Bi-ideals and Weak Bi-ideals of Near Left Almost Rings

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Abstract. In this paper, we define bi-ideals and weak bi-ideals of nLA-ring. We investigate the properties of bi-ideals and weak bi-ideals of nLA-ring.

1. Introduction

M.A. Kazim and MD. Naseeruddin defined LA-semigroup as the following; a groupoid $S$ is called a left almost semigroup, abbreviated as LA-semigroup if

$$(ab)c = (cb)a, \quad \forall a, b, c \in S$$

M.A. Kazim and MD. Naseeruddin [1, Proposition 2.1] asserted that, in every LA-semigroups $G$ a medial law hold

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall a, b, c, d \in G.$$ 

Q. Mushtaq and M. Khan [3, p.322] asserted that, in every LA-semigroups $G$ with left identity

$$(a \cdot b) \cdot (c \cdot d) = (d \cdot b) \cdot (c \cdot a), \quad \forall a, b, c, d \in G.$$ 

Further M. Khan, Faisal, and V. Amjid [2], asserted that, if a LA-semigroup $G$ with left identity the following law holds

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall a, b, c \in G.$$ 

M. Sarwar (Kamran) [5] defined LA-group as the following; a groupoid $G$ is called a left almost group, abbreviated as LA-group, if $(i)$ there exists $e \in G$ such that $ea = a$ for all $a \in G$, $(ii)$ for every $a \in G$ there exists $a' \in G$ such that, $a'a = e$, $(iii) (ab)c = (cb)a$ for every $a, b, c \in G$.

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Let \( \langle G, \cdot \rangle \) be an LA-group and \( S \) be a non-empty subset of \( G \) and \( S \) is itself and LA-group under the binary operation induced by \( G \), the \( S \) is called an LA-subgroup of \( G \), then \( S \) is called an LA-subgroup of \( \langle G, \cdot \rangle \).

S.M. Yusuf in [7, p.211] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set \( R \) with two binary operations “+” and “·” is called a left almost ring, if \( \langle R, + \rangle \) is an LA-group, \( \langle R, \cdot \rangle \) is an LA-semigroup and distributive laws of “·” over “+” holds. T. Shah and I. Rehman [7, p.211] asserted that a commutative ring \( \langle R, +, \cdot \rangle \), we can always obtain an LA-ring \( \langle R, \oplus, \cdot \rangle \) by defining, for \( a, b, c \in R \), \( a \oplus b = b - a \) and \( a \cdot b \) is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring \( \langle R, +, \cdot \rangle \) is said to be LA-integral domain if \( a \cdot b = 0 \), \( a, b \in R \), then \( a = 0 \) or \( b = 0 \). Let \( \langle R, +, \cdot \rangle \) be an LA-ring and \( S \) be a non-empty subset of \( R \) and \( S \) is itself and LA-ring under the binary operation induced by \( R \), the \( S \) is called an LA-subring of \( R \), then \( S \) is called a left ideal of \( R \) if \( RS \subseteq S \). Right and two-sided ideals are defined in the usual manner.

By [4] a near-ring is a non-empty set \( N \) together with two binary operations “+” and “·” such that \( \langle N, + \rangle \) is a group (not necessarily abelian), \( \langle N, \cdot \rangle \) is a semigroup and one sided distributive (left or right) of “·” over “+” holds.

By [8] If a subgroup \( B \) of \( \langle N, + \rangle \) is said to be a bi-ideal of \( N \) if \( BNB \cap (BN) \ast B \subseteq B \). If \( N \) has a zero symmetric near-ring a subgroup \( B \) of \( \langle N, + \rangle \) is a bi-ideal if and only if \( BNB \subseteq B \).

By [9] A subgroup \( B \) of \( \langle N, + \rangle \) is said to be a weak bi-ideal of \( N \) if \( B^3 \subseteq B \). In this paper we will define bi-ideal of near-ring has a zero symmetric.

2. Near Left Almost Rings

T. Shah, F. Rehman and M. Raees [6, pp.1103-1111] introduces the concept of a near left almost ring (nLA-ring).

**Definition 2.1.** [6]. A non-empty set \( N \) with two binary operation “+” and “·” is called a near left almost ring (or simply an nLA-ring) if and only if

1. \( \langle N, + \rangle \) is an LA-group.
2. \( \langle N, \cdot \rangle \) is an LA-semigroup.
3. Left distributive property of \( \cdot \) over + holds, that is \( a \cdot (b + c) = a \cdot b + a \cdot c \) for all \( a, b, c \in N \).

