ON THE DEGREE OF APPROXIMATION OF A FUNCTION BY 
\((C,1)(E,q)\) MEANS OF ITS FOURIER-LAGUERRE SERIES

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Abstract. In this note a theorem on the degree of approximation of a function by \((C,1)(E,q)\) means of its Fourier-Laguerre series at the frontier point \(x = 0\) is proved.

1. Introduction

Let us consider the infinite series \(\sum_{n=0}^{\infty} u_n\) with the sequence of its \(n\)-th partial sums \(s := \{s_n\}\).

If for \(q > 0\)

\[ E^q_n(s) = \frac{1}{(1 + q)^n} \sum_{k=0}^{n} \binom{n}{k} q^k s_k \rightarrow s_1 \quad \text{as} \quad n \rightarrow \infty, \]

then it is said that \(s := \{s_n\}\) is summable by \((E,q)\) means (see Hardy [3]), and we write \(s_n \rightarrow s_1(E,q)\).

The Fourier-Laguerre expansion of a function \(f(x) \in L(0, \infty)\) is given by

\[ f(x) \sim \sum_{n=0}^{\infty} a_n L^{(\alpha)}_n(x), \]

where

\[ a_n = \frac{1}{\Gamma(\alpha + 1) \binom{n+\alpha}{n}} \int_{0}^{\infty} e^{-y} y^\alpha L^{(\alpha)}_n(y) dy, \]

\(L^{(\alpha)}_n(x)\) denotes the \(n\)-th Laguerre polynomial of order \(\alpha > -1\), defined by generating function

\[ \sum_{n=0}^{\infty} L^{(\alpha)}_n(x) \omega^n = \frac{e^{\frac{x}{1-\omega}}}{(1 - \omega)^{\alpha+1}}, \]

and it is assumed that the integral (1.3) exists.

In 1971, D. P. Gupta [2] estimated the order of the function by Cesáro means of series (1.2) at the point \(x = 0\), after replacing the continuity condition in Szegő’s theorem [6] by a much lighter condition. He proved the following theorem.

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Theorem 1.1 ([2]). If
\[ F(t) = \int_0^t \frac{|f(y)|}{y} \, dy = o \left( \log \left( \frac{1}{t} \right) \right)^{1+p}, \quad t \to 0, -1 < p < \infty, \]
and
\[ \int_1^{\infty} e^{-y/2} y^{(3\alpha - 3k - 1)/3} |f(y)| \, dy < \infty, \]
are fulfilled, then
\[ \sigma_k^n(0) = o \left( \log n \right)^{p+1} \]
provided that \( k > \alpha + 1/2 \), \( \alpha > -1 \), with \( \sigma_k^n(0) \) being the \( n \)-th Cesàro mean of order \( k \).

Further, we use the notation
\[ (1.5) \quad \phi(y) = e^{-y} y^\alpha \left[ f(y) - f(0) \right] / \Gamma(\alpha + 1), \]
and denote by \( t_n \) harmonic means of the series (1.2). T. Singh [5] estimated the deviation \( t_n(x) - f(x) \) at the point \( x = 0 \) by some weaker conditions than those of Theorem 1.1. Namely, he verified the following theorem.

Theorem 1.2 ([5]). For \( \alpha \in (-5/6, -1/2) \)
\[ t_n(0) - f(0) = o \left( \log n \right)^{p+1} \]
provided that \( k > \alpha + 1/2 \), \( \alpha > -1 \), with \( \sigma_k^n(0) \) being the \( n \)-th Cesàro mean of order \( k \).

Very recently, Nigam and Sharma [4] proved a theorem of such type using \((E, 1)\) means which is entirely different from \((C, k)\) and harmonic means of the series (1.2), they employed a condition which is weaker than condition (1.6), and increased the range of \( \alpha \) to \((-1, -1/2)\) which is more appropriate for applications. In their paper they established the following statement.

Theorem 1.3 ([4]). If
\[ (1.7) \quad E_n^1 = \frac{1}{2\pi} \sum_{k=0}^n \binom{n}{k} s_k \to \infty \quad \text{as} \quad n \to \infty, \]
then the degree of approximation of Fourier-Laguerre expansion (1.2) at the point \( x = 0 \) by \((E, 1)\) means \( E_n^1 \) is given by
\[ (1.8) \quad E_n^1(0) - f(0) = o \left( \xi(n) \right) \]
provided that
\[ (1.9) \quad \Phi(t) = \int_0^t |\phi(y)| \, dy = o \left( \xi^{1+1} \left( \frac{1}{t} \right) \right), \quad t \to 0, \]
where \( \delta \) is a fixed positive constant.
\( \int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| \, dy = o \left( n^{-(2\alpha+1)/4} \xi(n) \right), \)

and
\( \int_{n}^{\infty} e^{y/2} y^{-1/3} |\phi(y)| \, dy = o(\xi(n)), \quad n \to \infty, \)

where \( \delta \) is a fixed positive constant, \( \alpha \in (-1, -1/2) \), and \( \xi(t) \) is a positive monotonic increasing function of \( t \) such that \( \xi(n) \to \infty \) as \( n \to \infty \).

