Some Properties of Generalized \((\Lambda, \alpha)\)-Closed Sets

Chawalit Boonpok, Montri Thongmoon∗

Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

*Corresponding author: montri.t@msu.ac.th

Abstract. The aim of this paper is to introduce the concept of generalized \((\Lambda, \alpha)\)-closed sets. Moreover, we investigate some characterizations of \(\Lambda_{\alpha}T_{1/2}\)-spaces, \((\Lambda, \alpha)\)-normal spaces and \((\Lambda, \alpha)\)-regular spaces by utilizing generalized \((\Lambda, \alpha)\)-closed sets.

1. Introduction

The concept of generalized closed sets was first introduced by Levine [7]. Moreover, Levine defined a separation axiom called \(T_{1/2}\) between \(T_0\) and \(T_1\). Dontchev and Ganster [3] introduced the notion of \(T_{1/2}\)-spaces which are situated between \(T_1\) and \(T_{1/2}\) and showed that the digital line or the Khalimsky line [5] \((\mathbb{Z}, \kappa)\) lies between \(T_1\) and \(T_{1/2}\). As a modification of generalized closed sets, Palaniappan and Rao [10] introduced and studied the notion of regular generalized closed sets. As the further modification of regular generalized closed sets, Noiri and Popa [9] introduced and investigated the concept of regular generalized \(\alpha\)-closed sets. Park et al. [11] obtained some characterizations of \(T_{1/2}\)-spaces. Dungthaisong et al. [4] characterized \(\mu_{(m,n)}T_{1/2}\)-spaces by utilizing the concept of \(\mu_{(m,n)}\)-closed sets. Torton et al. [12] introduced and studied the notions of \(\mu_{(m,n)}\)-regular spaces and \(\mu_{(m,n)}\)-normal spaces. Buadong et al. [1] introduced and investigated the notions of \(T_1\)-GTMS spaces and \(T_2\)-GTMS spaces. Caldas et al. [2] by considering the concepts of \(\alpha\)-open sets and \(\alpha\)-closed sets, introduced and investigated \(\Lambda_{\alpha}\)-sets, \((\Lambda, \alpha)\)-closed sets, \((\Lambda, \alpha)\)-open sets and the \((\Lambda, \alpha)\)-closure operator. Khampakdee and Boonpok [6] studied some properties of \((\Lambda, \alpha)\)-open sets. In the present paper, we introduce the concept of generalized \((\Lambda, \alpha)\)-closed sets. Furthermore, some properties of

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generalized $(\Lambda, \alpha)$-closed sets are discussed. In particular, several characterizations of $\Lambda_\alpha T^{\frac{1}{2}}$-spaces, $(\Lambda, \alpha)$-normal spaces and $(\Lambda, \alpha)$-regular spaces are established.

2. Preliminaries

Let $A$ be a subset of a topological space $(X, \tau)$. The closure of $A$ and the interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A subset $A$ of a topological space $(X, \tau)$ is said to be $\alpha$-open [8] if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$. The complement of an $\alpha$-open set is called $\alpha$-closed. The family of all $\alpha$-open sets in a topological space $(X, \tau)$ is denoted by $\alpha(X, \tau)$. A subset $\Lambda_\alpha(A)$ [2] is defined as follows:

$$\Lambda_\alpha(A) = \cap\{O \in \alpha(X, \tau) | A \subseteq O\}.$$

**Lemma 2.1.** [2] For subsets $A$, $B$ and $A_i (i \in I)$ of a topological space $(X, \tau)$, the following properties hold:

1. $A \subseteq \Lambda_\alpha(A)$.
2. If $A \subseteq B$, then $\Lambda_\alpha(A) \subseteq \Lambda_\alpha(B)$.
3. $\Lambda_\alpha(\Lambda_\alpha(A)) = \Lambda_\alpha(A)$.
4. $\Lambda_\alpha(\cap\{A_i | i \in I\}) \subseteq \cap\{\Lambda_\alpha(A_i) | i \in I\}$.
5. $\Lambda_\alpha(\cup\{A_i | i \in I\}) = \cup\{\Lambda_\alpha(A_i) | i \in I\}$.

Recall that a subset $A$ of a topological space $(X, \tau)$ is said to be a $\Lambda_\alpha$-set [2] if $A = \Lambda_\alpha(A)$.

**Lemma 2.2.** [2] For subsets $A$ and $A_i (i \in I)$ of a topological space $(X, \tau)$, the following properties hold:

1. $\Lambda_\alpha(A)$ is a $\Lambda_\alpha$-set.
2. If $A$ is $\alpha$-open, then $A$ is a $\Lambda_\alpha$-set.
3. If $A_i$ is a $\Lambda_\alpha$-set for each $i \in I$, then $\cap_{i \in I} A_i$ is a $\Lambda_\alpha$-set.
4. If $A_i$ is a $\Lambda_\alpha$-set for each $i \in I$, then $\cup_{i \in I} A_i$ is a $\Lambda_\alpha$-set.

A subset $A$ of a topological space $(X, \tau)$ is called $(\Lambda, \alpha)$-closed [2] if $A = T \cap C$, where $T$ is a $\Lambda_\alpha$-set and $C$ is an $\alpha$-closed set. The complement of a $(\Lambda, \alpha)$-closed set is called $(\Lambda, \alpha)$-open. The collection of all $(\Lambda, \alpha)$-open (resp. $(\Lambda, \alpha)$-closed) sets in a topological space $(X, \tau)$ is denoted by $\Lambda_\alpha O(X, \tau)$ (resp. $\Lambda_\alpha \text{C}(X, \tau)$). Let $A$ be a subset of a topological space $(X, \tau)$. A point $x \in X$ is called a $(\Lambda, \alpha)$-cluster point of $A$ [2] if for every $(\Lambda, \alpha)$-open set $U$ of $X$ containing $x$ we have $A \cap U \neq \emptyset$. The set of all $(\Lambda, \alpha)$-cluster points of $A$ is called the $(\Lambda, \alpha)$-closure of $A$ and is denoted by $A^{(\Lambda, \alpha)}$.

