GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A QUASILINEAR PARABOLIC EQUATION WITH ABSORPTION AND NONLINEAR BOUNDARY CONDITION

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Abstract. This paper deals with the evolution $r$-Laplacian equation with absorption and nonlinear boundary condition. By using differential inequality techniques, global existence and blow-up criteria of nonnegative solutions are determined. Moreover, upper bound of the blow-up time for the blow-up solution is obtained.

1. Introduction

In this paper, we investigate the global existence and finite time blow-up of nonnegative solutions for the following initial-boundary value problem

$$
\begin{align*}
\begin{cases}
  u_t = \text{div}(|-u|^{r-2} \nabla u) - f(u), & (x, t) \in \Omega \times (0, t^*), \\
  |u|^{r-2} \frac{\partial u}{\partial n} = g(u), & (x, t) \in \partial \Omega \times (0, t^*), \\
  u(x, 0) = u_0(x) > 0, & x \in \Omega,
\end{cases}
\end{align*}
$$

(1.1)

where $r \geq 2$, $\frac{\partial u}{\partial n}$ is the outward normal derivative of $u$ on the boundary $\partial \Omega$ assumed sufficiently smooth, $\Omega$ is a bounded star-shaped region in $\mathbb{R}^N$ ($N \geq 2$) and $t^*$ is the blow-up time if blow-up occurs, or else $t^* = \infty$. It is well known that the functions $f$ and $g$ may greatly affect the behavior of the solution $u(x, t)$ with the development of time. From the physical standpoint, $-f$ is the cold source function, $g$ is the heat-conduction function transmitting into interior of $\Omega$ from the boundary of $\Omega$.

The global existence and blow-up for nonlinear parabolic equations have been extensively investigated by many authors in the last decades (see [1–6] and the references therein). In recent years, many authors have also studied bounds for the blow-up time in nonlinear parabolic problems by using differential inequality techniques (see [7–12]). In particular, Payne et al. [13] considered the following semilinear heat equation with nonlinear boundary condition

$$
\begin{align*}
\begin{cases}
  u_t = \Delta u - f(u), & (x, t) \in \Omega \times (0, t^*), \\
  \frac{\partial u}{\partial n} = g(u), & (x, t) \in \partial \Omega \times (0, t^*), \\
  u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\end{align*}
$$

(1.2)

2010 Mathematics Subject Classification. 35K55, 35K65.

Key words and phrases. Global existence; Blow-up; Quasilinear parabolic equation; Nonlinear boundary condition.
and established sufficient conditions on the nonlinearities to guarantee that the solution \( u(x, t) \) exists for all time \( t > 0 \) or blows up in finite time \( t^* \). Moreover, an upper bound for \( t^* \) was derived. Under more restrictive conditions, a lower bound for \( t^* \) was also obtained.

Moreover, in [14], Payne et al. also studied the following initial-boundary problem

\[
\begin{aligned}
  &u_t = \nabla(|\nabla u|^{2p} \nabla u), & (x, t) &\in \Omega \times (0, t^*), \\
  &|\nabla u|^{2p} \frac{\partial u}{\partial n} = f(u), & (x, t) &\in \partial \Omega \times (0, t^*), \\
  &u(x, 0) = u_0(x), & x &\in \Omega,
\end{aligned}
\]

and obtained upper and lower bounds for the blow-up time under some conditions when blow-up does occur at some finite time.

In the present work, by using differential inequality techniques, we give some sufficient conditions on the functions \( f \) and \( g \) for the global existence and blow-up of nonnegative solutions to problem (1.1). Our main results are stated as follows.

**Theorem 1.1.** (Conditions for global existence). Let \( u(x, t) \) be the solution of problem (1.1) and assume that the non-negative functions \( f \) and \( g \) satisfy the following conditions

\[
\begin{align*}
  f(\xi) &\geq k_1 \xi^p, & \xi &\geq 0, \\
  g(\xi) &\leq k_2 \xi^q, & \xi &\geq 0,
\end{align*}
\]

for some non-negative constants \( k_1 \) and \( k_2 \). Moreover suppose that the positive constants \( p \) and \( q \) satisfy the following conditions

\[
p > q > r - 1 \quad \text{and} \quad rq < (r - 1)(p + 1).
\]

Then the non-negative solution \( u(x, t) \) of problem (1.1) exists globally for all time \( t > 0 \).

