BEST APPROXIMATION OF THE DUNKL MULTIPLIER OPERATORS $T_{k,\ell,m}$

FETHI SOLTANI

Abstract. We study some class of Dunkl multiplier operators $T_{k,\ell,m}$; and we give for them an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators $T_{k,\ell,m}$ on a Hilbert space $H_{s_{\ell}}$.

1. Introduction

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and norm $|y| := \sqrt{\langle y, y \rangle}$. For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $\sigma_\alpha$ be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to $\alpha$:

$$\sigma_\alpha x := x - \frac{2\langle \alpha, x \rangle}{|\alpha|^2} \alpha.$$

A finite set $\mathcal{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system, if $\mathcal{R} \cap \mathbb{R} \cdot \alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \mathcal{R} = \mathcal{R}$ for all $\alpha \in \mathcal{R}$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \mathcal{R}$. For a root system $\mathcal{R}$, the reflections $\sigma_\alpha$, $\alpha \in \mathcal{R}$, generate a finite group $G$. The Coxeter group $G$ is a subgroup of the orthogonal group $O(d)$. All reflections in $G$, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in \mathcal{R}} H_\alpha$, we fix the positive subsystem $\mathcal{R}_+ := \{\alpha \in \mathcal{R} : \langle \alpha, \beta \rangle > 0\}$. Then for each $\alpha \in \mathcal{R}$ either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$. Let $k, \ell : \mathcal{R} \to \mathbb{C}$ be two multiplicity functions on $\mathcal{R}$ (a functions which are constants on the orbits under the action of $G$). As an abbreviation, we introduce the index $\gamma_k := \sum_{\alpha \in \mathcal{R}_+} k(\alpha)$ and $\gamma_\ell := \sum_{\alpha \in \mathcal{R}_+} \ell(\alpha)$.

Throughout this paper, we will assume that $k(\alpha), \ell(\alpha) \geq 0$ for all $\alpha \in \mathcal{R}$, and $\gamma_\ell \geq \gamma_k$. Moreover, let $w_k$ denote the weight function $w_k(x) := \prod_{\alpha \in \mathcal{R}_+} |\langle \alpha, x \rangle|^{2k(\alpha)}$, for all $x \in \mathbb{R}^d$, which is $G$-invariant and homogeneous of degree $2\gamma_k$.

Let $c_k$ be the Mehta-type constant given by

$$c_k := \left( \int_{\mathbb{R}^d} e^{-|x|^2/2} w_k(x) dx \right)^{-1}.$$
We denote by $\mu_k$ the measure on $\mathbb{R}^d$ given by $d\mu_k(x) := c_k w_k(x) dx$; and by $L^p(\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \left( \int_{\mathbb{R}^d} |f(x)|^p d\mu_k(x) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mu_k)} := \text{ess sup}_{x \in \mathbb{R}^d} |f(x)| < \infty.$$

For $f \in L^1(\mu_k)$ the Dunkl transform is defined (see [2]) by

$$F_k(f)(y) := \int_{\mathbb{R}^d} E_k(-ix, y) f(x) d\mu_k(x), \quad y \in \mathbb{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel (for more details, see the next section).

Let $s > 0$. We consider the Hilbert $H^s_{k\ell}$ consisting of functions $f \in L^2(\mu_k)$ such that $e^{s|x|^2/2} F_k(f) \in L^2(\mu_k)$. The space $H^s_{k\ell}$ is endowed with the inner product

$$(f, g)_{H^s_{k\ell}} := \int_{\mathbb{R}^d} e^{s|x|^2} F_k(f)(z) \overline{F_k(g)(z)} d\mu_k(z).$$

Let $m$ be a function in $L^2(\mu_k)$. The Dunkl multiplier operators $T_{k,\ell,m}$, are defined for $f \in H^s_{k\ell}$ by

$$T_{k,\ell,m} f(x, a) := F_k^{-1} (m(a) F_k(f))(x), \quad (x, a) \in \mathcal{K} := \mathbb{R}^d \times (0, \infty).$$

These operators are studied in [14] where the author established some applications (Calderon’s reproducing formulas, best approximation formulas, extremal functions...). In particular, when $k = \ell$ these operators are studied in [13].

