } (t, x) \in Q_T, \tag{2.12}$$

plays an important role in the study of semi-linear systems (in our case, hypothesis (2.12) is satisfied since by (2.4), $F \leq 0$ whenever $w, z \geq 0$). Indeed, if (2.7) or (2.12) does not hold, Pierre and Schmitt [27] have proved blow up in finite time of the solutions to some semi-linear reaction-diffusion systems. When F and G are depend on the gradient, Boudiba [8] has solved the case where the triangular structure is satisfied and the growth of F and G with respect to $|\nabla w|$, $|\nabla z|$ is sub-quadratic

$$\begin{cases} \exists\ 1 \leq m < 2,\ C : [0, \infty)^2 \to [0, \infty) \text{ non-decreasing such that} \\ |F(w, z, \nabla w, \nabla z)| + |G(w, z, \nabla w, \nabla z)| \leq C(|w|, |z|) \left[1 + |\nabla w|^m + |\nabla z|^m \right]. \end{cases}$$

About the critical growth with respect to the gradient ($m = 2$), we recall that for the case of a single equation ($\lambda_1 = \lambda_2$ and $F = G$), existence results have been proved for the elliptic case in [4,6,20]. The corresponding parabolic equations have also been studied by many authors, see for instance [1,7,11,21].

3. Statement of the result

First, we have to clarify in which sense we want to solve problem (2.1).

Definition 3.1. *We say that (w,z) is a solution of (2.1) if*

$$\begin{cases} w,z\in C\left([0,T];L^1(\Omega)\right)\cap L^1\left(0,T;W_0^{1,1}(\Omega)\right)\\ F(t,x,w,z,\nabla w,\nabla z)\ \text{ and } G(t,x,w,z,\nabla w,\nabla z)\in L^1(Q_T)\\ w(t)=S_{\lambda_1}(t)\,w_0+\int_0^t S_{\lambda_1}(t-s)\,F(s,.,w(s),z(s),\nabla w(s),\nabla z(s))\,ds,\ \forall t\geq 0\\ z(t)=S_{\lambda_2}(t)\,z_0+\int_0^t S_{\lambda_2}(t-s)\,G(s,.,w(s),z(s),\nabla w(s),\nabla z(s))\,ds,\ \forall t\geq 0 \end{cases} \tag{3.1}$$

where $S_{\lambda_1}(t)$ and $S_{\lambda_2}(t)$ denote the semigroups in $L^1(\Omega)$ generated by $-\lambda_1\Delta$ and $-\lambda_2\Delta$ with Dirichlet or Neumann boundary conditions.

Example 3.1. *A typical example where the result of this paper can be applied is*

$$\begin{cases} \frac{\partial w}{\partial t}-\lambda_1\Delta w=-w\varphi(z)\left|\nabla w\right|^2 & \text{in } Q_T\\ \frac{\partial z}{\partial t}-\lambda_2\Delta z=w\varphi(z)\left|\nabla w\right|^2 & \text{in } Q_T\\ w=z=0 \text{ or } \frac{\partial w}{\partial \eta}=\frac{\partial z}{\partial \eta}=0 & \text{on } \Sigma_T\\ w(0,x)=w_0(x),\ z(0,x)=z_0(x) & \text{in } \Omega, \end{cases}$$

where φ is a bounded function.

3.1. **Main Result.**

Theorem 3.1. *Assume that (2.4)-(2.11) hold. If $w_0,z_0\in L^2(\Omega)$, then there exists a positive global solution (w,z) of system (2.1). Moreover, $w,z\in L^2\left(0,T;H_0^1(\Omega)\right)$.*

Before giving the proof of the above theorem, let us define the following functions. Given a real positive number k, we set

$$T_k(s)=\max\{-k,\min(k,s)\},\ \text{ and } G_k(s)=s-T_k(s).$$

We note that

$$\begin{cases} T_k(s)=s & \text{for } 0\leq s\leq k,\\ T_k(s)=k & \text{for } s>k. \end{cases}$$

$0\leq s\leq k$, $T_k(s)=s$ and $T_k(s)=k\ \ s>k$.

4. PROOF OF THEOREM 3.1

To prove Theorem 3.1, we will use the results which we will present in this section.

4.1. **Preliminaries.**

Theorem 4.1. *Let Ω is an open bounded domain in $\mathbb{R}^n$, and $X = L^1(\Omega) \cap H^2(\Omega)$. The operator A defined by*

$$\begin{cases} D(A) = \left\{ u \in L^1(\Omega) \cap H^2(\Omega) \;,\; \dfrac{\partial u}{\partial \eta} = 0 \;\; or \;\; u = 0 \quad on\ \partial\Omega \right\} \\ Au = \Delta u \;\;,\;\; for\ all\ u \in D(A) \end{cases}$$

is $m-$dissipative in $L^1(\Omega) \cap H^2(\Omega)$.

An important result of functional analysis which ensures the local existence of the solution is the following lemma:

Lemma 4.1. *Let A be a $m-$dissipative operator of the dense domain in the Banach space X and $S(t)$ a semigroup engendered by A, F a function locally Lipchitz. Then for any $w_0 \in X$ it exists $T(w_0) = T_{\max}$ such that the problem*

$$\begin{cases} w \in C([0,T], D(A)) \cap C^1([0,T], X), \\ \dfrac{dw}{dt} - Aw = F(s, ., w(s), \nabla w(s)), \\ w(0) = w_0, \end{cases}$$

admits a unique solution w verifying

$$w(t) = S(t)w_0 + \int_0^t S(t-s) F(s, ., w(s), \nabla w(s))\, ds,\ \forall t \in [0, T_{\max}].$$

4.1.1. *Compactness result.* In this subsection we will give a compactness result of the operator L defining the solution of the problem (2.1) where the initial value is equal to zero, i.e.

$$L(F)(t) = w(t) = \int_0^t S(t-s) F(s, ., w(s), \nabla w(s))\, ds, \quad \forall t \in [0, T].$$

Theorem 4.2. *For all $t > 0$, if the operators $S(t)$ are compact, then L are compact of $L^1([0,T], X)$ in $L^1([0,T], X)$.*

Proof. **Step 1:** We show that $S(\lambda) L : F \to S(\lambda) L(F)$ is compact in $L^1([0,T], X)$, i.e. show that the set $\{S(\lambda) L(F)(t),\ \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0,T], X)$.
Since $S(t)$ is compact then, the application $t \mapsto S(t)$ is continuous of $]0, +\infty[$ in $\mathcal{L}(X)$, therefore

$$\begin{aligned} \forall \varepsilon &> 0,\ \forall \delta > 0,\ \exists \eta > 0,\ \forall 0 \leq h \leq \eta, \\ \forall t &\geq \delta,\ \|S(t+h) - S(t)\|_{\mathcal{L}(X)} \leq \varepsilon \end{aligned}$$

We choose $\lambda = \delta$, we have for $0 \leq t \leq T - h$

$$S(\lambda) w(t+h) - S(\lambda) w(t) = \int_0^{t+h} S(\lambda + t + h - s) F(s, ., w(s), \nabla w(s))\, ds$$

$$\begin{aligned}
&\quad -\int_0^t S(\lambda+t-s)\, F(s,.,w(s),\nabla w(s))\, ds \\
&= \int_t^{t+h} S(\lambda+t+h-s)\, F(s,.,w(s),\nabla w(s))\, ds \\
&\quad +\int_0^t \left(S(\lambda+t+h-s)-S(\lambda+t-s)\right) F(s,.,w(s),\nabla w(s))\, ds,
\end{aligned}$$

from where

$$\begin{aligned}
&\left\|S(\lambda)\, w(t+h)-S(\lambda)\, w(t)\right\|_X \\
&\leq \int_t^{t+h} \left\|F(s,.,w(s),\nabla w(s))\right\|_X ds+\varepsilon\int_0^t \left\|F(s,.,w(s),\nabla w(s))\right\|_X ds.
\end{aligned}$$