**Theorem 1.2.** (Conditions for blow-up in finite time). Let \( u(x, t) \) be the solution of problem (1.1) and assume that the non-negative functions \( f \) and \( g \) satisfy the following conditions

\[
\begin{align*}
  \xi f(\xi) &\leq r F(\xi), & \xi &\geq 0, \\
  \xi g(\xi) &\geq r G(\xi), & \xi &\geq 0,
\end{align*}
\]

with

\[
F(\xi) = \int_0^\xi f(\eta) d\eta, \quad G(\xi) = \int_0^\xi g(\eta) d\eta.
\]

Moreover suppose that \( \Psi(0) > 0 \), where

\[
\Psi(t) = r \int_{\partial \Omega} G(u) ds - \int_\Omega |\nabla u|^r dx - r \int_\Omega F(u) dx.
\]

Then the solution \( u(x, t) \) of problem (1.1) blows up at time \( t^* < T \) with

\[
T = \frac{\Phi(0)}{(r - 2)\Psi(0)}, \quad \text{for} \ r > 2,
\]
where $\Phi(t) = \int_\Omega u^2 dx$. If $r = 2$, we have $T = \infty$.

This paper is organized as follows. In Section 2, we establish the conditions on the functions $f$ and $g$, which guarantee that $u(x,t)$ exists globally, and prove Theorem 1.1. In Section 3, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time $t^*$.

2. Conditions for global existence

In this section, we establish the sufficient conditions on the functions $f$ and $g$, which guarantee that $u(x,t)$ exists globally, and prove Theorem 1.1. To do this, we need the following Lemma.

**Lemma 2.1.** Let $\Omega$ be a bounded star-shaped domain in $\mathbb{R}^N$, $N \geq 2$. Then for any non-negative $C^1$ function $u$ and $\gamma > 0$, we have

$$
\int_{\partial \Omega} u^\gamma ds \leq \frac{N}{\rho_0} \int_\Omega u^\gamma dx + \frac{\gamma d}{\rho_0} \int_\Omega u^{\gamma-1} |\nabla u| dx,
$$

where

$$
\rho_0 = \min_{x \in \partial \Omega} (x \cdot n) \quad \text{and} \quad d = \max_{x \in \partial \Omega} |x|.
$$

**Proof.** As $\Omega$ is a bounded star-shaped domain, it is easy to see that $\rho_0 > 0$. Integrating the identity

$$
\text{div}(u^\gamma x) = Nu^\gamma + \gamma u^{\gamma-1}(x \cdot \nabla u)
$$

over $\Omega$, it follows from the divergence theorem that

$$
\int_{\partial \Omega} u^\gamma (x \cdot n) ds = N \int_\Omega u^\gamma dx + \gamma \int_\Omega u^{\gamma-1}(x \cdot \nabla u) dx.
$$

By the definition of $\rho_0$ and $d$, we obtain

$$
\rho_0 \int_{\partial \Omega} u^\gamma ds \leq \int_{\partial \Omega} u^\gamma (x \cdot n) ds \leq N \int_\Omega u^\gamma dx + \gamma d \int_\Omega u^{\gamma-1} |\nabla u| dx,
$$

which implies the desired conclusion.

**Proof of Theorem 1.1.** Setting

$$
\Phi(t) = \int_\Omega u^2 dx,
$$

then it follows from (1.1), (1.4) and (1.5) that

$$
\Phi'(t) = 2 \int_\Omega uu_t dx = 2 \int_\Omega u [\text{div}(|\nabla u|^{r-2} \nabla u) - f(u)] dx
$$

$$
= 2 \int_{\partial \Omega} u |\nabla u|^{r-2} \frac{\partial u}{\partial n} ds - 2 \int_\Omega |\nabla u|^r dx - 2 \int_\Omega uf(u) dx
$$

$$
= 2 \int_{\partial \Omega} ug(u) ds - 2 \int_\Omega |\nabla u|^r dx - 2 \int_\Omega uf(u) dx
$$

$$
\leq 2k_2 \int_{\partial \Omega} u^{q+1} ds - 2 \int_\Omega |\nabla u|^r dx - 2k_1 \int_\Omega u^{p+1} dx.
$$
By Lemma 2.1, we have

\begin{equation}
\int_{\partial \Omega} u^{q+1} ds \leq \frac{N}{\rho_0} \int_{\Omega} u^{q+1} dx + \frac{(q+1)d}{\rho_0} \int_{\Omega} u^q |\nabla u| dx,
\end{equation}

where \( \rho_0 \) and \( d \) are given by (2.2). Combining (2.7) with (2.8), we obtain

\begin{equation}
\Phi'(t) \leq \frac{2k_2 N}{\rho_0} \int_{\Omega} u^{q+1} dx + \frac{2k_2(q+1)d}{\rho_0} \int_{\Omega} u^q |\nabla u| dx - 2k_1 \int_{\Omega} u^{p+1} dx.
\end{equation}

