ON A TYPE OF PROJECTIVE SEMI-SYMMETRIC CONNECTION

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Abstract. In the present paper, we have studied some properties of curvature tensors of special projective semi-symmetric connection. We have shown that curvature tensor of such a connection satisfies Bianchi’s identities.

1. Introduction

The idea of semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten [2] in 1924. In 1932, H. A. Hayden [4] studied semi-symmetric metric connection. It was K. Yano [10] who started systematic study of semi-symmetric metric connection and this was further studied by T. Imai [6], R. S. Mishra and S. N. Pandey [9], U. C. De and B. K. De [1] and several other mathematicians ([7], [11]). In 2001, P. Zhao and H. Song [12] studied a semi-symmetric connection which is projectively equivalent to Levi-Civita connection and such a connection is called as projective semi-symmetric connection. They found an invariant under the transformation of projective semi-symmetric connection and showed that this invariant could degenerate into the Weyl projective curvature tensor under certain conditions. After this various papers ([3], [5], [13]) on projective semi-symmetric metric connection have appeared.

The organization of the paper is as follows. After introduction we give some preliminary results in section 2. In sections 3, we present a brief account of special projective semi-symmetric connection. Section 4 is devoted to the study of special projective semi symmetric connection with recurrent curvature tensor.

2. Preliminaries

Let \( M^n \) be an \( n \)-dimensional \((n > 2)\) Riemannian manifold equipped with a Riemannian metric \( g \) and \( \nabla \) be the Levi-Civita connection associated with metric \( g \). A linear connection \( \nabla \) on \( M^n \) is called the semi symmetric metric connection [10], if the torsion tensor \( \tilde{T} \) of the connection \( \nabla \), given by

\[
\tilde{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]
\]

satisfies the condition

\[
\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y
\]
and
\begin{equation}
(\bar{\nabla}_X g)(Y, Z) = 0,
\end{equation}
where \( \pi \) is a 1-form on \( M^n \) associated with vector field \( \rho \), i.e.,
\begin{equation}
\pi(X) = g(X, \rho).
\end{equation}
If the geodesic with respect to \( \bar{\nabla} \) are always consistent with those of \( \nabla \), then \( \bar{\nabla} \) is called a connection projectively equivalent to \( \nabla \). If \( \bar{\nabla} \) is projective equivalent connection to \( \nabla \) as well as the semi-symmetric, then \( \bar{\nabla} \) is called projective semi-symmetric connection. We also call \( \bar{\nabla} \) as projective semi-symmetric transformation.

In this paper, we study a type of projective semi-symmetric connection \( \bar{\nabla} \) introduced by P. Zhao and H. Song [12]. The connection is given by
\begin{equation}
\bar{\nabla}_X Y = \nabla_X Y + \psi(Y)X + \psi(X)Y + \phi(Y)X - \phi(X)Y,
\end{equation}
where 1-forms \( \phi \) and \( \psi \) are given as
\begin{equation}
\phi(X) = \frac{1}{2} \pi(X) \quad \text{and} \quad \psi(X) = \frac{n-1}{2(n+1)} \pi(X).
\end{equation}
It is easy to observe that torsion tensor of projective semi-symmetric transformation is same as given by the equation (2.2) and also that
\begin{equation}
(\bar{\nabla}_X g)(Y, Z) = \frac{1}{n+1}[2\pi(X)g(Y, Z) - n\pi(Y)g(Z, X) - n\pi(Z)g(X, Y)],
\end{equation}
i.e., the connection \( \bar{\nabla} \) is a non metric one.

Let \( \bar{R} \) and \( R \) be the curvature tensor of the manifold relative to the projective semi-symmetric connection \( \bar{\nabla} \) and Levi-Civita connection \( \nabla \) respectively. It is known that [12]
\begin{equation}
\bar{R}(X, Y, Z) = R(X, Y, Z) + \beta(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X,
\end{equation}
where \( \beta(X, Y) \) and \( \alpha(X, Y) \) are given by the following relations
\begin{equation}
\beta(X, Y) = \Psi'(X, Y) - \Psi'(Y, X) + \Phi'(Y, X) - \Phi'(X, Y),
\end{equation}
\begin{equation}
\alpha(X, Y) = \Psi'(X, Y) + \Phi'(Y, X) - \psi(X)\phi(Y) - \phi(X)\psi(Y),
\end{equation}
and
\begin{equation}
\Phi'(X, Y) = (\nabla_X \psi)(Y) - \psi(X)\psi(Y).
\end{equation}
Contracting \( X \) in the equation (2.8), we get a relation between Ricci tensors \( \bar{Ric}(Y, Z) \) and \( Ric(Y, Z) \) of manifold with respect to connections \( \bar{\nabla} \) and \( \nabla \) respectively
\begin{equation}
\bar{Ric}(Y, Z) = Ric(Y, Z) + \beta(Y, Z) - (n-1)\alpha(Y, Z).
\end{equation}
If \( \bar{r} \) and \( r \) are scalar curvatures of manifold with respect to connection \( \bar{\nabla} \) and \( \nabla \) respectively, then from the equation (2.13), we get
\begin{equation}
\bar{r} = r + b - (n-1)a,
\end{equation}
where
\[ b = \sum_{i=1}^{n} \beta(e_i, e_i) \quad \text{and} \quad a = \sum_{i=1}^{n} \alpha(e_i, e_i). \]