We define $z(t)$ by

$$z(t)=\begin{cases} w(t) & \text{if } 0\leq t\leq T \\ 0 & \text{if not} \end{cases}$$

Therefore

$$\left\|S(\lambda)\, z(t+h)-S(\lambda)\, z(t)\right\|_1 \leq (h+\varepsilon T)\left\|F(s,.,w(s),\nabla w(s))\right\|_1,$$

which implies that all $\{S(\lambda)\, z,\ \|F\|_1\leq 1\}$ is equi-integrable, then it is conventional that all $\{S(\lambda)\, L(F)(t),\ \|F\|_1\leq 1\}$ is relatively compact in $L^1([0,T],X)$, this way $S(\lambda)\, L$ is compact.

Step 2: We show that $S(\lambda)\, L\to L$ when $\lambda\to 0$, in $L^1([0,T],X)$. We have

$$\begin{aligned}
&S(\lambda)\, w(t)-w(t) \\
&= \int_0^t S(\lambda+t-s)\, F(s,.,w(s),\nabla w(s))\, ds-\int_0^t S(t-s)\, F(s,.,w(s),\nabla w(s))\, ds.
\end{aligned}$$

So for $t\geq\delta$, we have

$$\begin{aligned}
\left\|S(\lambda)\, w(t)-w(t)\right\| &\leq \int_\delta^t \left\|S(\lambda+s)-S(s)\right\|_{\mathcal{L}(X)} \left\|F(s,.,w(s),\nabla w(s))\right\| ds \\
&\quad +2\int_{t-\delta}^t \left\|F(s,.,w(s),\nabla w(s))\right\| ds.
\end{aligned}$$

We choose $0<\lambda<\eta$, then

$$\left\|S(\lambda)\, w(t)-w(t)\right\|\leq\varepsilon\int_\delta^t \left\|F(s,.,w(s),\nabla w(s))\right\| ds+2\int_{t-\delta}^t \left\|F(s,.,w(s),\nabla w(s))\right\| ds,$$

and for $0\leq t<\delta$, we have

$$\left\|S(\lambda)\, w(t)-w(t)\right\|\leq 2\int_0^t \left\|F(s,.,w(s),\nabla w(s))\right\| ds.$$

Since $F\in L^1(0,T,X)$, from where

$$\left\|S(\lambda)\, w(t)-w(t)\right\|\leq(\varepsilon T+2\delta)\left\|F(s,.,w(s),\nabla w(s))\right\|_1.$$

Therefore, if $\lambda\to 0$ then $S(\lambda)\, w\to w$ into $L^1([0,T],X)$, where the operator L is a uniform limit with compact linear operator between two Banach spaces, then L is compact in $L^1([0,T],X)$. $\square$

Remark 4.1. *The semigroup $S(t)$ generated by the operator Δ is compact in $L^1(\Omega)$.*

4.2. **Approximating Scheme.** For every function h defined from $\mathbb{R}^+ \times \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N}$ into $\mathbb{R}$, we associate $\hat{h}$ such that

$$\hat{h}(t,x,w,z,p,q) = \begin{cases} h(t,x,w,z,p,q) & \text{if } w,z \geq 0 \\ h(t,x,w,0,p,q) & \text{if } w \geq 0, z \leq 0 \\ h(t,x,0,z,p,q) & \text{if } u \leq 0, z \geq 0 \\ h(t,x,0,0,p,q) & \text{if } w,z \leq 0, \end{cases}$$

and consider the system

$$\begin{cases} \frac{\partial w}{\partial t} - \lambda_1 \Delta w = \hat{F}(t,x,w,z,\nabla w,\nabla z) & \text{in } Q_T \\ \frac{\partial z}{\partial t} - \lambda_2 \Delta z = \hat{G}(t,x,w,z,\nabla w,\nabla z) & \text{in } Q_T \\ w = z = 0 \text{ or } \frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0, & \text{on } \Sigma_T \\ w(0,x) = w_0(x), \; z(0,x) = z_0(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

It is obviously seen, by the structure of $\hat{F}$ and $\hat{G}$, that systems (2.1) and (4.1) are equivalent on the set where $w,z \geq 0$. Consequently, to prove theorem 3.1, we have to show that problem (4.1) has a weak solution which is positive.

To this end, we consider the truncated function ψ_n in $C_c^\infty(\mathbb{R})$ such that

$$0 \leq \psi_n \leq 1$$

and

$$\psi_n(r) = \begin{cases} 1 & \text{if } |r| \leq n \\ 0 & \text{if } |r| \geq n+1, \end{cases}$$

and the mollification with respect to (t,x) is defined as follows. Let $\rho \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right)$ such that $supp\rho \subset B(0,1)$, $\int \rho = 1$, $\rho \geq 0$ on $\mathbb{R} \times \mathbb{R}^N$ and $\rho_n(y) = n^N \rho(ny)$. One can see that $\rho_n \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right)$, $supp\rho_n \subset B\left(0, \frac{1}{n}\right)$, $\int \rho_n = 1$ and $\rho_n \geq 0$ on $\mathbb{R} \times \mathbb{R}^N$.

For all $n > 0$, we define the functions w_{n_0} and z_{n_0} by

$$w_{n_0} = \min\{w_0, n\} \in C_c^\infty(\Omega) \quad \text{and} \quad z_{n_0} = \min\{z_0, n\} \in C_c^\infty(\Omega)$$

It is clear that w_{n_0} and z_{n_0} are non-negative sequences and

$$w_{n_0} \to w_0, \; z_{n_0} \to z_0, \qquad \text{in } L^2(\Omega),$$

and define for all (t,x,w,z,p,q) in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N}$;

$$\begin{aligned} F_n(t,x,w,z,p,q) &= [\psi_n(|w| + |z| + \|p\| + \|q\|) F(.,.,w,z,p,q)] * \rho_n(t,x) \\ G_n(t,x,w,z,p,q) &= [\psi_n(|w| + |z| + \|p\| + \|q\|) G(.,.,w,z,p,q)] * \rho_n(t,x) \end{aligned}$$

Note that these functions enjoy the same properties as F and G, moreover they are Hölder continuous with respect to t, x and $|F_n|, |G_n| \leq M_n$, where M_n is a constant depending only on n (these estimates can be derived from (2.9), the properties of the convolution product, and the fact that $\int \rho_n = 1$.