**Lemma 2.3.** [2] Let $A$ and $B$ be subsets of a topological space $(X, \tau)$. For the $(\Lambda, \alpha)$-closure, the following properties hold:

1. $A \subseteq A^{(\Lambda, \alpha)}$ and $[A^{(\Lambda, \alpha)}]^{(\Lambda, \alpha)} = A^{(\Lambda, \alpha)}$.
2. $A^{(\Lambda, \alpha)} = \cap\{F | A \subseteq F \text{ and } F \text{ is } (\Lambda, \alpha)\text{-closed}\}$.
3. If $A \subseteq B$, then $A^{(\Lambda, \alpha)} \subseteq B^{(\Lambda, \alpha)}$. 
(4) \(A\) is \((\Lambda, \alpha)\)-closed if and only if \(A = A^{(\Lambda, \alpha)}\).

(5) \(A^{(\Lambda, \alpha)}\) is \((\Lambda, \alpha)\)-closed.

**Definition 2.1.** [6] Let \(A\) be a subset of a topological space \((X, \tau)\). The union of all \((\Lambda, \alpha)\)-open sets of \(X\) contained in \(A\) is called the \((\Lambda, \alpha)\)-interior of \(A\) and is denoted by \(A_{(\Lambda, \alpha)}\).

**Lemma 2.4.** [6] Let \(A\) and \(B\) be subsets of a topological space \((X, \tau)\). For the \((\Lambda, \alpha)\)-interior, the following properties hold:

1. \(A_{(\Lambda, \alpha)} \subseteq A\) and \([A_{(\Lambda, \alpha)}]_{(\Lambda, \alpha)} = A_{(\Lambda, \alpha)}\).
2. If \(A \subseteq B\), then \(A_{(\Lambda, \alpha)} \subseteq B_{(\Lambda, \alpha)}\).
3. \(A\) is \((\Lambda, \alpha)\)-open if and only if \(A_{(\Lambda, \alpha)} = A\).
4. \(A_{(\Lambda, \alpha)}\) is \((\Lambda, \alpha)\)-open.
5. \([X - A]_{(\Lambda, \alpha)} = X - A_{(\Lambda, \alpha)}\).
6. \([X - A]_{(\Lambda, \alpha)} = X - A^{(\Lambda, \alpha)}\).

3. Generalized \((\Lambda, \alpha)\)-closed sets

In this section, we introduce the notion of generalized \((\Lambda, \alpha)\)-closed sets. Moreover, some properties of generalized \((\Lambda, \alpha)\)-closed sets are discussed.

**Definition 3.1.** A subset \(A\) of a topological space \((X, \tau)\) is said to be generalized \((\Lambda, \alpha)\)-closed (briefly \(g\)-\((\Lambda, \alpha)\)-closed) if \(A_{(\Lambda, \alpha)} \subseteq U\) and \(U\) is \((\Lambda, \alpha)\)-open in \((X, \tau)\). The complement of a generalized \((\Lambda, \alpha)\)-closed set is said to be generalized \((\Lambda, \alpha)\)-open (briefly \(g\)-\((\Lambda, \alpha)\)-open).

**Definition 3.2.** A topological space \((X, \tau)\) is said to be \(\Lambda_{\alpha}\)-symmetric if for \(x\) and \(y\) in \(X\), \(x \in \{y\}^{(\Lambda, \alpha)}\) implies \(y \in \{x\}^{(\Lambda, \alpha)}\).

**Theorem 3.1.** A topological space \((X, \tau)\) is \(\Lambda_{\alpha}\)-symmetric if and only if \(\{x\}\) is \(g\)-\((\Lambda, \alpha)\)-closed for each \(x \in X\).

**Proof.** Assume that \(x \in \{y\}^{(\Lambda, \alpha)}\) but \(y \notin \{x\}^{(\Lambda, \alpha)}\). This implies that the complement of \(\{x\}^{(\Lambda, \alpha)}\) contains \(y\). Therefore, the set \(\{y\}\) is a subset of the complement of \(\{x\}^{(\Lambda, \alpha)}\). This implies that \(\{y\}^{(\Lambda, \alpha)}\) is a subset of the complement of \(\{x\}^{(\Lambda, \alpha)}\). Now the complement of \(\{x\}^{(\Lambda, \alpha)}\) contains \(x\) which is a contradiction.

Conversely, suppose that \(\{x\} \subseteq V \in \Lambda_{\alpha}O(X, \tau)\), but \(\{x\}^{(\Lambda, \alpha)}\) is not a subset of \(V\). This means that \(\{x\}^{(\Lambda, \alpha)}\) and the complement of \(V\) are not disjoint. Let \(y\) belongs to their intersection. Now, we have \(x \in \{y\}^{(\Lambda, \alpha)}\) which is a subset of the complement of \(V\) and \(x \notin V\). This is a contradiction. \(\square\)

**Theorem 3.2.** A subset \(A\) of a topological space \((X, \tau)\) is \(g\)-\((\Lambda, \alpha)\)-closed if and only if \(A^{(\Lambda, \alpha)} - A\) contains no nonempty \((\Lambda, \alpha)\)-closed set.
Proof. Let $F$ be a $(\Lambda, \alpha)$-closed subset of $A^{(\Lambda,\alpha)} - A$. Now, $A \subseteq X - F$ and since $A$ is g-$(\Lambda, \alpha)$-closed, we have $A^{(\Lambda,\alpha)} \subseteq X - F$ or $F \subseteq X - A^{(\Lambda,\alpha)}$. Thus, $F \subseteq A^{(\Lambda,\alpha)} \cap [X - A^{(\Lambda,\alpha)}] = \emptyset$ and hence $F$ is empty.