The Weyl-projective curvature tensor \( W \), conharmonic curvature tensor \( P \) and concircular curvature tensor \( I \) are given by [9]

\[ W(X, Y, Z) = R(X, Y, Z) + \frac{1}{n-1} \{ Ric(X, Z)Y - Ric(Y, Z)X \}, \tag{2.15} \]

\[ P(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-2} \left[ Ric(Y, Z)X - Ric(X, Z)Y \right. \]
\[ + g(Y, Z)QX - g(X, Z)QY \left]. \tag{2.16} \]

where
\[ g(QX, Y) = Ric(X, Y) \tag{2.17} \]

and
\[ I(X, Y, Z) = R(X, Y, Z) - \frac{r}{n-1} \{ g(Y, Z)X - g(X, Z)Y \}. \tag{2.18} \]

3. Special Projective Semi-Symmetric Connection

In this section, we consider a projective semi-symmetric connection \( \bar{\nabla} \) given by the equation (2.5) whose associated 1-form \( \pi \) is closed, i.e.,

\[ (\bar{\nabla}_X \pi)Y = (\bar{\nabla}_Y \pi)X. \tag{3.1} \]

Such a connection \( \bar{\nabla} \) is called special projective semi-symmetric connection [12]. It is easy to verify that both the 1-forms \( \phi \) and \( \psi \) are closed as the 1-form \( \pi \) is closed and also that the tensors \( \Phi' \) and \( \Psi' \) are symmetric. Consequently, we get

\[ \beta(X, Y) = 0 \tag{3.2} \]

and

\[ \alpha(X, Y) = \alpha(Y, X). \tag{3.3} \]

In view of the equations (3.1) and (3.2), the expressions (2.8), (2.13) and (2.14) reduces to

\[ \bar{R}(X, Y, Z) = R(X, Y, Z) + \alpha(X, Z)Y - \alpha(Y, Z)X, \tag{3.4} \]

\[ \bar{Ric}(Y, Z) = Ric(Y, Z) - (n-1)\alpha(Y, Z) \tag{3.5} \]

and

\[ \bar{r} = r - (n-1)a. \tag{3.6} \]

It is easy to observe that the Ricci tensor \( \bar{Ric}(Y, Z) \) is symmetric.

Now, we prove the following theorems:

**Theorem 3.1.** Curvature tensor of special projective semi-symmetric connection satisfies Bianchi’s first identity.
**Proof**: Writing two more equations by cyclic permutations of $X$, $Y$ and $Z$ from equation (3.4), we get

\[ \bar{R}(Y, Z, X) = R(Y, Z, X) + \alpha(Y, X)Z - \alpha(Z, X)Y, \]

and

\[ \bar{R}(Z, X, Y) = R(Z, X, Y) + \alpha(Z, Y)X - \alpha(X, Y)Z. \]

Adding these equations to the equation (3.4), we get result.

**Theorem 3.2.** Curvature tensor of special projective semi-symmetric connection satisfies Bianchi’s second identity if $\alpha$ is parallel tensor with respect to Levi-Civita connection $\nabla$.

**Proof**: Suppose $\alpha$ is a parallel tensor with respect to Levi-Civita connection $\nabla$, i.e., $\nabla \alpha = 0$. Now differentiating the equation (3.4) covariantly with respect to the connection $\nabla$, we have

\[ (\nabla X\bar{R})(Y, Z, U) = (\nabla X R)(Y, Z, U). \]

Writing two more equations by cyclic permutations of $X$, $Y$ and $Z$ in above equation, we get

\[ (\nabla Y\bar{R})(Z, X, U) = (\nabla Y R)(Z, X, U), \]

and

\[ (\nabla Z\bar{R})(X, Y, U) = (\nabla Z R)(X, Y, U). \]

Adding the equations (3.7), (3.8) and (3.9), we get

\[ (\nabla X\bar{R})(Y, Z, U) + (\nabla Y\bar{R})(Z, X, U) + (\nabla Z\bar{R})(X, Y, U) = 0. \]

This shows that the curvature tensor of special projective semi-symmetric connection satisfies Bianchi’s second identity.