Let us now consider the truncated system

$$\begin{cases} \dfrac{\partial w_n}{\partial t} - \lambda_1 \Delta w_n = F_n\left(t, x, w_n, z_n, \nabla w_n, \nabla z_n\right) & \text{in } Q_T \\ \dfrac{\partial z_n}{\partial t} - \lambda_2 \Delta z_n = G_n\left(t, x, w_n, z_n, \nabla w_n, \nabla z_n\right) & \text{in } Q_T \\ w_n = z_n = 0 \text{ or } \frac{\partial w_n}{\partial \eta} = \frac{\partial z_n}{\partial \eta} = 0, & \text{on } \Sigma_T \\ w_n\left(0, x\right) = w_{n_0}\left(x\right), \ z_n\left(0, x\right) = z_{n_0}\left(x\right) & \text{in } \Omega. \end{cases} \tag{4.2}$$

4.2.1. *Local existence of the solution of problem (4.2).* We transform the system (4.2) into a first order system in the Banach space $X = L^1(\Omega) \times L^1(\Omega)$, we obtain

$$\begin{cases} \dfrac{\partial \omega_n}{\partial t} = A\omega_n + \Psi\left(t, x, \omega_n, \nabla \omega_n\right) \quad , \ t > 0 \\ \omega_n\left(0\right) = \omega_{n_0} = \left(w_{n_0}, z_{n_0}\right) \in X. \end{cases} \tag{4.3}$$

Here $\omega_n = \text{col}(w_n, z_n)$, the operator A is defined as follows

$$A = \begin{pmatrix} \lambda_1 \Delta & 0 \\ 0 & \lambda_2 \Delta \end{pmatrix}$$

where

$$D\left(A\right) := \left\{\omega_n = \text{col}\left(w_n, z_n\right) \in X : \text{col}\left(\Delta w_n, \Delta z_n\right) \in X\right\}$$

and the function Ψ is defined by

$$\Psi\left(t, x, \omega_n, \nabla \omega_n\right) = \text{col}\left(F_n\left(t, x, \omega_n, \nabla \omega_n\right), G_n\left(t, x, \omega_n, \nabla \omega_n\right)\right)$$

with Dirichlet ($w_n = z_n = 0$) or Neumann ($\frac{\partial w_n}{\partial \eta} = \frac{\partial z_n}{\partial \eta} = 0$) boundary conditions.

Theorem 4.3. *There exist $T_M > 0$ and (w_n, z_n) a local solution of (4.3) for all $t \in [0, T_M]$.*

Proof. We know that $S_{\lambda_1}(t)$, $S_{\lambda_2}(t)$ are contraction semigroups and that Ψ is locally Lipschitz in ω_n, then there exists $T_M > 0$ such that (w_n, z_n) is a local solution of (4.3) on $[0, T_M]$. $\square$

It remains to show the positivity of the solutions

4.2.2. *Positivity of the solution of problem (4.2).* The positivity of the solution is preserved with time, which is ensured by 2.6.

Lemma 4.2. *Let (w_n, z_n) be a classical solution of (4.2) and suppose that $w_{n_0}, z_{n_0} \geq 0$. Then $w_n, z_n \geq 0$.*

Proof. Let $\bar{w}_n = e^{-\sigma t} w_n$ and $\bar{z}_n = e^{-\sigma t} z_n$ $\sigma > 0$. then

$$\begin{aligned} \frac{\partial w_n}{\partial t} &= e^{\sigma t}\left(\frac{\partial \bar{w}_n}{\partial t} + \sigma \bar{w}_n\right) \\ \frac{\partial z_n}{\partial t} &= e^{\sigma t}\left(\frac{\partial \bar{z}_n}{\partial t} + \sigma \bar{z}_n\right) \end{aligned}$$

Consequently By the problem (4.2), we have $(\bar{w}_n, \bar{z}_n)$ is a solution of the system

$$\begin{cases} \dfrac{\partial \bar{w}_n}{\partial t} + \sigma \bar{w}_n - \lambda_1 \Delta \bar{w}_n = e^{-\sigma t} F_n\left(t, x, \bar{w}_n, \bar{z}_n, \nabla \bar{w}_n, \nabla \bar{z}_n\right) & \text{in } Q_T \\ \dfrac{\partial \bar{z}_n}{\partial t} + \sigma \bar{z}_n - \lambda_2 \Delta \bar{z}_n = e^{-\sigma t} G_n\left(t, x, \bar{w}_n, \bar{z}_n, \nabla \bar{w}_n, \nabla \bar{z}_n\right) & \text{in } Q_T \\ \bar{w}_n = \bar{z}_n = 0 \text{ or } \frac{\partial \bar{w}_n}{\partial \eta} = \frac{\partial \bar{z}_n}{\partial \eta} = 0, & \text{on } \Sigma_T \\ \bar{w}_n(0,x) = w_{n_0}(x), \ \bar{z}_n(0,x) = z_{n_0}(x) & \text{in } \Omega, \end{cases} \tag{4.4}$$

Let $U_0 = (t_0, x_0)$ be the minimum of $\bar{w}_n$ on Q_T. We will show that $\bar{w}_n(U_0) \geq 0$ which will imply that $\bar{w}_n \geq 0$ on Q_T and then $w_n \geq 0$ on Q_T.

Suppose the contrary, namely $\bar{w}_n(U_0) < 0$.

By the properties of the minimum, we can ensure that $U_0 \in]0, T] \times \Omega$ and

$$\begin{aligned} &\frac{\partial \bar{w}_n}{\partial t}(U_0) = 0, \ \nabla \bar{w}_n(U_0) = 0, \ \Delta \bar{w}_n(U_0) \geq 0 \quad \text{if } 0 < t_0 < T \\ &\frac{\partial \bar{w}_n}{\partial t}(U_0) \leq 0, \ \nabla \bar{w}_n(U_0) = 0, \ \Delta \bar{w}_n(U_0) \geq 0 \quad \text{if } t_0 = T. \end{aligned}$$

Hence the first equation in (4.4) yields

$$\begin{aligned} \sigma \bar{w}_n(U_0) &= -\frac{\partial \bar{w}_n}{\partial t}(U_0) + \lambda_1 \Delta \bar{w}_n(U_0) + e^{-\sigma t_0} F_n\left(U_0, \bar{w}_n(U_0), \bar{z}_n(U_0), 0, \nabla \bar{z}_n(U_0)\right) \\ &\geq e^{-\sigma t_0} F_n\left(U_0, \bar{w}_n(U_0), \bar{z}_n(U_0), 0, \nabla \bar{z}_n(U_0)\right). \end{aligned}$$

Now we use the structure of $\bar{w}_n(U_0)$ and hypothesis (2.6) to write

$$F_n\left(U_0, \bar{w}_n(U_0), \bar{z}_n(U_0), 0, \nabla \bar{z}_n(U_0)\right) = F_n\left(U_0, 0, \bar{z}_n(U_0), 0, \nabla \bar{z}_n(U_0)\right) \geq 0.$$

This implies that $\bar{w}_n(U_0) \geq 0$ which is impossible by the hypotheses.
Arguing in the same way for the second component $\bar{z}_n$, we obtain the positivity of (w_n, z_n). ☐

4.2.3. *Global existence of the solution of problem (4.2).* The total mass of the components w, z is controlled with time, which is ensured by the following lemma.