As is pointed out in [1] the infinite series
\[ 1 - 4 \sum_{n=1}^{\infty} (-3)^{n-1} \]

is not \((E, 1)\) summable nor \((C, 1)\) summable. However, it is proved that the above series is \((C, 1)(E, 1)\) summable. Therefore the product summability \((C, 1)(E, 1)\) is more powerful than the individual methods \((C, 1)\) and \((E, 1)\). Thus, \((C, 1)(E, 1)\) mean gives an approximation for a wider class of Fourier-Laguerre series than the individual methods \((C, 1)\) and \((E, 1)\). The main aim of this paper is to prove the counterpart of the Theorem 1.3 using the product mean \((C, 1)(E, q)\), which obviously, based on what we discussed above, will give more general results. To achieve this aim we need an auxiliary result (see [6], page 175).

**Lemma 1.1.** Let \( \alpha \) be arbitrary and real, \( c \) and \( d \) be fixed positive constants, and let \( n \to \infty \). Then
\begin{align*}
L_n^{(\alpha)}(x) &= O \left( n^\alpha \right), \quad \text{if} \quad 0 \leq x \leq \frac{c}{n} \\
L_n^{(\alpha)}(x) &= O \left( x^{-(2\alpha+1)/4} n^{-(2\alpha-1)/4} \right), \quad \text{if} \quad \frac{c}{n} \leq x \leq d.
\end{align*}

**2. Main Result**

We prove the following theorem.

**Theorem 2.1.** The degree of approximation of Fourier-Laguerre expansion (1.2) at the point \( x = 0 \) by \((C, 1)(E, q)\), \( q \geq 1 \) means \([C, 1](E, q)\) is given by
\[ [[C, 1](E, q)]_n(0) - f(0) = o(\xi(n)) \]

provided that
\begin{align*}
\Phi(t) &= \int_{0}^{t} |\phi(y)| \, dy = o \left( t^{\alpha+1} \xi \left( \frac{1}{t} \right) \right), \quad t \to 0, \\
\int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| \, dy &= o \left( n^{-(2\alpha+1)/4} \xi(n) \right), \\
\int_{n}^{\infty} e^{y/2} y^{-1/3} |\phi(y)| \, dy &= o(\xi(n)), \quad n \to \infty,
\end{align*}

where \( \delta \) is a fixed positive constant, \( \alpha \in (-1, -1/2) \), and \( \xi(t) \) is a positive monotonic increasing function of \( t \) such that \( \xi(n) \to \infty \) as \( n \to \infty \).
Proof. Based on the equality

\[ L_n^{(\alpha)}(0) = \binom{n + \alpha}{\alpha}, \]

we obtain

\[
s_n(0) = \sum_{k=0}^{n} a_k L_n^{(\alpha)}(0) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty e^{-y^\alpha} f(y) \sum_{k=0}^{n} L_k^{(\alpha)}(y) dy \]

\[
\text{(2.5)}
\]

Thus,

\[
[(E, q)]_n(0) = \frac{1}{(1 + q)^n} \sum_{k=0}^{n} \binom{n}{k} q^k s_k(0) = \frac{1}{(1 + q)^n} \sum_{k=0}^{n} \binom{n}{k} q^k \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty e^{-y^\alpha} f(y) L_k^{(\alpha)}(y) dy,
\]

and

\[
[(C, 1)(E, q)]_n(0) = \frac{1}{n + 1} \sum_{\nu = 0}^{n} \frac{1}{(1 + q)^\nu} \sum_{k=0}^{\nu} \binom{\nu}{k} q^k \phi(y) L_k^{(\alpha)}(\nu)(y) dy.
\]