**Definition 2.2.** [6]. An nLA-ring \( \langle N, + \rangle \) with left identity 1, such that \( 1 \cdot a = a \) for all \( a \in N \), is called an nLA-ring with left identity.

**Definition 2.3.** [6]. A non-empty subset \( S \) of an nLA-ring \( N \) is said to be an nLA-subring if and only if \( S \) is itself an nLA-ring under the same binary operations as in \( N \).
Definition 2.4. [6]. An nLA-subring $I$ of an nLA-ring $N$ is called a left ideal of $N$ if $NI \subseteq I$, and $I$ is called a right ideal if for all $n, m \in N$ and $i \in I$ such that $(i + n)m - nm \in I$, and is called two sided ideal or simply ideal if it is both left and right ideal.

Definition 2.5. [6]. Let $(N, +, \cdot)$ be an nLA-ring. A non-zero element $a$ of $N$ is called a left zero divisor if there exists $0 \neq b \in N$ such that $a \cdot b = 0$. Similarly $a$ is a right zero divisor if $b \cdot a = 0$. If $a$ is both a left and a right zero divisor, then $a$ is called a zero divisor.

Definition 2.6. [6]. An nLA-ring $(D, +, \cdot)$ with left identity 1, is called an nLA-ring integral domain if it has no left zero divisor.

Definition 2.7. [6]. An nLA-ring $(F, +, \cdot)$ with left identity 1, is called a near almost field (n-almost field) if and only if each non-zero element of $F$ has inverse under “$\cdot$”.

3. Bi-ideals and Weak Bi-ideals of Near Left Almost Rings

Next we defines of a bi-ideals and weak bi-ideals in nLA-ring is defines the same as a bi-ideal and weak bi-ideal in near-ring in [8] and [9].

Definition 3.1. If a LA-subgroup $B$ of $(N, +)$ is said to be a bi-ideal of $N$ if $(BN)B \cap (BN)B \subseteq B$. If $N$ has a zero symmetric nLA-ring a LA-subgroup $B$ of $(N, +)$ is a bi-ideal if and only if $(BN)B \subseteq B$.

Lemma 3.1. Let $N$ be a zero symmetric nLA-ring. An LA-subgroup $B$ of $N$ is a bi-ideal if and only if $(BN)B \subseteq B$.

Proof. For an LA-subgroup $N$ of $(N, +)$ if $(BN)B \subseteq B$ then $B$ is a bi-ideal of $N$.
Conversely if $B$ is a bi-ideal, we have $(BN)B \cap (BN)B \subseteq B$. Since $N$ is a zero symmetric nLA-ring, $NB \subseteq N \cdot B$, we get

$$(BN)B = (BN)B \cap (BN)B \subseteq (BN)B \cap (BN)B \subseteq B.$$

Thus $(BN)B \subseteq B$. □

Definition 3.2. Let $N$ be an nLA-ring. An LA-subgroup $B$ of $(N, +)$ is a bi-ideal if $(BN)B \subseteq B$.

Theorem 3.1. If $B$ be a bi-ideal of a nLA-ring $N$ and $S$ is an nLA-subring of $N$. Then $B \cap S$ is a bi-ideal of $S$.

Proof. Since $B$ is a bi-ideal of $N$ we have $(BN)B \subseteq B$. Assume that $C := B \cap S$. Then $(CS)C \subseteq (SS)S \subseteq S$, since $S$ is a nLA-subring of $N$ and $C \subseteq S$.

On the other hand $(CS)C \subseteq (BS)B \subseteq (BN)B \subseteq B$. Hence $(CS)C \subseteq B \cap S = C$. Therefore $C$ is a bi-ideal of $S$. □

Theorem 3.2. Let $N$ be an nLA-ring and $A, B$ be bi-ideals of an nLA-ring $N$. Then $A \cap B$ is a bi-ideal of $N$. 
**Theorem 3.3.** The set of all bi-ideal of nLA-ring.

**Proof.** Let \( \{B_i\}_{i \in I} \) be a set of bi-ideal in \( N \) and \( B := \bigcap_{i \in I} B_i \). Then \((B_i)B \subseteq (\bigcap_{i \in I} B_i)\bigcap_{i \in I} B_i \subseteq B_i\) for every \( i \in I \). Thus \( B \) is a bi-ideal of \( N \). \( \square \)

**Definition 3.3.** Let \( N \) be an nLA-ring. An element \( d \) of \( N \) is called distributive if \((n + n')d = nd + n'd\) for all \( n, n' \in N \).