Lemma 4.3. *There exists a constant M depending on $\|w_0\|_{L^1(\Omega)}, \|z_0\|_{L^1(\Omega)}, L_1, T$ and $|\Omega|$ such that*

$$\|w_n(t) + z_n(t)\|_{L^1(\Omega)} \leq M, \qquad \forall t \in [0, T]. \tag{4.5}$$

Proof. Of the first and second equation of (4.2) with:

$$\frac{\partial}{\partial t}(w_n + z_n) - \Delta(\lambda_1 w_n + \lambda_2 z_n) = F_n + G_n.$$

The hypothesis (2.7) allowed the following estimate

$$\frac{\partial}{\partial t}\left(w_n+z_n\right)-\Delta\left(\lambda_1 w_n+\lambda_2 z_n\right)\leq L_1\left(w_n+z_n+1\right).$$

let us integrate on Ω and apply the formula of Green, then

$$\int_\Omega \Delta w_n=0,\ \text{ and } \int_\Omega \Delta z_n=0, \tag{4.6}$$

we find

$$\int_\Omega \frac{\partial}{\partial t}\left(w_n+z_n\right)\leq L_1\int_\Omega\left(w_n+z_n+1\right).$$

so

$$\frac{\frac{\partial}{\partial t}\int_\Omega\left(w_n+z_n\right)dx}{\int_\Omega\left(w_n+z_n+1\right)dx}\leq L_1,$$

Integrating this inequality on $[0,t]$, $\forall t\in\left]0,T\right]$ yields

$$\ln\int_\Omega\left(w_n+z_n+1\right)dx\Big|_0^t\leq L_1 t,$$

thus

$$\ln\frac{\int_\Omega\left(w_n(t)+z_n(t)+1\right)dx}{\int_\Omega\left(w_{n_0}+z_{n_0}+1\right)dx}\leq L_1 t,$$

which implies

$$\frac{\int_\Omega\left(w_n(t)+z_n(t)+1\right)dx}{\int_\Omega\left(w_{n_0}+z_{n_0}+1\right)dx}\leq \exp\left(L_1 t\right)$$

then we have

$$\int_\Omega\left(w_n(t)+z_n(t)+1\right)dx\leq \exp\left(L_1 t\right)\int_\Omega\left(w_{n_0}+z_{n_0}+1\right)dx$$

also

$$\begin{aligned}
\int_\Omega\left(w_n+z_n\right)(t)\,dx &\leq \int_\Omega\left(w_n(t)+z_n(t)+1\right)dx\\
&\leq \exp\left(L_1 t\right)\int_\Omega\left(w_{n_0}+z_{n_0}+1\right)dx\\
&= \exp\left(L_1 t\right)\left[\int_\Omega\left(w_{n_0}+z_{n_0}\right)dx+|\Omega|\right]\\
&\leq \exp\left(L_1 T\right)\left[\int_\Omega\left(w_0+z_0\right)dx+|\Omega|\right] \text{ as if } w_{n_0}\leq w_0,\ z_{n_0}\leq z_0\\
&\leq \exp\left(L_1 T\right)\left[\|w_0\|_{L^1(\Omega)}+\|z_0\|_{L^1(\Omega)}+|\Omega|\right].
\end{aligned}$$

This ends the proof of the lemma. □

We can conclude from this estimate that the solution (w_n, z_n) given by the Theorem 4.3 is a global solution.

Lemma 4.4. *There exists a constant R_1 depending on T, $\|w_0\|_{L^1(\Omega)}$, $\|z_0\|_{L^1(\Omega)}$, L_1, L_2 and $|\Omega|$ such that*

$$\int_{Q_T} |F_n(t,x,w_n,z_n,\nabla w_n,\nabla z_n)| + |G_n(t,x,w_n,z_n,\nabla w_n,\nabla z_n)| \le R_1$$

Proof. Considering the equations satisfied by w_n and z_n, we can write

$$-F_n = -\frac{\partial w_n}{\partial t} + \lambda_1 \Delta w_n, \text{ and } -G_n = -\frac{\partial z_n}{\partial t} + \lambda_2 \Delta z_n.$$

Integrating on Q_T and using (4.6), the positivity of the solutions yield

$$-\int_{Q_T} F_n \le \int_\Omega w_{n_0}.$$

Hence by hypothesis (2.4)

$$\int_{Q_T} |F_n| = -\int_{Q_T} F_n \le \int_\Omega w_0. \tag{4.7}$$

Similarly, we get

$$-\int_{Q_T} G_n \le \int_\Omega z_0. \tag{4.8}$$

Integrating on Q_T and by hypothesis (2.7) we get

$$\int_{Q_T} G_n \le -\int_{Q_T} F_n + \int_{Q_T} L_1(w_n + z_n + 1)$$

Moreover, by (4.5) and (4.7) we have

$$\int_{Q_T} G_n \le L_1 T(M + |\Omega|) + \int_\Omega w_0. \tag{4.9}$$

By (4.8) and (4.9) we conclude that

$$\int_{Q_T} |G_n| \le L_1 T(M + |\Omega|) + \int_\Omega w_0. \tag{4.10}$$

By (4.7) and (4.10) we get

$$\int_{Q_T} |F_n(t,x,w_n,z_n,\nabla w_n,\nabla z_n)| + |G_n(t,x,w_n,z_n,\nabla w_n,\nabla z_n)| \le R_1$$

Let us put:

$$R_1 = L_1 T(M + |\Omega|) + 2\|w_0\|_{L^1(\Omega)}$$

□

Lemma 4.5. *(i) There exists a constant R_2 depending on λ_1, $\|w_0\|_{L^2(\Omega)}$ such that*

$$\int_{Q_T} |\nabla w_n|^2 \le R_2, \quad \int_{Q_T} |\nabla T_k(w_n)|^2 \le R_2.$$

(ii) There exists a constant R_3 depending on λ_1, λ_2, L_1, $\|w_0\|_{L^2(\Omega)}$, $\|z_0\|_{L^2(\Omega)}$, $|\Omega|$ such that

$$\int_{Q_T} |\nabla z_n|^2 \le R_3, \quad \int_{Q_T} |\nabla T_k(z_n)|^2 \le R_3.$$

(iii) There exists a constant R_4 depending on λ_1, λ_2, T, $\|w_0\|_{L^2(\Omega)}$, $\|z_0\|_{L^2(\Omega)}$, L_1, $|\Omega|$ such that

$$\int_{Q_T} (2w_n + z_n) \left(|F_n (t, x, w_n, z_n, \nabla w_n, \nabla z_n)| + |G_n (t, x, w_n, z_n, \nabla w_n, \nabla z_n)| \right) \leq R_4.$$

Proof. (i) We multiply the first equation in the truncated problem by w_n and we integrate on Q_T. We obtain

$$\int_{Q_T} w_n \frac{\partial w_n}{\partial t} - \lambda_1 \int_{Q_T} w_n \Delta w_n = \int_{Q_T} F_n w_n.$$

Since, by hypothesis (2.4), $w_n F_n \leq 0$, we have

$$\int_{Q_T} |\nabla w_n|^2 \leq \frac{1}{\lambda_1} \int_{\Omega} (w_{n_0})^2 \leq \frac{1}{\lambda_1} \int_{\Omega} (w_0)^2 .$$

Then

$$\int_{Q_T} |\nabla w_n|^2 \leq R_2, \text{ where } R_2 \geq \frac{1}{\lambda_1} \|w_0\|^2_{L^2(\Omega)} .$$

We have

$$\int_{Q_T} |\nabla w_n|^2 = \int_{[w_n < k]} |\nabla w_n|^2 + \int_{[w_n \geq k]} |\nabla w_n|^2 \leq R_2$$

Then, since $\int_{[w_n \geq k]} |\nabla w_n|^2 \geq 0$

$$\int_{Q_T} |\nabla T_k (w_n)|^2 \leq R_2 \tag{4.11}$$

(ii) Of the first and second equation of (4.2) we obtain

$$\frac{\partial}{\partial t} (w_n + z_n) - \Delta (\lambda_1 w_n + \lambda_2 z_n) = F_n + G_n.$$

i.e

$$\frac{\partial (w_n + z_n)}{\partial t} - \Delta (\lambda_1 w_n + \lambda_2 z_n) - \lambda_2 \Delta w_n + \lambda_2 \Delta w_n = F_n + G_n.$$

we use hypothesis (2.7). We get

$$\frac{\partial (w_n + z_n)}{\partial t} - \lambda_2 \Delta (w_n + z_n) + (\lambda_2 - \lambda_1) \Delta w_n \leq L_1 (w_n + z_n) + L_1. \tag{4.12}$$

multiply (4.12) by $\exp(-L_1 t)$. We obtain

$$\exp(-L_1 t) \frac{\partial (w_n + z_n)}{\partial t} - L_1 \exp(-L_1 t) (w_n + z_n)$$
$$-\lambda_2 \exp(-L_1 t) \Delta (w_n + z_n) + (\lambda_2 - \lambda_1) \exp(-L_1 t) \Delta w_n \leq L_1 \exp(-L_1 t).$$