Therefore, using (1.5) we have

\[
(C, 1)(E_n^q)(0) - f(0) = \left( \int_{0}^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{n} + \int_{n}^{\infty} \right) \frac{1}{n + 1} \sum_{\nu = 0}^{n} \frac{1}{(1 + q)^\nu} \sum_{k=0}^{\nu} \binom{\nu}{k} q^k \phi(y) L_k^{(\alpha)}(\nu)(y) dy.
\]

\[
\text{(2.7)} := \sum_{m=0}^{4} r_m.
\]
Using the property of the orthogonality, condition (2.1) and Lemma 1.1, we obtain

\[ r_1 = \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k O(k^{\alpha+1}) \int_0^{1/n} |\phi(y)| dy \]

\[ = \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k O(n^{\alpha+1}) o\left(\frac{\xi(n)}{n^{\alpha+1}}\right) \]

\[ = o\left(\frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k \xi(n)\right) \]

(2.8)

since

\[ \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k = n+1. \]

Again, using the property of the orthogonality and Lemma 1.1, we have

\[ r_2 = \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1+q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k O\left(k^{(2\alpha+1)/4}\right) \int_{1/n}^{d} y^{(2\alpha+3)/4} |\phi(y)| dy. \]

Since

\[ \sum_{k=0}^{v} \binom{v}{k} q^k k^{(2\alpha+1)/4} = \sum_{k=0}^{\lceil \frac{v}{2} \rceil} \binom{v}{k} q^k k^{(2\alpha+1)/4} + \sum_{k=\lceil \frac{v}{2} \rceil + 1}^{v} \binom{v}{k} q^k k^{(2\alpha+1)/4} \]

\[ \leq \sum_{k=0}^{v} \binom{v}{k} q^k k^{(2\alpha+1)/4} + \binom{v}{\lceil \frac{v}{2} \rceil} \sum_{k=\lceil \frac{v}{2} \rceil + 1}^{v} q^k k^{(2\alpha+1)/4} \]

\[ \leq (1+q)^v v^{(2\alpha+1)/4} + \binom{v}{\lceil \frac{v}{2} \rceil} v^{(2\alpha+5)/4} q^v \]

\[ = (1+q)^v v^{(2\alpha+1)/4} + \binom{v}{\lceil \frac{v}{2} \rceil} v^{(2\alpha+1)/4} q^v \quad q \geq 1. \]

and

\[ (1+q)^v = \sum_{k=0}^{v} \binom{v}{k} q^k \]

\[ = \binom{v}{0} q^0 + \binom{v}{1} q^1 + \cdots + \binom{v}{\lceil \frac{v}{2} \rceil} q^{\lceil \frac{v}{2} \rceil} + \binom{v}{\lceil \frac{v}{2} \rceil + 1} q^{\lceil \frac{v}{2} \rceil + 1} + \cdots + \binom{v}{v} q^v \]

\[ \geq \binom{v}{\lceil \frac{v}{2} \rceil} q^{\lceil \frac{v}{2} \rceil} + \binom{v}{\lceil \frac{v}{2} \rceil + 1} q^{\lceil \frac{v}{2} \rceil + 1} + \cdots + \binom{v}{v} q^v \]

\[ \geq \left[ \binom{v}{\lceil \frac{v}{2} \rceil} + \binom{v}{\lceil \frac{v}{2} \rceil + 1} + \cdots + \binom{v}{v} \right] q^{\lceil \frac{v}{2} \rceil} \]

\[ \geq K \left( \binom{v}{\lceil \frac{v}{2} \rceil} + 1 \right) \binom{v}{\lceil \frac{v}{2} \rceil} q^{v} \geq K \frac{v}{2} \binom{v}{\lceil \frac{v}{2} \rceil} q^v, \quad \text{(for } K \leq 1/q). \]
then
\[ \frac{1}{(1 + q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k k^{(2\alpha+1)/4} \leq \left( 1 + \frac{2}{K} \right) v^{(2\alpha+1)/4}. \]

and moreover,
\[ \frac{1}{n+1} \sum_{v=0}^{n} \frac{1}{(1 + q)^v} \sum_{k=0}^{v} \binom{v}{k} q^k k^{(2\alpha+1)/4} = O\left(n^{(2\alpha+1)/4}\right). \]

Using latter estimation, and doing the same reasoning as in [4] page 6, we obtain
\[ (2.9) \quad r_2 = O\left(n^{(2\alpha+1)/4}\right) \int_{1/n}^{\delta} y^{(2\alpha+3)/4} |\phi(y)| \, dy = O\left(\xi(n)\right). \]

Further we estimate \( r_3 \):
\[ (2.10) \quad o\left(\xi(n)\right). \]

Finally, we have
\[ (2.11) \quad o\left(\xi(n)\right). \]

Now, putting estimations (2.8)-(2.11) into (2.7) we obtain
\[ [(C, 1)(E, q)]_n(0) - f(0) = O\left(\xi(n)\right). \]

The proof of the theorem is completed. \( \square \)

References


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