For $m \in L^2(\mu_k)$ satisfying the admissibility condition: $\int_0^\infty |m(ax)|^2 \frac{da}{a} = 1$, a.e. $x \in \mathbb{R}^d$, then the operators $T_{k,\ell,m}$ satisfy, for $f \in H^s_{k\ell}$:

$$\|T_{k,\ell,m} f\|_{L^2(\Omega_k)}^2 = \|F_k(f)\|_{L^2(\mu_k)}^2,$$

where $\Omega_k$ is the measure on $\mathcal{K}$ given by $d\Omega_k(x, a) := \frac{da}{a} d\mu_k(x)$.

Building on the ideas of Matsura et al. [5], Saitoh [9, 11] and Yamada et al. [18], and using the theory of reproducing kernels [8], we give best approximation of the operator $T_{k,\ell,m}$ on the Hilbert spaces $H^s_{k\ell}$. More precisely, for all $\lambda > 0$, $g \in L^2(\Omega_k)$, the infimum

$$\inf_{f \in H^s_{k\ell}} \left\{ \lambda \|f\|_{H^s_{k\ell}}^2 + \|g - T_{k,\ell,m} f\|_{L^2(\Omega_k)}^2 \right\},$$

is attained at one function $F^*_{\lambda, g}$, called the extremal function, and given by

$$F^*_{\lambda, g}(y) = \int_{\mathbb{R}^d} \frac{E_k(iy, z)}{1 + \lambda e^{m|z|^2}} \left[ \int_0^\infty \frac{m(bz) F_k(g(., b))(z) db}{b} \right] d\mu_k(z).$$

Next we show for $F^*_{\lambda, g}$ the following properties.

(i) $\|F^*_{\lambda, g}\|_{H^s_{k\ell}} \leq \frac{1}{2\sqrt{\lambda}} \|g\|_{L^2(\Omega_k)}$.

(ii) $T_{k,\ell,m} F^*_{\lambda, g}(y, a) = \int_{\mathbb{R}^d} \frac{m(az) E_k(iy, z)}{1 + \lambda e^{m|z|^2}} \left[ \int_0^\infty \frac{m(bz) F_k(g(., b))(z) db}{b} \right] d\mu_k(z)$.

In the Dunkl setting, the extremal functions are studied in several directions [12, 13, 14, 15, 16].

This paper is organized as follows. In section 2 we define and study the Dunkl multiplier operators $T_{k,\ell,m}$ on the Hilbert space $H^s_{k\ell}$. The last section of this paper is
devoted to give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators $T_{k,t,m}$ on the Hilbert space $H^s_{k,t}$.

2. DUNKL TYPE MULTIPLIER OPERATORS

The Dunkl operators $D_j; j = 1, \ldots, d$, on $\mathbb{R}^d$ associated with the finite reflection group $G$ and multiplicity function $k$ are given, for a function $f$ of class $C^1$ on $\mathbb{R}^d$, by

$$D_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}^+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle x, \alpha \rangle}.$$  

For $y \in \mathbb{R}^d$, the initial problem $D_j u(j, y) = y_j u(x, y)$, $j = 1, \ldots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel [1, 3]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$ (see [7]). In our case (see [1, 2]),

$$|E_k(ix, y)| \leq 1, \quad x, y \in \mathbb{R}^d. \tag{2.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [2], where already many basic properties were established. Dunkl’s results were completed and extended later by De Jeu [3]. The Dunkl transform of a function $f$ in $L^1(\mu_k)$, is defined by

$$F_k(f)(y) := \int_{\mathbb{R}^d} E_k(-ix, y) f(x) d\mu_k(x), \quad y \in \mathbb{R}^d.$$  

We notice that $F_0$ agrees with the Fourier transform $F$ that is given by

$$F(f)(y) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i(y, x)} f(x) dx, \quad x \in \mathbb{R}^d.$$  

Some of the properties of Dunkl transform $F_k$ are collected below (see [2, 3]).