Set $x_n = \exp(-L_1 t)(w_n + z_n)$ we get

$$\frac{\partial x_n}{\partial t} - \lambda_2 \Delta x_n + (\lambda_2 - \lambda_1) \exp(-L_1 t) \Delta w_n \leq L_1 \exp(-L_1 t) \leq L_1.$$

Now, multiply by x_n and integrate on Q_T and by simple use of Green's formula we have

$$\int_{Q_T} x_n \frac{\partial x_n}{\partial t} + \lambda_2 \int_{Q_T} |\nabla x_n|^2 + (\lambda_1 - \lambda_2) \int_{Q_T} \exp(-2L_1 t) \nabla w_n \nabla (w_n + z_n) \leq L_1 \int_{Q_T} x_n.$$

hence, by Lemma 4.3

$$\lambda_2 \int_{Q_T} |\nabla x_n|^2 + (\lambda_1 - \lambda_2) \int_{Q_T} \exp(-2L_1 t) \nabla w_n \nabla (w_n + z_n) \leq \frac{1}{2} \int_{\Omega} x_n^2 (0) + L_1 M T.$$

Using Young's inequality $|ab| \leq \frac{a^p}{p} + \frac{b^q}{q}$ (where $a = \exp(-L_1 t)|\nabla w_n|$, $b = |\nabla x_n| = \exp(-L_1 t)|\nabla(w_n + z_n)|$, $p = q = 2$), we have

$$\begin{aligned}\lambda_2 \int_{Q_T} |\nabla x_n|^2 &\leq \frac{\lambda_2 - \lambda_1}{2} \int_{Q_T} |\nabla x_n|^2 + \\ &\frac{\lambda_2 - \lambda_1}{2} \int_{Q_T} \exp(-2L_1 t)|\nabla w_n|^2 + \frac{1}{2}\int_\Omega (w_n + z_n)^2(0) + L_1 MT.\end{aligned}$$

Then, by (i)

$$(\lambda_2 + \lambda_1) \int_{Q_T} |\nabla x_n|^2 \leq (\lambda_2 - \lambda_1) R_2 + \int_\Omega (w_0 + z_0)^2 + 2L_1 MT.$$

Now, using Young's inequality another time and the fact that $\exp(-2L_1 t) \geq \exp(-2L_1 T)$, for all $t \in [0, T]$, we end the proof of (ii).

Similarly, by the same method of proof 4.11 given

$$\int_{Q_T} |\nabla T_k(z_n)|^2 \leq R_3 \tag{4.13}$$

(iii) Set

$$R_n = L_1 + L_1(w_n + z_n) - F_n - G_n \geq 0, \tag{4.14}$$

by hypothesis (2.7). Combining the equations of system (4.2), we have

$$\frac{\partial(2w_n + z_n)}{\partial t} - \Delta(2\lambda_1 w_n + \lambda_2 z_n) + |F_n| + R_n = L_1 + L_1(w_n + z_n)$$

Multiplying by $(2w_n + z_n)$ and integrating on Q_T and by simple use of Green's formula we have

$$\frac{1}{2}\int_\Omega (2w_n + z_n)^2(T) + \int_{Q_T} \nabla(2\lambda_1 w_n + \lambda_2 z_n)\nabla(2w_n + z_n)$$

$$+ \int_{Q_T} (2w_n + z_n)(|F_n| + R_n) = I + J,$$

where

$$I = L_1 \int_{Q_T} (2w_n + z_n) + L_1 \int_{Q_T} (2w_n + z_n)(w_n + z_n)$$

and

$$J = \frac{1}{2}\int_\Omega (2w_n + z_n)^2(0)$$

We have by Lemma 4.3

$$\begin{aligned}&L_1 \int_{Q_T} (2w_n + z_n) + L_1 \int_{Q_T} (2w_n + z_n)(w_n + z_n) \\ \leq\ & 2L_1 \int_{Q_T} (w_n + z_n) + 2L_1 \int_{Q_T} (w_n + z_n)^2 \\ \leq\ & 2L_1 MT + 2L_1 \int_{Q_T} (w_n + z_n)^2,\end{aligned}$$

then

$$I \leq 2L_1 MT + 2L_1 \int_{Q_T} (w_n + z_n)^2 \tag{4.15}$$

and

$$J \leq \frac{1}{2} \int_{\Omega} (2w_0 + z_0)^2 \tag{4.16}$$

since $\int_{Q_T} |\nabla w_n|^2$ and $\int_{Q_T} |\nabla z_n|^2$ are uniformly bounded with respect to n by (i) and (ii), then $\int_{Q_T} |w_n|^2$, $\int_{Q_T} |z_n|^2$ are bounded too. Now, we investigate the second term of the inequality

$$\begin{aligned} & \int_{Q_T} \nabla (2\lambda_1 w_n + \lambda_2 z_n) \nabla (2w_n + z_n) \\ = \;& 4\lambda_1 \int_{Q_T} |\nabla w_n|^2 + \lambda_2 \int_{Q_T} |\nabla z_n|^2 + 2(\lambda_1 + \lambda_2) \int_{Q_T} \nabla w_n \nabla z_n \end{aligned}$$

hence

$$\begin{aligned} & \int_{Q_T} (2w_n + z_n)(|F_n| + R_n) \\ = \;& I + J - \frac{1}{2} \int_{\Omega} (2w_n + z_n)^2 (T) - \int_{Q_T} \nabla (2\lambda_1 w_n + \lambda_2 z_n) \nabla (2w_n + z_n), \\ = \;& I + J - \frac{1}{2} \int_{\Omega} (2w_n + z_n)^2 (T) - 4\lambda_1 \int_{Q_T} |\nabla w_n|^2 \\ & -\lambda_2 \int_{Q_T} |\nabla z_n|^2 - 2(\lambda_1 + \lambda_2) \int_{Q_T} \nabla w_n \nabla z_n \end{aligned}$$

using **(**4.15**)** and **(**4.16**)** yields

$$\begin{aligned} & \int_{Q_T} (2w_n + z_n)(|F_n| + R_n) \\ \leq \;& 2L_1 MT + 2L_1 \int_{Q_T} (w_n + z_n)^2 + \frac{1}{2} \int_{\Omega} (2w_0 + z_0)^2 - 2(\lambda_1 + \lambda_2) \int_{Q_T} \nabla w_n \nabla z_n, \end{aligned}$$

i.e.

$$\begin{aligned} & \int_{Q_T} (2w_n + z_n)(|F_n| + R_n) \\ \leq \;& 2L_1 MT + 2L_1 \int_{Q_T} (w_n + z_n)^2 + \frac{1}{2} \int_{\Omega} (2w_0 + z_0)^2 + 2(\lambda_1 + \lambda_2) \int_{Q_T} |\nabla w_n \nabla z_n| \end{aligned}$$