**Theorem 3.3.** The Weyl-projective curvature tensor of Riemannian manifold with respect to the special projective semi-symmetric connection $\bar{\nabla}$ satisfies

\[ \bar{W}(X, Y, Z) + \bar{W}(Y, Z, X) + \bar{W}(Z, X, Y) = 0. \]

**Proof**: The Weyl-projective curvature tensor of Riemannian Manifold with respect to special projective semi-symmetric connection $\bar{\nabla}$ is given by

\[ \bar{W}(X, Y, Z) = \bar{R}(X, Y, Z) - \frac{1}{n-1} [\bar{Ric}(Y, Z)X - \bar{Ric}(X, Z)Y]. \]

Writing two more equations by cyclic permutations of $X$, $Y$ and $Z$ in above equation, we get

\[ \bar{W}(Y, Z, X) = \bar{R}(Y, Z, X) - \frac{1}{n-1} [\bar{Ric}(Z, X)Y - \bar{Ric}(X, Z)Y], \]

\[ \bar{W}(Z, X, Y) = \bar{R}(Z, X, Y) - \frac{1}{n-1} [\bar{Ric}(X, Y)Z - \bar{Ric}(Z, Y)X]. \]

Adding the equations (3.10), (3.11) and (3.12), we get

\[ \bar{W}(X, Y, Z) + \bar{W}(Y, Z, X) + \bar{W}(Z, X, Y) = 0. \]
4. Special Projective Semi-Symmetric Connection with Recurrent Curvature Tensor

In this section, we consider a special projective semi-symmetric connection \( \tilde{\nabla} \) whose curvature tensor \( \tilde{R} \) is recurrent with respect to the Levi-Civita connection \( \nabla \), i.e.,

\[
(4.1) \quad (\nabla_U \tilde{R})(X, Y, Z) = B(U)\tilde{R}(X, Y, Z),
\]

where \( B \) is a non-zero 1-form.

Differentiating the equation (3.4) covariantly with respect to the Levi-Civita connection \( \nabla \), we get

\[
(4.2) \quad (\nabla_U \tilde{R})(X, Y, Z) = (\nabla_U \tilde{R})(X, Y, Z) + (\nabla_U \alpha)(X, Z)Y - (\nabla_U \alpha)(Y, Z)X.
\]

Contracting \( X \) in above, we have

\[
(4.3) \quad (\nabla_U \tilde{Ric})(Y, Z) = (\nabla_U \tilde{Ric})(Y, Z) - (n - 1)(\nabla_U \alpha)(Y, Z).
\]

Putting \( Y = Z = e_i \) in the above equation and taking summation over \( i \), \( 1 \leq i \leq n \), we get

\[
(4.4) \quad (\nabla_U \tilde{r}) = (\nabla_U r) - (n - 1)(\nabla_U \alpha).
\]

Now the equations (3.4) and (4.2) together give

\[
(4.5) \quad (\nabla_U \tilde{R})(X, Y, Z) - B(U)\tilde{R}(X, Y, Z) = (\nabla_U \tilde{R})(X, Y, Z) - B(U)\tilde{R}(X, Y, Z)
\]

\[
+ [(\nabla_U \alpha)(X, Y) - B(U)\alpha(X, Z)]Y - [(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)]X,
\]

which, in view of the equation (4.1), reduces to

\[
(4.6) \quad (\nabla_U \tilde{R})(X, Y, Z) - B(U)\tilde{R}(X, Y, Z) = [\tilde{\nabla}_U \alpha](Y, Z) - B(U)\alpha(Y, Z)]X
\]

\[
- [\tilde{\nabla}_U \alpha](X, Z) - B(U)\alpha(X, Z)]Y.
\]

Contracting \( X \) in above, we get

\[
(4.7) \quad (\nabla_U \tilde{Ric})(Y, Z) - B(U)\tilde{Ric}(Y, Z) = (n - 1)[(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)].
\]

Further, we obtain

\[
(4.8) \quad (\nabla_U r) - B(U)r = (n - 1)[(\nabla_U \alpha) - B(U)a].
\]