**Theorem 2.1** (i) $L^1 - L^\infty$-boundedness. For all $f \in L^1(\mu_k)$, $F_k(f) \in L^\infty(\mu_k)$ and

$$\|F_k(f)\|_{L^\infty(\mu_k)} \leq \|f\|_{L^1(\mu_k)}.$$  

(ii) Inversion theorem. Let $f \in L^1(\mu_k)$, such that $F_k(f) \in L^1(\mu_k)$. Then

$$f(x) = F_k(F_k(f))(-x), \quad \text{a.e. } x \in \mathbb{R}^d.$$  

(iii) Plancherel theorem. The Dunkl transform $F_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,

$$\|F_k(f)\|_{L^2(\mu_k)} = \|f\|_{L^2(\mu_k)}.$$  

Let $s > 0$. We define the Hilbert space $H^s_{k,t}$, as the set of all $f \in L^2(\mu_k)$ such that $e^{s|z|^2/2} F_k(f) \in L^2(\mu_k)$. The space $H^s_{k,t}$ provided with the inner product

$$\langle f, g \rangle_{H^s_{k,t}} := \int_{\mathbb{R}^d} e^{s|z|^2} F_k(f)(z) F_k(g)(z) d\mu_k(z),$$  

and the norm $\|f\|_{H^s_{k,t}} = \sqrt{\langle f, f \rangle_{H^s_{k,t}}}$. The space $H^s_{k,t}$ satisfies the following properties.

(i) The $H^s_{k,t}$ has the reproducing kernel

$$h^s_{k,t}(x, y) = \frac{c_k}{c_k} \int_{\mathbb{R}^d} e^{-s|z|^2} E_k(ix, z) E_k(-iy, z) w_{t-k}(z) d\mu_k(z).$$
If \( k = \ell \), then \( h_{kk}^s \) is the Dunkl-type heat kernel \([6, 12]\) and this kernel is given by

\[
h_{kk}^s(x, y) = \frac{1}{(2\pi)^{d+1}} e^{-\frac{(|x|^2 + |y|^2)}{2}} E_k\left(\frac{x}{\sqrt{2s}}, \frac{y}{\sqrt{2s}}\right).
\]

(ii) The space \( H_{k\ell}^s \) is continuously contained in \( L^2(\mu_k) \) and

\[
\|f\|_{L^2(\mu_k)}^2 \leq \frac{c_\ell}{c_k} \left( \frac{2}{\pi} \right)^{\gamma_{\ell-k} - \gamma_k} \left( \frac{\gamma_{\ell-k}}{s} \right)^{\gamma_{\ell-k}} \|f\|_{H_{k\ell}^s}^2.
\]

(iii) If \( f \in H_{k\ell}^s \) then \( \mathcal{F}_\ell(f) \in L^1(\mu_\ell) \) and \( \|\mathcal{F}_\ell(f)\|_{L^1(\mu_\ell)} \leq C_{k,\ell}\|f\|_{H_{k\ell}^s} \), where

\[
C_{k,\ell} = \left( \frac{c_\ell}{c_k} \int_{\mathbb{R}^d} e^{-s|x|^2} w_{\ell-k}(z) d\mu_\ell(z) \right)^{1/2}. \tag{2.2}
\]

(iv) If \( f \in H_{k\ell}^s \), then \( \mathcal{F}_\ell(f) \in L^1 \cap L^2(\mu_\ell) \) and

\[
f(x) = \int_{\mathbb{R}^d} E_\ell(ix, z) \mathcal{F}_\ell(f)(z) d\mu_\ell(z), \quad \text{a.e. } x \in \mathbb{R}^d.
\]

Let \( \lambda > 0 \). We denote by \( \langle \cdot, \cdot \rangle_{\lambda, H_{k\ell}^s} \) the inner product defined on the space \( H_{k\ell}^s \) by

\[
\langle f, g \rangle_{\lambda, H_{k\ell}^s} := \lambda \langle f, g \rangle_{H_{k\ell}^s} + \langle \mathcal{F}_\ell(f), \mathcal{F}_\ell(g) \rangle_{L^2(\mu_\ell)}, \tag{2.3}
\]

and the norm \( \|f\|_{\lambda, H_{k\ell}^s} := \sqrt{\langle f, f \rangle_{\lambda, H_{k\ell}^s}} \). On \( H_{k\ell}^s \) the two norms \( \|\cdot\|_{H_{k\ell}^s} \) and \( \|\cdot\|_{\lambda, H_{k\ell}^s} \) are equivalent. This \( (H_{k\ell}^s, \langle \cdot, \cdot \rangle_{\lambda, H_{k\ell}^s}) \) is a Hilbert space with reproducing kernel given by

\[
K_{k\ell}^s(x, y) = \frac{c_\ell}{c_k} \int_{\mathbb{R}^d} E_\ell(ix, z) E_\ell(-iy, z) w_{\ell-k}(z) d\mu_\ell(z). \tag{2.4}
\]