By using Young’s inequality with \( \varepsilon > 0 \), we derive

\begin{equation}
\int_{\Omega} u^q |\nabla u| dx \leq \frac{1}{r} \int_{\Omega} |\nabla u|^r dx + \frac{r-1}{r} \varepsilon^{\frac{1}{r}} \int_{\Omega} u^{\frac{r}{r-1}} dx,
\end{equation}

where \( \varepsilon = \frac{k_2(q+1)d}{r\rho_0} > 0 \). It follows from (2.9) and (2.10) that

\begin{equation}
\Phi'(t) \leq \frac{2k_2 N}{\rho_0} \int_{\Omega} u^{q+1} dx + 2(r-1) \left( \frac{k_2(q+1)d}{r\rho_0} \right)^{\frac{r}{r-1}} \int_{\Omega} u^{\frac{r}{r-1}} dx - 2k_1 \int_{\Omega} u^{p+1} dx.
\end{equation}

By Hölder’s inequality, we have

\begin{equation}
\int_{\Omega} u^{\frac{r}{r-1}} dx \leq \left( \int_{\Omega} u^{q+1} dx \right)^{\alpha} \left( \int_{\Omega} u^{p+1} dx \right)^{1-\alpha},
\end{equation}

where \( \alpha = \frac{(r-1)(p+1)-q}{r(p-q)} \in (0,1) \), due to (1.6). By using the fundamental inequality

\begin{equation}
a_1^2 + a_2^2 \leq r_1 a_1 + r_2 a_2, \quad a_1, a_2 > 0, \quad r_1, r_2 \geq 0 \quad \text{and} \quad r_1 + r_2 = 1,
\end{equation}

it follows from (2.12) that

\begin{equation}
\int_{\Omega} u^{\frac{r}{r-1}} dx \leq \left( \kappa^{\frac{a-1}{a}} \int_{\Omega} u^{q+1} dx \right)^{\alpha} \left( \kappa \int_{\Omega} u^{p+1} dx \right)^{1-\alpha}
\leq \alpha \kappa^{\frac{a-1}{a}} \int_{\Omega} u^{q+1} dx + (1-\alpha) \kappa \int_{\Omega} u^{p+1} dx,
\end{equation}

where

\begin{equation}
0 < \kappa < \frac{k_1}{(r-1)(1-\alpha)} \left( \frac{k_2(q+1)d}{r \rho_0} \right)^{\frac{r}{r-1}}.
\end{equation}

Combining (2.11) with (2.14), we obtain

\begin{equation}
\Phi'(t) \leq K_1 \int_{\Omega} u^{q+1} dx - K_2 \int_{\Omega} u^{p+1} dx,
\end{equation}

where

\begin{equation}
K_1 = \frac{2k_2 N}{\rho_0} + 2(r-1) \alpha \kappa^{\frac{a-1}{a}} \left( \frac{k_2(q+1)d}{r \rho_0} \right)^{\frac{r}{r-1}} > 0,
\end{equation}

and

\begin{equation}
K_2 = 2k_1 - 2(r-1)(1-\alpha) \kappa \left( \frac{k_2(q+1)d}{r \rho_0} \right)^{\frac{r}{r-1}} > 0,
\end{equation}

due to (2.15). According to Hölder’s inequality, we derive

\begin{equation}
\int_{\Omega} u^{q+1} dx \leq \left( \int_{\Omega} u^{p+1} dx \right)^{\frac{\alpha}{\alpha-1}} \left| \Omega \right|^\frac{p+1}{p+1},
\end{equation}

\end{proof}
where $|\Omega| = \int_{\Omega} dx$ is the $N$-volume of $\Omega$. It follows from (2.16) and (2.19) that

\[
(2.20) \quad \Phi'(t) \leq \left( \int_{\Omega} u^{p+1} dx \right)^{\frac{p+1}{p+1}} \left[ K_1 |\Omega|^{\frac{p-q}{p+1}} - K_2 \left( \int_{\Omega} u^{p+1} dx \right)^{\frac{p-q}{p+1}} \right].
\]