**Theorem 3.4.** Let \( N \) be an nLA-ring. If \( B \) is a bi-ideal of \( N \) then \( Bn \) and \( n'B \) are bi-ideal of \( N \) where \( n, n' \in N \) and \( n' \) is a distributive element in \( N \).

**Proof.** Since \( B \) is a bi-ideal we have \( Bn \) and \( n'B \) are an LA-subgroup \( \langle N, + \rangle \). Thus

\[
((Bn)N)(Bn) \subseteq (BN)(Bn) = (BN)Bn \subseteq Bn.
\]

Hence \( Bn \) is a bi-ideal of \( N \).

Again

\[
((n'B)N)(n'B) \subseteq ((n'B)N)B = (n'BN)B \subseteq n'B.
\]

Thus \( n'B \) are bi-ideal of \( N \). \( \square \)

**Corollary 3.1.** If \( B \) is a bi-ideal of nLA-ring. For \( b, c \in B \), if \( b \) is a distributive element in \( N \), then \( bBc \) is a bi-ideal of \( N \).

**Proof.** Let \( B \) be a bi-ideal of nLA-ring and \( b \) is a distributive element in \( N \). Then \( b(n + n') = bn + dn' \) for all \( n, n' \in N \). Since \( B \) is a bi-ideal we have \( bBc \) is an LA-subgroup \( \langle N, + \rangle \) then \((bBc)N)(bBc) \subseteq (BN)B \subseteq B \). \( \square \)

**Definition 3.4.** An nLA-ring \( N \) is said to be \( B \)-simple if it has no proper bi-ideals.

**Theorem 3.5.** Let \( N \) be an nLA-ring with more than one element. If \( N \) is a near almost field. Then \( N \) is a \( B \)-simple.

**Proof.** Let \( N \) be a near almost field then \( \{0\} \) and \( N \) are the only bi-ideals of \( N \). For if \( 0 \neq B \) is a bi-ideal of \( N \), then for \( 0 \neq b \in B \) we get \( N = Nb \) and \( N = bN \).

Now \( N = N^2 = (bN)(Nb) \subseteq bNb \subseteq B \), since \( B \) is a bi-ideal of \( N \). Then \( N = B \). Thus \( N \) is a \( B \)-simple. \( \square \)

The following we defined weak bi-ideal and study properties it.

**Definition 3.5.** An LA-subgroup \( B \) of \( \langle N, + \rangle \) is said to be a weak bi-ideal of \( N \) if \( B^3 \subseteq B \).
Theorem 3.6. Every bi-ideal of an nLA-ring is a weak bi-ideal.

Proof. Since \( B^3 = (BB)B \subseteq (BN)B \subseteq B \) we have every bi-ideal is a weak bi-ideal. \( \Box \)

Theorem 3.7. If \( B \) is a weak bi-ideal of a nLA-ring \( N \) and \( S \) is a nLA-subring of \( N \). Then \( B \cap S \) is a weak bi-ideal of \( N \).

Proof. Assume that \( C := B \cap S \). Then

\[
C^3 = ((B \cap S)(B \cap S))(B \cap S)
\]

\[
= ((B \cap S)(B \cap S))B \cap ((B \cap S)(B \cap S))S
\]

\[
\subseteq (BB)B \cap SSS
\]

\[
= B^3 \cap SSS
\]

\[
\subseteq B^3 \cap S
\]

\[
\subseteq B \cap S
\]

\[
= C.
\]

Thus \( C^3 \subseteq C \). Hence \( C \) is a weak bi-ideal of \( N \). \( \Box \)

Theorem 3.8. Let \( N \) be an nLA-ring. If \( B \) is a weak bi-ideal of \( N \) then \( Bn \) and \( n'B \) are bi-ideal of \( N \) where \( n, n' \in N \) and \( n' \) is a distributive element in \( N \).

Proof. Since \( B \) is a weak bi-ideal we have \( Bn \) and \( n'B \) are an LA-subgroup \( \langle N, + \rangle \). Thus

\[
(Bn)^3 = (BnBn)Bn \subseteq (BB)Bn \subseteq B^3 n = Bn.
\]

Hence \( Bn \) is a weak bi-ideal of \( N \).

Again

\[
(n'B)^3 = (n'Bn'B)n'B \subseteq (n'BBB)n'B = n'B^3 \subseteq n'B.
\]

Thus \( n'B \) is a weak bi-ideal of \( N \). \( \Box \)

Corollary 3.2. If \( B \) is a weak bi-ideal of nLA-ring. For \( b, c \in B \), if \( b \) is a distributive element in \( N \), then \( bBc \) is a weak bi-ideal of \( N \).

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