Let \( m \) be a function in \( L^2(\mu_k) \). The Dunkl multiplier operators \( T_{k,\ell,m} \), are defined for \( f \in H_{k\ell}^s \) by

\[
T_{k,\ell,m} f(x, a) := \mathcal{F}_k^{-1}(m(a) \mathcal{F}_\ell(f))(x), \quad (x, a) \in K. \tag{2.5}
\]

We denote by \( \Omega_k \) the measure on \( K \) given by \( d\Omega_k(x, a) := \frac{da}{a} d\mu_k(x) \); and by \( L^2(\Omega_k) \), the space of measurable functions \( F \) on \( K \), such that

\[
\|F\|_{L^2(\Omega_k)} := \left( \int_{\mathbb{R}^d} \int_0^\infty |F(x, a)|^2 d\Omega_k(x, a) \right)^{1/2} < \infty.
\]

Let \( m \) be a function in \( L^2(\mu_k) \) satisfying the admissibility condition

\[
\int_0^\infty |m(ax)|^2 \frac{da}{a} = 1, \quad \text{a.e. } x \in \mathbb{R}^d. \tag{2.6}
\]

Then from Theorem 2.1 (iii), for \( f \in H_{k\ell}^s \), we have

\[
\|T_{k,\ell,m} f\|_{L^2(\Omega_k)} = \|\mathcal{F}_\ell(f)\|_{L^2(\mu_\ell)} \leq \|f\|_{H_{k\ell}^s}. \tag{2.7}
\]
3. Extremal functions for the operators $T_{k,t,m}$

In this section, by using the theory of extremal function and reproducing kernel of Hilbert space [8, 9, 10, 11] we study the extremal function associated to the Dunkl multiplier operators $T_{k,t,m}$. In the particular case when $k = t$ this function is studied in [16, 17]. The main result of this section can be stated as follows.

**Theorem 3.1.** Let $m \in L^2(\mu_k)$ satisfying (2.6). For any $g \in L^2(\Omega_k)$ and for any $\lambda > 0$, there exists a unique function $F_{\lambda,g}$, where the infimum

$$\inf_{f \in H_{k,t}^*} \left\{ \lambda \|f\|^2_{H_{k,t}^*} + \|g - T_{k,t,m}f\|^2_{L^2(\Omega_k)} \right\}$$

is attained. Moreover, the extremal function $F_{\lambda,g}^*$ is given by

$$F_{\lambda,g}^*(y) = \int_{\mathbb{R}^d} \int_0^\infty g(x,a)Q_s(x,y,a)d\Omega_k(x,a),$$

where

$$Q_s(x,y,a) = \int_{\mathbb{R}^d} \frac{m(az)E_k(-ix,z)E_\ell(iy,z)}{1 + \lambda e^{s|z|^2}}d\mu(z).$$

**Proof.** Let $s, \lambda > 0$. Since $m \in L^2(\mu_k)$ and satisfying (2.6), then by (2.7), the inner product $\langle \cdot, \cdot \rangle_{\lambda,H_{k,t}^*}$ defined by (2.3) is written by

$$\langle f, g \rangle_{\lambda,H_{k,t}^*} = \lambda \langle f, g \rangle_{H_{k,t}^*} + \langle T_{k,t,m}f, T_{k,t,m}g \rangle_{L^2(\Omega_k)}.$$  \(\square\)

Then, the existence and unicity of the extremal function $F_{\lambda,g}^*$ satisfying (3.1) is obtained in [4, 5, 10]. Especially, $F_{\lambda,0}^*$ is given by the reproducing kernel of $H_{k,t}^*$ with $\|\cdot\|_{\lambda,H_{k,t}^*}$ norm as

$$F_{\lambda,0}^*(y) = \langle g, T_{k,t,m}(K_{k,t}^*(\cdot,y)) \rangle_{L^2(\Omega_k)},$$

where $K_{k,t}^*$ is the kernel given by (2.4). Then, we obtain the result by Theorem 2.1 (ii) and the fact that

$$\mathcal{F}_\ell(K_{k,t}^*(\cdot,y))(z) = \frac{c_\ell E_\ell(-iy, z)}{c_k (1 + \lambda e^{s|z|^2})^{\gamma_{\ell - k}}} \nu_{\gamma_{\ell - k}}(z), \quad z \in \mathbb{R}^d. \tag{3.3}$$