Conversely, suppose that $A \subseteq U$ and $U$ is $(\Lambda, \alpha)$-open. If $A^{(\Lambda,\alpha)} \not\subseteq U$, then $A^{(\Lambda,\alpha)} \cap (X - U)$ is a nonempty $(\Lambda, \alpha)$-closed subset of $A^{(\Lambda,\alpha)} - A$.

Definition 3.3. Let $A$ be a subset of a topological space $(X, \tau)$. The $(\Lambda, \alpha)$-frontier of $A$, $\Lambda_{\alpha}Fr(A)$, is defined as follows: $\Lambda_{\alpha}Fr(A) = A^{(\Lambda,\alpha)} \cap [X - A]^{(\Lambda,\alpha)}$.

Theorem 3.3. Let $A$ be a subset of a topological space $(X, \tau)$. If $A$ is g-$(\Lambda, \alpha)$-closed and

$$A \subseteq V \in \Lambda_{\alpha}O(X, \tau),$$

then $\Lambda_{\alpha}Fr(V) \subseteq [X - A]^{(\Lambda,\alpha)}$.

Proof. Let $A$ be g-$(\Lambda, \alpha)$-closed and $A \subseteq V \subseteq \Lambda_{\alpha}O(X, \tau)$. Then, $A^{(\Lambda,\alpha)} \subseteq V$. Suppose that $x \in \Lambda_{\alpha}Fr(V)$. Since $V \subseteq \Lambda_{\alpha}O(X, \tau)$, $\Lambda_{\alpha}Fr(V) = V^{(\Lambda,\alpha)} - V$. Therefore, $x \not\in V$ and $x \not\in A^{(\Lambda,\alpha)}$. Thus, $x \in [X - A]^{(\Lambda,\alpha)}$ and hence $\Lambda_{\alpha}Fr(V) \subseteq [X - A]^{(\Lambda,\alpha)}$.

Theorem 3.4. Let $(X, \tau)$ be a topological space. For each $x \in X$, either $\{x\}$ is $(\Lambda, \alpha)$-closed or g-$(\Lambda, \alpha)$-open.

Proof. Suppose that $\{x\}$ is not $(\Lambda, \alpha)$-closed. Then, $X - \{x\}$ is not $(\Lambda, \alpha)$-open and the only $(\Lambda, \alpha)$-open set containing $X - \{x\}$ is $X$ itself. Thus, $[X - \{x\}]^{(\Lambda,\alpha)} \subseteq X$ and hence $X - \{x\}$ is g-$(\Lambda, \alpha)$-closed. Therefore, $\{x\}$ is g-$(\Lambda, \alpha)$-open.

Theorem 3.5. Let $A$ be a subset of a topological space $(X, \tau)$. Then, $A$ is g-$(\Lambda, \alpha)$-open if and only if $F \subseteq A^{(\Lambda,\alpha)}$ whenever $F \subseteq A$ and $F$ is $(\Lambda, \alpha)$-closed.

Proof. Suppose that $A$ is g-$(\Lambda, \alpha)$-open. Let $F \subseteq A$ and $F$ be $(\Lambda, \alpha)$-closed. Then, we have

$$X - A \subseteq X - F \in \Lambda_{\alpha}O(X, \tau)$$

and $X - A$ is g-$(\Lambda, \alpha)$-closed. Thus, $X - A^{(\Lambda,\alpha)} = [X - A]^{(\Lambda,\alpha)} \subseteq X - F$ and hence $F \subseteq A^{(\Lambda,\alpha)}$.

Conversely, let $X - A \subseteq U$ and $U \in \Lambda_{\alpha}O(X, \tau)$. Then, $X - U \subseteq A$ and $X - U$ is $(\Lambda, \alpha)$-closed. By the hypothesis, $X - U \subseteq A^{(\Lambda,\alpha)}$ and hence $[X - A]^{(\Lambda,\alpha)} = X - A^{(\Lambda,\alpha)} \subseteq U$. This shows that $X - A$ is g-$(\Lambda, \alpha)$-closed. Thus, $A$ is g-$(\Lambda, \alpha)$-open.

Theorem 3.6. A subset $A$ of a topological space $(X, \tau)$ is g-$(\Lambda, \alpha)$-closed if and only if $A \cap \{x\}^{(\Lambda,\alpha)} \neq \emptyset$ for every $x \in A^{(\Lambda,\alpha)}$.

Proof. Let $A$ be a g-$(\Lambda, \alpha)$-closed set and suppose that there exists $x \in A^{(\Lambda,\alpha)}$ such that $A \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$. Therefore, $A \subseteq X - \{x\}^{(\Lambda,\alpha)}$ and so $A^{(\Lambda,\alpha)} \subseteq X - \{x\}^{(\Lambda,\alpha)}$. Hence $x \not\in A^{(\Lambda,\alpha)}$, which is a contradiction.
Conversely, suppose that the condition of the theorem holds and let $U$ be any $(\Lambda, \alpha)$-open set containing $A$. Let $x \in A^{(\Lambda, \alpha)}$. Then, by the hypothesis $A \cap A^{(\Lambda, \alpha)} \neq \emptyset$, so there exists $y \in A \cap \{x\}^{(\Lambda, \alpha)}$ and so $y \in A \subseteq U$. Thus, $\{x\} \cap U \neq \emptyset$. Hence $x \in U$, which implies that $A^{(\Lambda, \alpha)} \subseteq U$. This shows that $A$ is $g$-$(\Lambda, \alpha)$-closed.