Using Young's inequality (where $a = |\nabla w_n|$, $b = |\nabla z_n|$, $p = q = 2$) we conclude that

$$\begin{aligned} & \textstyle\int_{Q_T} (2w_n + z_n)(|F_n| + R_n) \leq 2L_1 MT + 2L_1 \int_{Q_T} (w_n + z_n)^2 \\ & \textstyle + \frac{1}{2} \int_{\Omega} (2w_0 + z_0)^2 + (\lambda_1 + \lambda_2) \left[\int_{Q_T} |\nabla w_n|^2 + \int_{Q_T} |\nabla z_n|^2 \right] \end{aligned}$$

and by (i), (ii) we obtain

$$\begin{aligned} & \textstyle\int_{Q_T} (2w_n + z_n)(|F_n| + R_n) \\ & \textstyle\leq 2L_1 MT + 2L_1 \int_{Q_T} (w_n + z_n)^2 + \frac{1}{2} \int_{\Omega} (2w_0 + z_0)^2 + (\lambda_1 + \lambda_2)(R_1 + R_2) \end{aligned} \tag{4.17}$$

by **(**4.14**)** we have

$$G_n = L_1 + L_1 (w_n + z_n) - F_n - R_n.$$

Then

$$|F_n| + |G_n| \leq R_n + 2|F_n| + L_1 + L_1 (w_n + z_n)$$

Now, multiply by $(2w_n + z_n)$ and integrate on Q_T using **(4.17)** yields

$$\begin{aligned}
& \int_{Q_T} (2w_n + z_n)(|F_n| + |G_n|) \\
\leq\ & \int_{Q_T} (2w_n + z_n)(R_n + 2|F_n| + L_1 + L_1 (w_n + z_n)) \\
=\ & \int_{Q_T} (2w_n + z_n)(R_n + 2|F_n|) + L_1 \int_{Q_T} (2w_n + z_n) + L_1 \int_{Q_T} (2w_n + z_n)(w_n + z_n) \\
\leq\ & 2\int_{Q_T} (2w_n + z_n)(R_n + |F_n|) + L_1 \int_{Q_T} (2w_n + z_n) + 2L_1 \int_{Q_T} (w_n + z_n)(w_n + z_n) \\
\leq\ & 4L_1MT + 4L_1 \int_{Q_T} (w_n + z_n)^2 + \int_\Omega (2w_0 + z_0)^2 + 2(\lambda_1 + \lambda_2)(R_1 + R_2) \\
& +2L_1MT + 2L_1 \int_{Q_T} (w_n + z_n)^2 \\
=\ & 6L_1MT + 6L_1 \int_{Q_T} (w_n + z_n)^2 + \int_\Omega (2w_0 + z_0)^2 + 2(\lambda_1 + \lambda_2)(R_1 + R_2)
\end{aligned}$$

since $\int_{Q_T} (w_n + z_n)^2$ is uniformly bounded with respect to n, we get the result where

$$R_4 \geq \int_\Omega (2w_0 + z_0)^2 + 2(\lambda_1 + \lambda_2)(R_1 + R_2) + 6L_1MT + 6L_1 \int_{Q_T} (w_n + z_n)^2 .$$

$\square$

Remark 4.2. *The estimates (4.11) and (4.13) enable us to study the convergence of the truncated problem.*

4.3. **Convergence.** Our objective is to show that (w_n, z_n) converges to some (w, z) solution of the problem (3.1). The sequences w_{n_0} and z_{n_0} are uniformly bounded in $L^1(\Omega)$ (since they converge in $L^2(\Omega)$, and by Lemma 4.4, the non-linearities F_n and G_n are uniformly bounded in $L^1(Q_T)$.
We define the application L by

$$L : (w_0, h) \mapsto S_d(t) w_0 + \int_0^t S_d(t-s) h(s)\, ds$$

where $S_d(t)$ is the contraction semigroup generated by the operator $d\Delta$. According to the previous Theorem 4.2 and as $S_d(t)$ is compact, then the application L is the addition of two compact applications in $L^1(Q_T)$, which shows that L is also compact from $L^1(Q_T) \times L^1(Q_T)$ in $L^1(Q_T)$.
Then the applications

$$(w_{n_0}, F_n) \to w_n, \text{ and } (z_{n_0}, G_n) \to z_n$$

are compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1\left(0, T; W_0^{1,1}(\Omega)\right)$

Therefore, we can extract a subsequence, still denoted by (w_n, z_n), such that

$$\begin{array}{ll}(w_n, z_n) \to (w, z) & \text{in } L^1\left(0, T; W_0^{1,1}(\Omega)\right) \\ (w_n, z_n) \to (w, z) & \text{a.e. in } Q_T \\ (\nabla w_n, \nabla z_n) \to (\nabla w, \nabla z) & \text{a.e.in } Q_T.\end{array}$$

since F_n and G_n are continuous, we have

$$\begin{array}{l}F_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n) \to F(t, x, w, z, \nabla w, \nabla z) \quad \text{a.e. in } Q_T \\ G_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n) \to G(t, x, w, z, \nabla w, \nabla z) \quad \text{a.e. in } Q_T\end{array}$$

This is not sufficient to ensure that (w, z) is a solution of (3.1). In fact, we have to prove that the previous convergences are in $L^1(Q_T)$. In view of the Vitali theorem, to show that $F_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n)$ (respectively $G_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n)$) converges to $F(t, x, w, z, \nabla w, \nabla z)$ (respectively to $G(t, x, w, z, \nabla w, \nabla z)$) in $L^1(Q_T)$, is equivalent to proving that $(F_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n))_n$ and
$(G_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n))_n$ are equi-integrable in $L^1(Q_T)$.

Lemma 4.6. *$(F_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n))_{n\in\mathbb{N}}$ and $(G_n(t, x, w_n, z_n, \nabla w_n, \nabla z_n))_{n\in\mathbb{N}}$ are equi-integrable in $L^1(Q_T)$.*

The proof of this lemma requires the following result based on some properties of two time-regularizations denoted by w_γ and $w_\sigma (\gamma, \sigma > 0)$ which we define for a function $w \in L^2\left(0, T; H_0^1(\Omega)\right)$ such that $w(0) = w_0 \in L^2(\Omega)$. In the following we will denote by $\omega(\varepsilon)$ a quantity that tends to zero as ε tends to zero, and $\omega^\sigma(\varepsilon)$ a quantity that tends to zero for every fixed σ as ε tends to zero.

Lemma 4.7. *Let $(w_n)_n$ be a sequence in $L^2\left(0, T; H_0^1(\Omega)\right) \cap C([0, T])$ such that $w_n(0) = w_{n_0} \in L^2(\Omega)$ and $\frac{\partial w_n}{\partial t} = \rho_{1,n} + \rho_{2,n}$ with $\rho_{1,n} \in L^2\left(0, T; H^{-1}(\Omega)\right)$ and $\rho_{2,n} \in L^1(Q_T)$. Moreover assume that w_n converges to w in $L^2(Q_T)$, and w_{n_0} converges to $w(0)$ in $L^2(\Omega)$.*

Let Ψ be a function in $C^1([0, T])$ such that $\Psi \geq 0$, $\Psi' \leq 0$, $\Psi(T) = 0$.

Let φ be a Lipschitz increasing function in $C^0(\mathbb{R})$ such that $\varphi(0) = 0$.