**Theorem 3.2.** Let $\lambda > 0$ and $g \in L^2(\Omega_k)$. The extremal function $F_{\lambda,g}^*$ satisfies

(i) $|F_{\lambda,g}^*(y)| \leq \frac{C_{k,t}}{2\sqrt{\lambda}} \|g\|_{L^2(\Omega_k)}$,

where $C_{k,t}$ is the constant given by (2.2).

(ii) $\|F_{\lambda,g}^*\|_{L^2(\mu_k)}^2 \leq \frac{D_{k,t}}{\lambda} \|m\|_{L^2(\mu_k)}^2 \int_{\mathbb{R}^d} \int_0^\infty |g(x,a)|^2 \frac{\gamma_{\ell - k}}{\frac{(2\gamma_k - 2\gamma_{\ell - k})}{(2\gamma_k - 2\gamma_{\ell - k})} + 1}d\Omega_k(x,a),$

where

$$D_{k,t} = \frac{c_k \sqrt{\pi}}{4c_\ell \sqrt{2\gamma_k + d}} \left( \frac{2}{s} \right)^{\gamma_{\ell} - \gamma_k} \left( \frac{\gamma_{\ell} - \gamma_k}{s} \right)^{\gamma_{\ell} - \gamma_k}. \tag{3.4}$$

**Proof.** (i) From (2.7) and (3.2), we have

$$|F_{\lambda,g}^*(y)| \leq \|g\|_{L^2(\Omega_k)} \|T_{k,t,m}(K_{k,t}^*(\cdot,y))\|_{L^2(\Omega_k)} \leq \|g\|_{L^2(\Omega_k)} \|\mathcal{F}_\ell(K_{k,t}^*(\cdot,y))\|_{L^2(\mu_k)}.$$  \(\square\)

Then, by (3.3) we deduce

$$|F_{\lambda,g}^*(y)| \leq \|g\|_{L^2(\Omega_k)} \left( \frac{c_\ell}{c_k} \int_{\mathbb{R}^d} \frac{w_{\ell-k}(z)d\mu_k(z)}{(1 + \lambda e^{s|z|^2})^{\gamma_{\ell - k}}} \right)^{1/2}. \tag{3.5}$$
Theorem 3.3. Let $s, \lambda > 0$. For every $g \in L^2(\Omega_k)$, we have

(i) $F_{\lambda,g}^*(y) = \int_{\mathbb{R}^d} \frac{E_1(igz)}{1 + \lambda e^{s|z|^2}} \left[ \int_0^\infty \frac{m(bz)}{1 + \lambda e^{s|z|^2}} |F_k(g(\cdot, b))(z)| \frac{db}{b} \right] d\mu(\epsilon)$. 

(ii) $F_\lambda(F_{\lambda,g}^*)(z) = \frac{1}{1 + \lambda e^{s|z|^2}} \left[ \int_0^\infty \frac{m(bz)}{1 + \lambda e^{s|z|^2}} |F_k(g(\cdot, b))(z)| \frac{db}{b} \right]$. 

(iii) $\|F_{\lambda,g}^*\|_{L^2(\mu_k)} \leq \frac{1}{2\sqrt{\lambda}} \|g\|_{L^2(\Omega_k)}$. 

Proof. (i) From (3.2) we have

$$F_{\lambda,g}^*(y) = \int_{\mathbb{R}^d} \int_0^\infty g(x, b) \overline{T_{k,\ell, m}(K_{k,\ell}^*(\cdot, y))}(x, b) d\Omega_k(x, b).$$