□

**Definition 3.4.** A subset $A$ of a topological space $(X, \tau)$ is said to be locally $(\Lambda, \alpha)$-closed if $A = U \cap F$, where $U \in \Lambda_\alpha O(X, \tau)$ and $F$ is a $(\Lambda, \alpha)$-closed set.

**Theorem 3.7.** For a subset $A$ of a topological space $(X, \tau)$, the following properties are equivalent:

1. $A$ is locally $(\Lambda, \alpha)$-closed;
2. $A = U \cap A^{(\Lambda, \alpha)}$ for some $U \in \Lambda_\alpha O(X, \tau)$;
3. $A^{(\Lambda, \alpha)} - A$ is $(\Lambda, \alpha)$-closed;
4. $A \cup [X - A^{(\Lambda, \alpha)}] \in \Lambda_\alpha O(X, \tau)$;
5. $A \subseteq [A \cup [X - A^{(\Lambda, \alpha)}]]_{(\Lambda, \alpha)}$.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $A = U \cap F$, where $U \in \Lambda_\alpha O(X, \tau)$ and $F$ is a $(\Lambda, \alpha)$-closed set. Since $A \subseteq F$, we have $A^{(\Lambda, \alpha)} \subseteq F^{(\Lambda, \alpha)} = F$. Since $A \subseteq U$, $A \subseteq U \cap A^{(\Lambda, \alpha)} \subseteq U \cap F = A$. Thus, $A = U \cap A^{(\Lambda, \alpha)}$ for some $U \in \Lambda_\alpha O(X, \tau)$.

(2) $\Rightarrow$ (3): Suppose that $A = U \cap A^{(\Lambda, \alpha)}$ for some $U \in \Lambda_\alpha O(X, \tau)$. Then, we have

$$A^{(\Lambda, \alpha)} - A = [X - U \cap A^{(\Lambda, \alpha)}] \cap A^{(\Lambda, \alpha)} = (X - U) \cap A^{(\Lambda, \alpha)}.$$}

Since $(X - U) \cap A^{(\Lambda, \alpha)}$ is $(\Lambda, \alpha)$-closed, $A^{(\Lambda, \alpha)} - A$ is $(\Lambda, \alpha)$-closed.

(3) $\Rightarrow$ (4): Since $X - [A^{(\Lambda, \alpha)} - A] = [X - A^{(\Lambda, \alpha)}] \cup A$ and by (3), $A \cup [X - A^{(\Lambda, \alpha)}] \in \Lambda_\alpha O(X, \tau)$.

(4) $\Rightarrow$ (5): By (4), we obtain $A \subseteq A \cup [X - A^{(\Lambda, \alpha)}] = [A \cup [X - A^{(\Lambda, \alpha)}]]_{(\Lambda, \alpha)}$.

(5) $\Rightarrow$ (1): We put $U = [A \cup [X - A^{(\Lambda, \alpha)}]]_{(\Lambda, \alpha)}$. Then, $U \in \Lambda_\alpha O(X, \tau)$ and

$$A = A \cap U \subseteq U \cap A^{(\Lambda, \alpha)} \subseteq [A \cup [X - A^{(\Lambda, \alpha)}]]_{(\Lambda, \alpha)} \cap A^{(\Lambda, \alpha)} = A \cap A^{(\Lambda, \alpha)} = A.$$}

Thus, $A = U \cap A^{(\Lambda, \alpha)}$, where $U \in \Lambda_\alpha O(X, \tau)$ and $A^{(\Lambda, \alpha)}$ is a $(\Lambda, \alpha)$-closed set. This shows that $A$ is locally $(\Lambda, \alpha)$-closed.

□

**Theorem 3.8.** A subset $A$ of a topological space $(X, \tau)$ is $(\Lambda, \alpha)$-closed if and only if $A$ is locally $(\Lambda, \alpha)$-closed and $g$-$(\Lambda, \alpha)$-closed.

**Proof.** Let $A$ be $(\Lambda, \alpha)$-closed. Then, $A$ is $g$-$(\Lambda, \alpha)$-closed. Since $X \in \Lambda_\alpha O(X, \tau)$ and $A = X \cap A$, $A$ is locally $(\Lambda, \alpha)$-closed.

Conversely, suppose that $A$ is locally $(\Lambda, \alpha)$-closed and $g$-$(\Lambda, \alpha)$-closed. Since $A$ is locally $(\Lambda, \alpha)$-closed, by Theorem 3.7, $A \subseteq [A \cup [X - A^{(\Lambda, \alpha)}]]_{(\Lambda, \alpha)}$. Since $[A \cup [X - A^{(\Lambda, \alpha)}]]_{(\Lambda, \alpha)} \in \Lambda_\alpha O(X, \tau)$ and $A$ is $g$-$(\Lambda, \alpha)$-closed, $A^{(\Lambda, \alpha)} \subseteq [A \cup [X - A^{(\Lambda, \alpha)}]]_{(\Lambda, \alpha)} \subseteq A \cup [X - A^{(\Lambda, \alpha)}]$ and hence $A^{(\Lambda, \alpha)} = A$. Thus, by Lemma 2.3, $A$ is $(\Lambda, \alpha)$-closed.

□
4. Applications of generalized $(\Lambda, \alpha)$-closed sets

We begin this section by introducing the concept of $\Lambda_\alpha-T_{\frac{1}{2}}$-spaces.

**Definition 4.1.** A topological space $(X, \tau)$ is called a $\Lambda_\alpha-T_{\frac{1}{2}}$-space if every $g$-$(\Lambda, \alpha)$-closed set of $X$ is $(\Lambda, \alpha)$-closed.

**Lemma 4.1.** Let $(X, \tau)$ be a topological space. For each $x \in X$, the singleton $\{x\}$ is $(\Lambda, \alpha)$-closed or $X - \{x\}$ is $g$-$(\Lambda, \alpha)$-closed.