Then for all $k, \gamma > 0$,

$$\begin{array}{l}\left\langle \rho_{1,n}, \Psi\varphi\left(T_k(w_n) - T_k(w_m)_\gamma\right)\right\rangle + \int_{Q_T} \rho_{2,n}\Psi\varphi\left(T_k(w_n) - T_k(w_m)_\gamma\right) \\ \geq \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \int_\Omega \Psi(0)\,\Phi\left(T_k(w) - T_k(w)_\gamma\right)(0)\,dx \\ - \int_\Omega G_k(w)(0)\,\Psi(0)\,\varphi\left(T_k(w) - T_k(w)_\gamma\right)(0)\,dx\end{array}$$

where $\Phi(t) = \int_0^t \varphi(s)ds$ and $G_k(s) = s - T_k(s)$.

Proof. See N. Alaa and I. Mounir [3]. ∎

Proof of Lemma 4.6. Let K be a measurable subset of Q_T. We have

$$\begin{aligned}
&\int_K |F_n(t,x,w_n,z_n,\nabla w_n,\nabla z_n)| \\
&= \int_{K\cap[w_n>k]} |F_n| + \int_{K\cap[w_n\le k,z_n>k]} |F_n| + \int_{K\cap[w_n\le k,z_n\le k]} |F_n| \\
&= I_1 + I_2 + I_3
\end{aligned}$$

Using Lemma 4.5, we obtain for k large enough

$$\begin{aligned}
I_1 &\le \frac{R_4}{k} \le \frac{\varepsilon}{3} \\
I_2 &\le \frac{R_4}{k} \le \frac{\varepsilon}{3}
\end{aligned}$$

Now, using hypothesis (2.10), we write

$$I_3 \le \int_{K\cap[w_n\le k,z_n\le k]} C_1(|w_n|)\left[F_1(t,x) + |\nabla w_n|^2 + |\nabla z_n|^\alpha\right]$$

Then

$$I_3 \le C_1(k)\left[\int_K F_1(t,x) + \int_{K\cap[w_n\le k,z_n\le k]} |\nabla w_n|^2 + \int_{K\cap[w_n\le k,z_n\le k]} |\nabla z_n|^\alpha\right]$$

The third integral can be controlled by using Hölder's inequality for $\alpha < 2$

$$\int_{K\cap[w_n\le k,z_n\le k]} |\nabla z_n|^\alpha \le \left[\int_K |\nabla z_n|^{2\frac{\alpha}{2}} |K|^{\frac{2-\alpha}{2}} \le R_3^{\frac{\alpha}{2}} |K|^{\frac{2-\alpha}{2}}\right],$$

where in the last inequality we used Lemma 4.5. Therefore

$$I_3 \le C_1(k)\left[\int_K F_1(t,x) + R_3^{\frac{\alpha}{2}} |K|^{\frac{2-\alpha}{2}} + \int_K |\nabla T_k(w_n)|^2\right]$$

Similarly by hypothesis (2.11), we get

$$\begin{aligned}
\int_K |G_n| &\le \frac{1}{k}\int_{Q_T} w_n |G_n| + \frac{1}{k}\int_{Q_T} z_n |G_n| + \int_{K\cap[w_n\le k,z_n\le k]} |G_n| \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + C_2(k,k)\left[\int_K G_1(t,x) + R_3^{\frac{\alpha}{2}} |K|^{\frac{2-\alpha}{2}} + \int_K |\nabla T_k(w_n)|^2\right].
\end{aligned}$$

For the remaining term, we must prove that $\left(|\nabla T_k(w_n)|^2\right)_n$ is equi-integrable in $L^1(Q_T)$. To do this we will show that $T_k(w_n)$ converges to $T_k(w)$ in $L^2\left(0,T;H_0^1(\Omega)\right)$; more precisely we will show that

$$\lim_{n\to\infty} \int_{Q_T} |\nabla T_k(w_n) - \nabla T_k(w)|^2 = 0$$

Let k and γ be positive real numbers, let $m \in \mathbb{N}$, and choose Ψ a test function as in Lemma 4.7, define φ by $\varphi(s) = s\exp\left(\beta s^2\right)$, with β to be fixed later. We will use a technique introduced by Boccardo *et al.* [7], we will multiply the first equation in the truncated problem (4.2) by the function test $\Psi\varphi\left(T_k(w_n) - T_k(w_m)_\gamma\right)$, then we will integrate on Q_T. Finally we will use Lemma 4.7 to get the result.

Since $\frac{\partial w_n}{\partial t}=\rho_{1,n}+\rho_{2,n}$, where $\rho_{1,n}=\lambda_1\Delta w_n\in L^2\left(0,T;H^{-1}\left(\Omega\right)\right)$ and $\rho_{2,n}=F_n\in L^1\left(Q_T\right)$, we have by Lemma 4.7

$$\begin{aligned}&\textstyle\int_{Q_T}\frac{\partial w_n}{\partial t}\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\\&\geq\omega^{\gamma,n}\left(\frac{1}{m}\right)+\omega^\gamma\left(\frac{1}{n}\right)-\textstyle\int_\Omega\Psi\left(0\right)\Phi\left(T_k\left(w\right)-T_k\left(w\right)_\gamma\right)dx\\&-\textstyle\int_\Omega G_k\left(w\right)\left(0\right)\Psi\left(0\right)\varphi\left(T_k\left(w\right)-T_k\left(w\right)_\gamma\right)\left(0\right)dx\end{aligned}$$

Hence

$$\begin{aligned}&\lambda_1\int_{Q_T}\nabla w_n\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\nabla\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\\&-\int_{Q_T}F_n\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\\\leq\;&\omega^{\gamma,n}\left(\frac{1}{m}\right)+\omega^\gamma\left(\frac{1}{n}\right)+\int_\Omega\Psi\left(0\right)\Phi\left(T_k\left(w\right)-T_k\left(w\right)_\gamma\right)\\&+\int_\Omega G_k\left(w\right)\left(0\right)\Psi\left(0\right)\varphi\left(T_k\left(w\right)-T_k\left(w\right)_\gamma\right)\left(0\right)\\\leq\;&\omega^{\gamma,n}\left(\frac{1}{m}\right)+\omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right),\end{aligned}$$

since $T_k\left(w\right)_\gamma\to T_k\left(w\right)$ strongly in $L^2\left(0,T;H_0^1\left(\Omega\right)\right)$. We have

$$\begin{aligned}&I=\lambda_1\textstyle\int_{Q_T}\nabla w_n\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\nabla\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\\&J=-\textstyle\int_{Q_T}F_n\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right).\end{aligned}$$

The term I can be written as

$$\begin{aligned}I\;=\;&\lambda_1\int_{Q_T}\nabla T_k\left(w_n\right)\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\nabla\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\\&+\lambda_1\int_{[w_n\geq k]}\nabla w_n\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\nabla\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\\=\;&I_1+I_2\end{aligned}$$

For I_2, we have

$$\begin{aligned}I_2\;=\;&-\lambda_1\int_{Q_T}\nabla w_n\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w_m\right)_\gamma\right)\nabla\left(T_k\left(w_m\right)_\gamma\right)\chi_{[w_n\geq k]}\\=\;&\omega^{\gamma,n}\left(\frac{1}{m}\right)-\lambda_1\int_{Q_T}\nabla w_n\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\nabla\left(T_k\left(w\right)_\gamma\right)\chi_{[w_n\geq k]}\\=\;&\omega^{\gamma,n}\left(\frac{1}{m}\right)-\lambda_1\int_{Q_T}\nabla w_n\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\\&\times\nabla\left(T_k\left(w\right)_\gamma\right)\chi_{[w_n\geq k]}X_{[\chi\geq k]}\\&-\lambda_1\int_{Q_T}\nabla w_n\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\nabla\left(T_k\left(w\right)_\gamma\right)\chi_{[w_n\geq k]}\chi_{[w<k]}\\=\;&\omega^{\gamma,n}\left(\frac{1}{m}\right)+I_{2.1}+I_{2.2}.\end{aligned}$$