By Hölder’s inequality again, we have

\[
(2.21) \quad \Phi(t) = \int_{\Omega} u^2 dx \leq \left( \int_{\Omega} u^{p+1} dx \right)^{\frac{2}{p+1}} |\Omega|^{\frac{p-2}{p+1}}.
\]

Therefore, we deduce from (2.20) and (2.21) that

\[
(2.22) \quad \Phi'(t) \leq \left( \int_{\Omega} u^{p+1} dx \right)^{\frac{p+1}{p+1}} \left[ K_1 |\Omega|^{\frac{p-q}{p+1}} - K_2 |\Omega|^{\frac{(1-p)(p-q)}{2(p+1)}} \Phi^{\frac{p-q}{2}} \right].
\]

Hence, we infer from (2.22) that $\Phi(t)$ is decreasing in each time interval on which we have

\[
(2.23) \quad \Phi(t) > \left( \frac{K_1}{K_2} \right)^{\frac{2}{p-q}} |\Omega|,
\]

so that $\Phi(t)$ remains bounded for all time under the conditions stated in Theorem 1.1, which completes the proof. □

3. CONDITIONS FOR BLOW-UP IN FINITE TIME

In this section, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time $t^*$.  

**Proof of Theorem 1.2.** Using Green formula and the assumptions stated in Theorem 1.2, we have

\[
(3.1) \quad \Phi'(t) = 2 \int_{\Omega} u u_t dx
\]

\[
= 2 \int_{\Omega} u |\nabla u|^r \nabla u - f(u)|dx
\]

\[
= 2 \int_{\partial \Omega} u |\nabla u|^r \frac{\partial u}{\partial n} ds - 2 \int_{\Omega} |\nabla u|^r dx - 2 \int_{\Omega} u f(u) dx
\]

\[
= 2 \int_{\partial \Omega} u g(u) ds - 2 \int_{\Omega} |\nabla u|^r dx - 2 \int_{\Omega} u f(u) dx
\]

\[
\geq 2r \int_{\partial \Omega} G(u) ds - 2 \int_{\Omega} |\nabla u|^r dx - 2r \int_{\Omega} F(u) dx
\]

\[
\geq 2 \Psi(t).
\]

Differentiating (1.10), we obtain

\[
(3.2) \quad \Psi'(t) = r \int_{\partial \Omega} u_t g(u) ds - \int_{\Omega} (|\nabla u|^r)_t dx - r \int_{\Omega} u_t f(u) dx
\]

\[
= r \int_{\Omega} u_t |\nabla u|^r \nabla u dx - r \int_{\Omega} u_t f(u) dx
\]

\[
= r \int_{\Omega} (u_t)^2 dx \geq 0.
\]
As $\Psi(0) > 0$, then $\Psi(t) > 0$ for all $t \in (0, t^*)$. By using Hölder’s inequality, we derive

$$
(\Phi'(t))^2 = 4 \left( \int_\Omega uu_t dx \right)^2 \leq 4 \int_\Omega u^2 dx \int_\Omega (u_t)^2 dx = \frac{4}{r} \Phi(t) \Psi'(t).
$$

(3.3)

It follows from (3.1) and (3.3) that

$$
\Phi(t) \Psi'(t) \geq \frac{r}{4}(\Phi'(t))^2 \geq \frac{r}{2} \Phi'(t) \Psi(t),
$$

that is

$$
(\Phi^\frac{r}{2} \Psi)'(t) \geq 0.
$$

Integrating from 0 to $t$, we have

$$
\Phi^\frac{r}{2} (t) \Psi(t) \geq \Phi^\frac{r}{2} (0) \Psi(0) =: K > 0.
$$

Therefore, we deduce from (3.1) that

$$
\Phi'(t) \geq 2 \Psi \geq 2K \Phi^\frac{r}{2} (t).
$$

(3.4)

If $r > 2$, it follows from integrating over $(0, t)$ that

$$
\Phi(t) \geq \left[ \frac{\Phi^\frac{2-r}{2} (0)}{K (r-2)t} \right]^{-\frac{2}{r-2}},
$$

which implies $\Phi(t) \to +\infty$ as $t \to T = \frac{\Phi^\frac{2-r}{2} (0)}{K (r-2)}$. Hence, for $r > 2$, we have

$$
t^* \leq \frac{\Phi(0)}{(r-2) \Psi(0)}.
$$

(3.5)

If $r = 2$, we infer from (3.7) that

$$
\Phi(t) \geq \Phi(0) e^{2Kt}, \quad \text{for all} \quad t > 0,
$$

which implies $t^* = \infty$, this completes the proof. \qed

References


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