**Proof.** Let $x \in X$ and the singleton $\{x\}$ be not $(\Lambda, \alpha)$-closed. Then, $X - \{x\}$ is not $(\Lambda, \alpha)$-open and $X$ is the only $(\Lambda, \alpha)$-open set which contains $X - \{x\}$ and $X - \{x\}$ is $g$-$(\Lambda, \alpha)$-closed.

Let $A$ be a subset of a topological space $(X, \tau)$. A subset $\Lambda_{(\Lambda, \alpha)}(A)$ [6] is defined as follows:

$$\Lambda_{(\Lambda, \alpha)}(A) = \bigcap \{U \mid A \subseteq U, U \in \Lambda_\alpha O(X, \tau)\}.$$ 

**Lemma 4.2.** [6] For subsets $A, B$ of a topological space $(X, \tau)$, the following properties hold:

(1) $A \subseteq \Lambda_{(\Lambda, \alpha)}(A)$.
(2) If $A \subseteq B$, then $\Lambda_{(\Lambda, \alpha)}(A) \subseteq \Lambda_{(\Lambda, \alpha)}(B)$.
(3) $\Lambda_{(\Lambda, \alpha)}[\Lambda_{(\Lambda, \alpha)}(A)] = \Lambda_{(\Lambda, \alpha)}(A)$.
(4) If $A$ is $(\Lambda, \alpha)$-open, $\Lambda_{(\Lambda, \alpha)}(A) = A$.

A subset $A$ of a topological space $(X, \tau)$ is called a $\Lambda_{(\Lambda, \alpha)}$-set if $A = \Lambda_{(\Lambda, \alpha)}(A)$. The family of all $\Lambda_{(\Lambda, \alpha)}$-sets of $(X, \tau)$ is denoted by $\Lambda_{(\Lambda, \alpha)}(X, \tau)$ (or simply $\Lambda_{(\Lambda, \alpha)}$).

**Definition 4.2.** A subset $A$ of a topological space $(X, \tau)$ is called a generalized $\Lambda_{(\Lambda, \alpha)}$-set (briefly $g$-$\Lambda_{(\Lambda, \alpha)}$-set) if $\Lambda_{(\Lambda, \alpha)}(A) \subseteq F$ whenever $A \subseteq F$ and $F$ is $(\Lambda, \alpha)$-closed.

**Lemma 4.3.** Let $(X, \tau)$ be a topological space. For each $x \in X$, the singleton $\{x\}$ is $(\Lambda, \alpha)$-open or $X - \{x\}$ is $g$-$\Lambda_{(\Lambda, \alpha)}$-set.

**Proof.** Let $x \in X$ and the singleton $\{x\}$ be not $(\Lambda, \alpha)$-open. Then, $X - \{x\}$ is not $(\Lambda, \alpha)$-closed and $X$ is the only $(\Lambda, \alpha)$-closed set which contains $X - \{x\}$ and $X - \{x\}$ is $g$-$\Lambda_{(\Lambda, \alpha)}$-set.

**Theorem 4.1.** For a topological space $(X, \tau)$, the following properties are equivalent:

(1) $(X, \tau)$ is a $\Lambda_\alpha-T_{\frac{1}{2}}$-space.
(2) For each $x \in X$, the singleton $\{x\}$ is $(\Lambda, \alpha)$-open or $(\Lambda, \alpha)$-closed.
(3) Every $g$-$\Lambda_{(\Lambda, \alpha)}$-set is a $\Lambda_{(\Lambda, \alpha)}$-set.

**Proof.** (1) $\Rightarrow$ (2): By Lemma 4.1, for each $x \in X$, the singleton $\{x\}$ is $(\Lambda, \alpha)$-closed or $X - \{x\}$ is $g$-$(\Lambda, \alpha)$-closed. Since $(X, \tau)$ is a $\Lambda_\alpha-T_{\frac{1}{2}}$-space, we have $X - \{x\}$ is $(\Lambda, \alpha)$-closed and hence $\{x\}$ is $(\Lambda, \alpha)$-open in the latter case. Thus, the singleton $\{x\}$ is $(\Lambda, \alpha)$-open or $(\Lambda, \alpha)$-closed.
(2) $\Rightarrow$ (3): Suppose that there exists a $g\Lambda_{\Lambda, \alpha}$-set $A$ which is not a $\Lambda_{\Lambda, \alpha}$-set. Then, there exists $x \in \Lambda_{\Lambda, \alpha}(A)$ such that $x \notin A$. In case the singleton $\{x\}$ is $(\Lambda, \alpha)$-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is $(\Lambda, \alpha)$-closed. Since $A$ is a $g\Lambda_{\Lambda, \alpha}$-set, $\Lambda_{\Lambda, \alpha}(A) \subseteq X - \{x\}$. This is a contradiction. In case the singleton $\{x\}$ is $(\Lambda, \alpha)$-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is $(\Lambda, \alpha)$-open. By Lemma 4.2,

$$\Lambda_{\Lambda, \alpha}(A) \subseteq \Lambda_{\Lambda, \alpha}(X - \{x\}) = X - \{x\}.$$

This is a contradiction. Therefore, every $g\Lambda_{\Lambda, \alpha}$-set is a $\Lambda_{\Lambda, \alpha}$-set.