Also, from the equation (2.17), we have

\[
(4.9) \quad g((\nabla_U \tilde{Q})X, Y) = (\nabla_U \tilde{Ric})(X, Y),
\]

which can be written as

\[
(4.10) \quad g((\nabla_U \tilde{Q})X - B(U)\tilde{Q}X, Y) = (\nabla_U \tilde{Ric})(X, Y) - B(U)\tilde{Ric}(X, Y).
\]

Now we prove following theorems:

**Theorem 4.1.** If the curvature tensor of special projective semi-symmetric connection on a Riemannian manifold \( M^n \) is recurrent with respect to the Levi-Civita connection then manifold \( M^n \) is projectively recurrent.

**Proof:** Differentiating the projective curvature tensor \( W \) given by (2.15) covariantly with respect to Levi-Civita connection \( \nabla \), we have

\[
(4.11) \quad (\nabla_U W)(X, Y, Z) = (\nabla_U \tilde{R})(X, Y, Z) + \frac{1}{n - 1} [(\nabla_U \tilde{Ric})(X, Z)Y - (\nabla_U \tilde{Ric})(Y, Z)X].
\]
The above equation gives
\begin{equation}
(\nabla_{U}W)(X,Y,Z) - B(U)W(X,Y,Z) = (\nabla_{U}R)(X,Y,Z) - B(U)R(X,Y,Z)
\end{equation}
\begin{align*}
&+ \frac{1}{n-1}\{[(\nabla_{U}Ric)(X,Y,Z) - B(U)Ric(X,Y,Z)]Y
\neg{[(\nabla_{U}Ric)(Y,Z) - B(U)Ric(Y,Z)]X}.\end{align*}

Using equation (4.6) and (4.7) in above, we get
\begin{equation}
(\nabla_{U}W)(X,Y,Z) = B(U)W(X,Y,Z),
\end{equation}
which proves the statement.

**Theorem 4.2.** A Riemannian manifold $M^n$ admitting a special projective semi-symmetric connection whose curvature tensor and tensor $\alpha$ are recurrent with respect to the Levi-Civita connection, is conharmonically recurrent.

**Proof:** Differentiating covariantly the equation (2.16) with respect to the Levi-Civita connection, we get
\begin{equation}
(\nabla_{U}P)(X,Y,Z) = (\nabla_{U}R)(X,Y,Z) - \frac{1}{n-2}\{[(\nabla_{U}Ric)(Y,Z)X - (\nabla_{U}Ric)(X,Z)Y
\neg{g(Y,Z)(\nabla_{U}Q)X - g(X,Z)(\nabla_{U}Q)Y}],
\end{equation}
From above, we have
\begin{equation}
(\nabla_{U}P)(X,Y,Z) - B(U)P(X,Y,Z) = (\nabla_{U}R)(X,Y,Z) - B(U)R(X,Y,Z)
\end{equation}
\begin{align*}
&+ \frac{1}{n-2}\{[(\nabla_{U}Ric)(Y,Z) - B(U)Ric(Y,Z)]X
\neg{[(\nabla_{U}Ric)(X,Z) - B(U)Ric(X,Z)]Y
\neg{g(Y,Z)((\nabla_{U}Q)X - B(U)QX)
\neg{g(X,Z)((\nabla_{U}Q)Y - B(U)QY)].
\end{align*}
If the tensor $\alpha$ and the curvature tensor of the special projective semi-symmetric connection $\nabla$ are recurrent with respect to the Levi-Civita connection $\nabla$, then from the equations (4.6), (4.7) and (4.10), we get
\begin{equation}
(\nabla_{U}P)(X,Y,Z) = B(U)P(X,Y,Z),
\end{equation}
which shows that manifold is conharmonically recurrent.

**Theorem 4.3.** A Riemannian manifold $M^n$ admitting a special projective semi-symmetric connection whose curvature tensor and tensor $\alpha$ are recurrent with respect to Levi-Civita connection, is concircular recurrent.