For $I_{2.1}$, we have by Hölder's inequality

$$|I_{2.1}| \leq \lambda_1 \left\|\nabla w_n \varphi' \left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right\|_{L^2(Q_T)} \left\|\nabla \left(T_k\left(w\right)_\gamma\right) \chi_{[w \geq k]}\right\|_{L^2(Q_T)}$$

Using the fact that $\varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \leq \varphi'(2k)$, and Lemma 4.5, we obtain

$$|I_{2.1}| \leq \lambda_1 C \left\|\nabla \left(T_k\left(w\right)_\gamma\right) \chi_{[w \geq k]}\right\|_{L^2(Q_T)} = \omega\left(\frac{1}{\gamma}\right)$$

since $T_k\left(w\right)_\gamma \to T_k\left(w\right)$ in $L^2\left(0, T; H_0^1\left(\Omega\right)\right)$, and $\nabla T_k\left(w\right) \chi_{[w \geq k]} = 0$ a.e. in Q_T. Now we study the term $I_{2.2}$

$$\begin{aligned} I_{2.2} = \; & -\lambda_1 \textstyle\int_{Q_T} \nabla w_n \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ & \times \nabla\left(T_k\left(w\right)_\gamma\right) \chi_{[w_n \geq k]} \chi_{[w<k]} = \omega^\gamma\left(\tfrac{1}{n}\right) \end{aligned}$$

since $\chi_{[u_n \geq k]}\chi_{[u<k]} \to 0$ a.e. in Q_T. Thus

$$I_2 \geq \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right)$$

We investigate I_1

$$\begin{aligned} I_1 &= \omega^{\gamma,n}\left(\frac{1}{m}\right) + \lambda_1 \int_{Q_T} \nabla T_k\left(w_n\right) \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &\quad \times \nabla\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{m}\right) + \lambda_1 \int_{Q_T} \nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right) \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &\quad \times \nabla\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &\quad + \lambda_1 \int_{Q_T} \nabla T_k\left(w\right) \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \nabla\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \lambda_1 \int_{Q_T} \nabla T_k\left(w\right) \Psi \varphi'\left(T_k\left(w\right)-T_k\left(w\right)_\gamma\right) \\ &\quad \times \nabla\left(T_k\left(w\right)-T_k\left(w\right)_\gamma\right) \\ &\quad + \lambda_1 \int_{Q_T} \nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right) \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &\quad \times \nabla\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\ &\quad + \lambda_1 \int_{Q_T} \left|\nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right)\right|^2 \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &\quad + \lambda_1 \int_{Q_T} \nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right) \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &\quad \times \nabla\left(T_k\left(w\right)-T_k\left(w\right)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\ &\quad + \lambda \int_{Q_T} \left|\nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right)\right|^2 \Psi \varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \end{aligned}$$

Hence

$$\begin{aligned} I &\geq \omega^{\gamma,n}\left(\frac{1}{m}\right)+\omega^{\gamma}\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right) \\ &\quad+\lambda_1\int_{Q_T}\left|\nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right)\right|^2\Psi\varphi'\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \end{aligned}$$

For J, we have

$$\begin{aligned} J &= \omega^{\gamma,n}\left(\frac{1}{m}\right)-\int_{Q_T}F_n\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{m}\right)-\int_{[w_n>k]}F_n\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \\ &\quad-\int_{[w_n\leq k]}F_n\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right) \end{aligned}$$

Then

$$J\geq\omega^{\gamma,n}\left(\frac{1}{m}\right)-\int_{[w_n\leq k]}F_n\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)$$

since $\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\geq 0$ on $[w_n>k]$, $\Psi\geq 0$ and $-F_n\geq 0$ by hypotheses (2.4). On the other hand

$$\begin{aligned} &\left|\int_{[w_n\leq k]}F_n\Psi\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \\ &\leq C_1\left(k\right)\int_{[w_n\leq k]}F_1\left(t,x\right)\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \\ &+C_1\left(k\right)\int_{[w_n\leq k]}\left|\nabla z_n\right|^\alpha\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \\ &+C_1\left(k\right)\int_{[w_n\leq k]}\left|\nabla T_k\left(w_n\right)\right|^2\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \end{aligned}$$

We set

$$J_1=C_1(k)\int_{[w_n\leq k]}F_1\left(t,x\right)\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right|=\omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)$$

since $\alpha<2$, we have

$$J_2=C_1(k)\int_{[w_n\leq k]}\left|\nabla z_n\right|^\alpha\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right|=\omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)$$

and

$$\begin{aligned} J_3 &= C_1(k)\int_{[w_n\leq k]}\left|\nabla T_k\left(w_n\right)\right|^2\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \\ &= C_1(k)\int_{[w_n\leq k]}\left|\nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right)\right|^2\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \\ &\quad+2C_1(k)\int_{[w_n\leq k]}\nabla T_k\left(w_n\right)\nabla T_k\left(w\right)\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \\ &\quad-C_1(k)\int_{[w_n\leq k]}\left|\nabla T_k\left(w\right)\right|^2\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right| \\ &= \omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right) \\ &\quad+C_1(k)\int_{[w_n\leq k]}\left|\nabla\left(T_k\left(w_n\right)-T_k\left(w\right)\right)\right|^2\Psi\left|\varphi\left(T_k\left(w_n\right)-T_k\left(w\right)_\gamma\right)\right|. \end{aligned}$$

Thus

$$
\begin{aligned}
&-\int_{[w_n\leq k]} F_n \Psi \varphi\left(T_k(w_n)-T_k(w)_\gamma\right) \\
\geq\ & \omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)-C_1(k)\int_{[w_n\leq k]}\left|\nabla\left(T_k(w_n)-T_k(w)\right)\right|^2\Psi\left|\varphi\left(T_k(w_n)-T_k(w)_\gamma\right)\right|
\end{aligned}
$$

hence

$$
\begin{aligned}
J\ \geq\ & \omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right) \\
& -C_1(k)\int_{[w_n\leq k]}\left|\nabla\left(T_k(w_n)-T_k(w)\right)\right|^2\Psi\left|\varphi\left(T_k(w_n)-T_k(w)_\gamma\right)\right|
\end{aligned}
$$

Then

$$
I+J\leq \omega^{\gamma,n}\left(\frac{1}{m}\right)+\omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)
$$

We conclude that

$$
\begin{aligned}
& \int_{Q_T}\Psi\left|\nabla\left(T_k(w_n)-T_k(w)\right)\right|^2\lambda_1\varphi'\left(T_k(w_n)-T_k(w)_\gamma\right) \\
& -C_1(k)\left|\varphi\left(T_k(w_n)-T_k(w)_\gamma\right)\right| \\
\leq\ & \omega^{\gamma,n}\left(\frac{1}{m}\right)+\omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right).
\end{aligned}
$$

Now, choose β such that $\beta\geq C_1^2(k)/4\lambda_1^2$. Then we have

$$
\lambda_1\varphi'(s)-C_1(k)|\varphi(s)|>\frac{\lambda_1}{2},
$$

and this ends the proof. □

Consequently by (2.2), we proved the existence of global solutions to a reaction-diffusion system (1.1).

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