(3) $\Rightarrow$ (1): Suppose that $(X, \tau)$ is not a $\Lambda_{\alpha}-T_{1/2}$-space. There exists a $g(\Lambda, \alpha)$-closed set $A$ which is not $(\Lambda, \alpha)$-closed. Since $A$ is not $(\Lambda, \alpha)$-closed, there exists a point $x \in A^{(\Lambda, \alpha)}$ such that $x \notin A$. By Lemma 4.3, the singleton $\{x\}$ is $(\Lambda, \alpha)$-open or $X - \{x\}$ is a $g\Lambda_{\Lambda, \alpha}$-set. (a) In case $\{x\}$ is $(\Lambda, \alpha)$-open, since $x \in A^{(\Lambda, \alpha)}$, $\{x\} \cap A \neq \emptyset$ and $x \in A$. This is a contradiction. (b) In case $X - \{x\}$ is a $\Lambda_{\Lambda, \alpha}$-set, if $\{x\}$ is not $(\Lambda, \alpha)$-closed, $X - \{x\}$ is not $(\Lambda, \alpha)$-open and $\Lambda_{\Lambda, \alpha}(X - \{x\}) = X$. Thus, $X - \{x\}$ is not a $\Lambda_{\Lambda, \alpha}$-set. This contradicts (3). If $\{x\}$ is $(\Lambda, \alpha)$-closed, $A \subseteq X - \{x\} \subseteq \Lambda_{\alpha}O(X, \tau)$ and $A$ is $g(\Lambda, \alpha)$-closed. Thus, $A^{(\Lambda, \alpha)} \subseteq X - \{x\}$. This contradicts that $x \in A^{(\Lambda, \alpha)}$. Therefore, $(X, \tau)$ is a $\Lambda_{\alpha}-T_{1/2}$-space.

\[\square\]

**Definition 4.3.** A topological space $(X, \tau)$ is said to be $(\Lambda, \alpha)$-normal if for any pair of disjoint $(\Lambda, \alpha)$-closed sets $F$ and $H$, there exist disjoint $(\Lambda, \alpha)$-open sets $U$ and $V$ such that $F \subseteq U$ and $H \subseteq V$.

**Lemma 4.4.** Let $(X, \tau)$ be a topological space. If $U$ is a $(\Lambda, \alpha)$-open set, then $U^{(\Lambda, \alpha)} \cap A \subseteq [U \cap A]^{(\Lambda, \alpha)}$ for every subset $A$ of $X$.

**Theorem 4.2.** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $(\Lambda, \alpha)$-normal.
2. For every pair of $(\Lambda, \alpha)$-open sets $U$ and $V$ whose union is $X$, there exist $(\Lambda, \alpha)$-closed sets $F$ and $H$ such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.
3. For every $(\Lambda, \alpha)$-closed set $F$ and every $(\Lambda, \alpha)$-open set $G$ containing $F$, there exists a $(\Lambda, \alpha)$-open set $U$ such that $F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G$.
4. For every pair of disjoint $(\Lambda, \alpha)$-closed sets $F$ and $H$, there exist disjoint $(\Lambda, \alpha)$-open sets $U$ and $V$ such that $F \subseteq U$ and $H \subseteq V$ and $U^{(\Lambda, \alpha)} \cap V^{(\Lambda, \alpha)} = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2): Let $U$ and $V$ be a pair of $(\Lambda, \alpha)$-open sets such that $X = U \cup V$. Then, $X - U$ and $X - V$ are disjoint $(\Lambda, \alpha)$-closed sets. Since $(X, \tau)$ is $(\Lambda, \alpha)$-normal, there exist disjoint $(\Lambda, \alpha)$-open sets $G$ and $W$ such that $X - U \subseteq G$ and $X - V \subseteq W$. Put $F = X - G$ and $H = X - W$. Then, $F$ and $H$ are $(\Lambda, \alpha)$-closed sets such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.

(2) $\Rightarrow$ (3): Let $F$ be a $(\Lambda, \alpha)$-closed set and $G$ be a $(\Lambda, \alpha)$-open set containing $F$. Then, $X - F$ and $G$ are $(\Lambda, \alpha)$-open sets whose union is $X$. Then by (2), there exist $(\Lambda, \alpha)$-closed sets $M$ and $N$ such that $M \subseteq X - F$, $N \subseteq G$ and $M \cup N = X$. Then, $F \subseteq X - M$, $X - G \subseteq X - N$ and $(X - M) \cap (X - N) = \emptyset$. Put $U = X - M$ and $V = X - N$. Then $U$ and $V$ are disjoint $(\Lambda, \alpha)$-open
sets such that \( F \subseteq U \subseteq X - V \subseteq G \). As \( X - V \) is a \((\Lambda, \alpha)\)-closed set, we have \( U^{(\Lambda, \alpha)} \subseteq X - V \) and hence \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G \).

(3) \implies (4): Let \( F \) and \( H \) be two disjoint \((\Lambda, \alpha)\)-closed sets of \( X \). Then, \( F \subseteq X - H \) and \( X - H \) is \((\Lambda, \alpha)\)-open and hence there exists a \((\Lambda, \alpha)\)-open set \( U \) of \( X \) such that \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq X - H \). Put \( V = X - U^{(\Lambda, \alpha)} \). Then, \( U \) and \( V \) are disjoint \((\Lambda, \alpha)\)-open sets of \( X \) such that \( F \subseteq U \), \( H \subseteq V \) and \( U^{(\Lambda, \alpha)} \cap V^{(\Lambda, \alpha)} = \emptyset \).