Since

$$\int_{\mathbb{R}^d} \int_0^\infty |g(x, b)\overline{T_{k,\ell, m}(K_{k,\ell}^*(\cdot, y))}(x, b)| d\Omega_k(x, b) \leq \|g\|_{L^2(\Omega_k)} \|F_k(K_{k,\ell}^*(\cdot, y))\|_{L^2(\mu_k)} < \infty,$$
then, by Fubini’s theorem, Theorem 2.1 (iii) and (3.3) we obtain
\[
F^*_{\lambda,g}(y) = \int_0^\infty \int_{\mathbb{R}^d} g(x, b)T_{k,\ell,m}(K^*_k, y)(x, b) \, d\mu_k(x) \, \frac{db}{b} 
\]
\[
= \int_0^\infty \int_{\mathbb{R}^d} \frac{m(bz)F_k(g(.,b))(z)\mathcal{F}_\ell(K^*_k,.,y)(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \, \frac{db}{b} 
\]
\[
= \int_0^\infty \int_{\mathbb{R}^d} \frac{m(bz)F_k(g(.,b))(z)\mathcal{F}_\ell(\mathcal{M}^*_k,.,y)(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \, \frac{db}{b} 
\]

Since
\[
\int_0^\infty \int_{\mathbb{R}^d} \frac{m(bz)F_k(g(.,b))(z)\mathcal{F}_\ell(\mathcal{M}^*_k,.,y)(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \, \frac{db}{b} \leq \frac{C_{k,\ell}}{2\sqrt{\lambda}} \|g\|_{L^2(\Omega_k)} < \infty,
\]
then, by Fubini’s theorem we deduce that
\[
F^*_{\lambda,g}(y) = \int_{\mathbb{R}^d} \frac{E_k(iy, z)}{E_k(iy, z)} \left[ \int_0^\infty \frac{m(bz)F_k(g(.,b))(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] \, \frac{db}{b} 
\]

(ii) The function \( z \to \frac{1}{1 + \lambda \varepsilon |z|^2} \left[ \int_0^\infty \frac{m(bz)F_k(g(.,b))(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] \) belongs to \( L^1 \cap L^2(\mu_k) \). Then by Theorem 2.1 (ii) and (iii), it follows that \( F^*_{\lambda,g} \) belongs to \( L^2(\mu_k) \), and
\[
F_k(F^*_{\lambda,g})(z) = \frac{1}{1 + \lambda \varepsilon |z|^2} \left[ \int_0^\infty \frac{m(bz)F_k(g(.,b))(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] 
\]

(iii) From (ii), Hölder’s inequality and (2.6) we have
\[
|F_k(F^*_{\lambda,g})(z)|^2 \leq \frac{1}{1 + \varepsilon |z|^2} \left[ \int_0^\infty |F_k(g(.,b))(z)|^2 \, d\mu_k(z) \right] 
\]
Thus,
\[
\|F^*_{\lambda,g}\|_{L^2_k}^2 \leq \int_{\mathbb{R}^d} \frac{E_k(iy, z)}{E_k(iy, z)} \left[ \int_0^\infty \frac{|F_k(g(.,b))(z)|^2}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] \, \frac{db}{b} \leq \frac{1}{4\lambda} \int_{\mathbb{R}^d} \left[ \int_0^\infty \frac{|F_k(g(.,b))(z)|^2}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] \, \frac{db}{b} = \frac{1}{4\lambda} \|g\|_{L^2(\Omega_k)}^2,
\]
which ends the proof.

**Theorem 3.4.** Let \( s, \lambda > 0 \). For every \( g \in L^2(\Omega_k) \), we have
\[
T_{k,\ell,m}F^*_{\lambda,g}(y, a) = \int_{\mathbb{R}^d} \frac{m(az)E_k(iy, z)}{1 + \lambda \varepsilon |z|^2} \left[ \int_0^\infty \frac{m(bz)F_k(g(.,b))(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] \, \frac{db}{b} 
\]

**Proof.** From (2.5) and Theorem 3.3 (ii), we have
\[
T_{k,\ell,m}F^*_{\lambda,g}(y, a) = F_k^{-1} \left( \frac{m(az)}{1 + \lambda \varepsilon |z|^2} \left[ \int_0^\infty \frac{m(bz)F_k(g(.,b))(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] \right) (y).
\]
The function \( z \to \frac{m(az)}{1 + \lambda \varepsilon |z|^2} \left[ \int_0^\infty \frac{m(bz)F_k(g(.,b))(z)}{1 + \lambda \varepsilon |z|^2} \, d\mu_k(z) \right] \) belongs to \( L^1(\mu_k) \). Then by Theorem 2.1 (ii), we obtain the result.

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