**Proof:** Differentiating the concircular curvature tensor $I$ of $M^n$ given by the equation (2.18) covariantly with respect to the Levi-Civita connection $\nabla$, we have
\begin{equation}
(\nabla_{U}I)(X,Y,Z) = (\nabla_{U}R)(X,Y,Z) - \frac{\nabla_{UR}}{(n-1)}\{g(Y,Z)X - g(X,Z)Y].
\end{equation}
From this, we have
\[(\nabla_U I)(X, Y, Z) - B(U)I(X, Y, Z) = (\nabla_U R)(X, Y, Z) - B(U)R(X, Y, Z) \]
\[-\frac{\nabla_U r - B(U)r}{(n - 1)} \{g(Y, Z)X - g(X, Z)Y\}.\]

If the tensor \( \alpha \) and the curvature tensor of the special projective semi-symmetric connection \( \bar{\nabla} \) are recurrent with respect to the Levi-Civita connection \( \nabla \), then from the equations (4.6), (4.7) and (4.8), we get
\[(\nabla_U I)(X, Y, Z) = B(U)I(X, Y, Z).\]

**Theorem 4.4.** Let \( M^n \) be a Riemannian manifold admitting a special projective semi-symmetric connection whose Ricci-tensor is recurrent with respect to the Levi-Civita connection. If the manifold is projectively recurrent with respect to Levi-Civita connection, then the curvature tensor of the special projective semi-symmetric connection is recurrent.

**Proof:** Let the manifold \( M^n \) be projectively recurrent with respect to Levi Civita connection \( \nabla \). Then from the equation (4.12), we have
\[(\nabla_U R)(X, Y, Z) - B(U)R(X, Y, Z) = \frac{1}{n - 1}\{(\nabla_U \nabla)(Y, Z) - B(U)Ric(Y, Z)X\} \]
\[-\{(\nabla_U Ric)(X, Z) - B(U)Ric(X, Z)Y\}.\]

Now, from equations (3.5) and (4.3), we get
\[(\nabla_U \nabla)(Y, Z) - B(U)Ric(Y, Z) = (\nabla_U Ric)(Y, Z) - B(U)Ric(Y, Z)\]
\[-(n - 1)\{(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)\}.\]

Since the Ricci tensor of the special projective semi-symmetric connection \( \nabla \) is recurrent with respect to the Levi-Civita connection \( \nabla \), hence the above equation gives
\[(\nabla_U Ric)(Y, Z) - B(U)Ric(Y, Z) = (n - 1)\{(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)\}.\]

Thus, from the equations (4.17) and (4.19), we get
\[(\nabla_U R)(X, Y, Z) - B(U)R(X, Y, Z) = \{(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)\}X \]
\[-\{(\nabla_U \alpha)(X, Z) - B(U)\alpha(X, Z)\}Y,\]

which, on using in the equation (4.5), gives
\[(\nabla_U \bar{\nabla})(X, Y, Z) = B(U)\bar{\nabla}(X, Y, Z).\]

This proves the statement.

**Theorem 4.5.** Let \( M^n \) be a Riemannian manifold admitting a special projective semi-symmetric connection whose Ricci-tensor is recurrent with respect to the Levi-Civita connection. If the manifold is of constant curvature, then the curvature tensor of the special projective semi-symmetric connection is recurrent with respect to the Levi-Civita connection.
Proof: If the Riemannian manifold $M^n$ is of constant curvature, then we have [9]

\[ R(X, Y, Z) = \frac{1}{n-1} \{ \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \}. \]

(4.22)

Using the above equation in the equation (3.4), we have

\[ \bar{R}(X, Y, Z) = \frac{1}{n-1} \{ [\text{Ric}(Y, Z) - (n-1)\alpha(Y, Z)]X - [\text{Ric}(X, Z) - (n-1)\alpha(X, Z)]Y \}, \]

which, on using the equation (3.5), gives

\[ \bar{R}(X, Y, Z) = \frac{1}{n-1} \{ \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \}. \]

(4.23)

Differentiating the above equation covariantly with respect to the Levi-Civita connection, we have

\[ (\nabla_U \bar{R})(X, Y, Z) = \frac{1}{n-1} \{ (\nabla_U \text{Ric})(Y, Z)X - (\nabla_U \text{Ric})(X, Z)Y \}, \]

which can be written as

\[ (\nabla_U \bar{R})(X, Y, Z) - B(U)\bar{R}(X, Y, Z) = \frac{1}{n-1} \{ [(\nabla_U \text{Ric})(Y, Z) - B(U)\text{Ric}(Y, Z)]X - [(\nabla_U \text{Ric})(X, Z) - B(U)\text{Ric}(X, Z)]Y \}. \]

(4.25)

Since the Ricci tensor of special projective semi-symmetric connection is recurrent with respect to the Levi-Civita connection $\nabla$, hence from the above equation, we have

\[ (\nabla_U \bar{R})(X, Y, Z) = B(U)\bar{R}(X, Y, Z), \]

which proves the statement.

References

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