(4) \implies (1): The proof is obvious. \[\square\]

**Theorem 4.3.** For a topological space \((X, \tau)\), the following properties are equivalent:

\begin{enumerate}
\item \((X, \tau)\) is \((\Lambda, \alpha)\)-normal.
\item For every pair of disjoint \((\Lambda, \alpha)\)-closed sets \( F \) and \( H \) of \( X \), there exist disjoint \( g-(\Lambda, \alpha)\)-open sets \( U \) and \( V \) of \( X \) such that \( F \subseteq U \) and \( H \subseteq V \).
\item For each \((\Lambda, \alpha)\)-closed set \( F \) and each \((\Lambda, \alpha)\)-open set \( G \) containing \( F \), there exists a \( g-(\Lambda, \alpha)\)-open set \( U \) such that \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G \).
\item For each \((\Lambda, \alpha)\)-closed set \( F \) and each \( g-(\Lambda, \alpha)\)-open set \( G \) containing \( F \), there exists a \((\Lambda, \alpha)\)-open set \( U \) such that \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G_{(\Lambda, \alpha)} \).
\item For each \((\Lambda, \alpha)\)-closed set \( F \) and each \( g-(\Lambda, \alpha)\)-open set \( G \) containing \( F \), there exists a \((\Lambda, \alpha)\)-open set \( U \) such that \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G_{(\Lambda, \alpha)} \).
\item For each \((\Lambda, \alpha)\)-closed set \( F \) and each \((\Lambda, \alpha)\)-open set \( G \) containing \( F \), there exists a \((\Lambda, \alpha)\)-open set \( U \) such that \( F^{(\Lambda, \alpha)} \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G \).
\item For each \( g-(\Lambda, \alpha)\)-closed set \( F \) and each \((\Lambda, \alpha)\)-open set \( G \) containing \( F \), there exists a \((\Lambda, \alpha)\)-open set \( U \) such that \( F^{(\Lambda, \alpha)} \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G \).
\end{enumerate}

**Proof.** (1) \implies (2): The proof is obvious.

(2) \implies (3): Let \( F \) be a \((\Lambda, \alpha)\)-closed set and \( G \) be a \((\Lambda, \alpha)\)-open set containing \( F \). Then, we have \( F \) and \( X - G \) are two disjoint \((\Lambda, \alpha)\)-closed sets. Hence by (2), there exist disjoint \( g-(\Lambda, \alpha)\)-open sets \( U \) and \( V \) of \( X \) such that \( F \subseteq U \) and \( X - G \subseteq V \). Since \( V \) is \( g-(\Lambda, \alpha)\)-open and \( X - G \) is \((\Lambda, \alpha)\)-closed, by Theorem 3.5, \( X - G \subseteq V_{(\Lambda, \alpha)} \). Thus, \( [X - V]^{(\Lambda, \alpha)} = X - V_{(\Lambda, \alpha)} \subseteq G \) and hence \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G \).

(3) \implies (1): Let \( F \) and \( H \) be two disjoint \((\Lambda, \alpha)\)-closed sets of \( X \). Then, \( F \) is a \((\Lambda, \alpha)\)-closed set and \( X - H \) is a \((\Lambda, \alpha)\)-open set containing \( F \). Thus by (3), there exists a \( g-(\Lambda, \alpha)\)-open set \( U \) such that \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq X - H \). By Theorem 3.5, \( F \subseteq U_{(\Lambda, \alpha)} \), \( H \subseteq X - U^{(\Lambda, \alpha)} \), where \( U_{(\Lambda, \alpha)} \) and \( X - U^{(\Lambda, \alpha)} \) are two disjoint \((\Lambda, \alpha)\)-open sets.

(4) \implies (5) and (5) \implies (2): The proofs are obvious.

(6) \implies (7) and (7) \implies (3): The proofs are obvious.

(3) \implies (5): Let \( F \) be a \((\Lambda, \alpha)\)-closed set and \( G \) be a \( g-(\Lambda, \alpha)\)-open set containing \( F \). Since \( G \) is \( g-(\Lambda, \alpha)\)-open and \( F \) is \((\Lambda, \alpha)\)-closed, by Theorem 3.5, \( F \subseteq G_{(\Lambda, \alpha)} \) and by (3), there exists a \( g-(\Lambda, \alpha)\)-open set \( U \) such that \( F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G_{(\Lambda, \alpha)} \).
(5) ⇒ (6): Let $F$ be a $(\Lambda, \alpha)$-closed set and $G$ be a $(\Lambda, \alpha)$-open set containing $F$. Then, $F^{(\Lambda, \alpha)} \subseteq G$. Since $G$ is $g-(\Lambda, \alpha)$-open, by (6), there exists a $g-(\Lambda, \alpha)$-open set $U$ such that $F^{(\Lambda, \alpha)} \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G$. Since $U$ is $g-(\Lambda, \alpha)$-open and $F^{(\Lambda, \alpha)} \subseteq U$, by Theorem 3.5, $F^{(\Lambda, \alpha)} \subseteq U^{(\Lambda, \alpha)}$. Put $V = U^{(\Lambda, \alpha)}$. Then, $V$ is $(\Lambda, \alpha)$-open and $F^{(\Lambda, \alpha)} \subseteq V \subseteq V^{(\Lambda, \alpha)} = [U^{(\Lambda, \alpha)}]^{(\Lambda, \alpha)} \subseteq U^{(\Lambda, \alpha)} \subseteq G$.

(6) ⇒ (4): Let $F$ be a $(\Lambda, \alpha)$-closed set and $G$ be a $g-(\Lambda, \alpha)$-open set containing $F$. Then by Theorem 3.5, $F^{(\Lambda, \alpha)} = F \subseteq G^{(\Lambda, \alpha)}$. Since $F$ is $g-(\Lambda, \alpha)$-closed and $G^{(\Lambda, \alpha)}$ is $(\Lambda, \alpha)$-open, by (6), there exists a $(\Lambda, \alpha)$-open set $U$ such that $F^{(\Lambda, \alpha)} = F \subseteq U \subseteq U^{(\Lambda, \alpha)} \subseteq G^{(\Lambda, \alpha)}$. □

**Definition 4.4.** A topological space $(X, \tau)$ is said to be $(\Lambda, \alpha)$-regular if for each $(\Lambda, \alpha)$-closed set $F$ of $X$ not containing $x$, there exist disjoint $(\Lambda, \alpha)$-closed sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Theorem 4.4.** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $(\Lambda, \alpha)$-regular.
2. For each $x \in X$ and each $U \in \Lambda_\alpha O(X, \tau)$ with $x \in U$, there exists $V \in \Lambda_\alpha O(X, \tau)$ such that $x \in V \subseteq V^{(\Lambda, \alpha)} \subseteq U$.
3. For each $(\Lambda, \alpha)$-closed set $F$ of $X$, $\bigcap \{V^{(\Lambda, \alpha)} \mid F \subseteq V \in \Lambda_\alpha O(X, \tau)\} = F$.
4. For each subset $A$ of $X$ and each $U \in \Lambda_\alpha O(X, \tau)$ with $A \cap U \neq \emptyset$, there exists $V \in \Lambda_\alpha O(X, \tau)$ such that $A \cap V \neq \emptyset$ and $V^{(\Lambda, \alpha)} \subseteq U$.
5. For each nonempty subset $A$ of $X$ and each $(\Lambda, \alpha)$-closed set $F$ of $X$ with $A \cap F = \emptyset$, there exist $V, W \in \Lambda_\alpha O(X, \tau)$ such that $A \cap V \neq \emptyset$, $F \subseteq W$ and $V \cap W = \emptyset$.
6. For each $(\Lambda, \alpha)$-closed set $F$ of $X$ and $x \notin F$, there exist $U \in \Lambda_\alpha O(X, \tau)$ and a $g-(\Lambda, \alpha)$-open set $V$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.
7. For each subset $A$ of $X$ and each $(\Lambda, \alpha)$-closed set $F$ with $A \cap F = \emptyset$, there exist $U \in \Lambda_\alpha O(X, \tau)$ and a $g-(\Lambda, \alpha)$-open set $V$ such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

**Proof.** (1) ⇒ (2): Let $G \in \Lambda_\alpha O(X, \tau)$ and $x \notin X - G$. Then, there exist disjoint $U, V \in \Lambda_\alpha O(X, \tau)$ such that $X - G \subseteq U$ and $x \in V$. Thus, $V \subseteq X - U$ and so $x \in V \subseteq V^{(\Lambda, \alpha)} \subseteq X - U \subseteq G$.

(2) ⇒ (3): Let $X - F \in \Lambda_\alpha O(X, \tau)$ with $x \in X - F$. Then by (2), there exists $U \in \Lambda_\alpha O(X, \tau)$ such that $x \in U \subseteq U^{(\Lambda, \alpha)} \subseteq X - F$. Thus, $F \subseteq X - U^{(\Lambda, \alpha)} = V \in \Lambda_\alpha O(X, \tau)$ and hence $U \cap V = \emptyset$. Then, we have $x \notin V^{(\Lambda, \alpha)}$. This shows that $F \supseteq \bigcap \{V^{(\Lambda, \alpha)} \mid F \subseteq V \in \Lambda_\alpha O(X, \tau)\}$.

(3) ⇒ (4): Let $A$ be a subset of $X$ and $U \in \Lambda_\alpha O(X, \tau)$ such that $A \cap U \neq \emptyset$. Let $x \in A \cap U$. Then, $x \notin X - U$. Hence by (3), there exists $W \in \Lambda_\alpha O(X, \tau)$ such that $X - U \subseteq W$ and $x \notin W^{(\Lambda, \alpha)}$. Put $V = X - W^{(\Lambda, \alpha)}$ which is a $(\Lambda, \alpha)$-open set containing $x$ and $A \cap V \neq \emptyset$. Now, $V \subseteq X - W$ and so $V^{(\Lambda, \alpha)} \subseteq X - W \subseteq U$.

(4) ⇒ (5): Let $A$ be a nonempty subset of $X$ and $F$ be a $(\Lambda, \alpha)$-closed set such that $A \cap F = \emptyset$. Then, $X - F \in \Lambda_\alpha O(X, \tau)$ with $A \cap (X - F) \neq \emptyset$ and hence by (4), there exists $V \in \Lambda_\alpha O(X, \tau)$ such that $A \cap V \neq \emptyset$ and $V^{(\Lambda, \alpha)} \subseteq X - F$. If we put $W = X - V^{(\Lambda, \alpha)}$, then $F \subseteq W$ and $W \cap V = \emptyset$. 


(5) ⇒ (1): Let $F$ be a $(\Lambda, \alpha)$-closed set not containing $x$. Then, $F \cap \{x\} = \emptyset$. Thus by (5), there exist $V, W \in \Lambda_\alpha O(X, \tau)$ such that $x \in V$, $F \subseteq W$ and $V \cap W = \emptyset$.

(1) ⇒ (6): The proof is obvious.

(6) ⇒ (7): Let $A$ be a subset of $X$ and $F$ be a $(\Lambda, \alpha)$-closed set such that $A \cap F = \emptyset$. Then, for $x \in A$, $x \notin F$ and by (6), there exist $U \in \Lambda_\alpha O(X, \tau)$ and a $g$-$(\Lambda, \alpha)$-open set $V$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Thus, $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(7) ⇒ (1): Let $F$ be a $(\Lambda, \alpha)$-closed set such that $x \notin F$. Since $\{x\} \cap F = \emptyset$, by (7), there exist $U \in \Lambda_\alpha O(X, \tau)$ and a $g$-$(\Lambda, \alpha)$-open set $W$ such that $x \in U$, $F \subseteq V$ and $U \cap W = \emptyset$. Since $W$ is $g$-$(\Lambda, \alpha)$-open, by Theorem 3.5, we have $F \subseteq W(\Lambda, \alpha) = V \in \Lambda_\alpha O(X, \tau)$ and hence $U \cap V = \emptyset$. This shows that $(X, \tau)$ is $(\Lambda, \alpha)$-regular. □

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References


