ON THE INTEGRAL REPRESENTATION OF STRICTLY CONTINUOUS SET-VALUED MAPS

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Abstract. Let $T$ be a completely regular topological space and $C(T)$ be the space of bounded, continuous real-valued functions on $T$. $C(T)$ is endowed with the strict topology (the topology generated by seminorms determined by continuous functions vanishing at infinity). R. Giles ([13], p. 472, Theorem 4.6) proved in 1971 that the dual of $C(T)$ can be identified with the space of regular Borel measures on $T$. We prove this result for positive, additive set-valued maps with values in the space of convex weakly compact non-empty subsets of a Banach space and we deduce from this result the theorem of R. Giles ([13], theorem 4.6, p.473).

1. Introduction

The strict topology $\beta$ was for the first time introduced by R. C. Buck ([1], [2]) on the space $C(T)$ of all bounded continuous functions on a locally compact space $T$. He has proved among others that the dual space of $(C(T), \beta)$ is the space of all finite signed regular Borel measures on $T$. After a large number of papers have appeared in the literature concerned with extending the results contained in Buck’s paper [1]( see e.g. [4], [5], [6], [7], [8], [12],[14], [15], [17], [18], [19], [22], [25] and [27]). R. Giles has generalized this notion of the strict topology introduced by Buck for completely regular space $T$ and has proved Buck’s results, particulary the theorem 2 in [1] for an arbitrary (not necessarily Hausdorff) completely regular space $T$. In this paper we generalize Giles’s result ([13], theorem 4.6, p.473) to additive, positive, positively homogeneous and strictly continuous set-valued maps defined on $C_+(T)$ with values in the space $cc(E)$ of all convex weakly compact non-empty subsets of a Banach space $E$. We deduce from this result the theorem of R. Giles.

2. Notations and definitions

Let $T$ be a completely regular topological space and let $B(T)$ be the Borel $\sigma$-algebra of $T$ and let $C(T)$ be the space of bounded continuous real-valued functions on $T$. Let $C_0(T)$ be the subspace of $C(T)$ consisting of functions $f$ vanishing at infinity i.e. for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset T$ such that $|f(x)| < \varepsilon$ for $x \in T \setminus K_\varepsilon$. We denote by $C_+(T)$ the subspace of $C(T)$ consisting of non-negative functions and by $1_A$ the characteristic function of each $A \subset T$. For all $f \in C(T)$, we put $f^+ = \sup(f, 0), f^- = \sup(-f, 0)$ and $\|f\|_\infty = \sup\{|f(t)|; t \in T\}$. We denote

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by $\mathbb{R}$ the set of real numbers. Let $E$ be a Banach space, $E'$ its dual and $cc(E)$ be the space of all non-empty, convex weakly compact subsets of $E$; we denote by $\|\cdot\|$ the norm on $E$ and $E'$. If $X$ and $Y$ are subsets of $E$ we shall denote by $X + Y$ the family of all elements of the form $x + y$ with $x \in X$ and $y \in Y$. The support function of $X$ is the function $\delta^*(\cdot|X)$ from $E'$ to $[-\infty; +\infty]$ defined by $\delta^*(y|X) = \sup\{y(x), x \in X\}$. We endow $cc(E)$ with a Hausdorff distance, denoted by $\delta$. For all $K \in cc(E)$ and for all $K' \in cc(E), \delta(K, K') = \sup\{\|\delta^*(y|K) - \delta^*(y|K')\|: y \in E', \|y\| \leq 1\}$. Recall that $(cc(E), \delta)$ is a complete metric space ([16], theorem 9, p.185) and ([21], theorem 15, p.2-2).

**Definition 2.1.** (1) let $m : B(T) \to \mathbb{R}$ be a positive countable additive measure. We say that $m$ is:

(i) inner regular if for all $A \in B(T)$ and $\varepsilon > 0$, there exists a compact $K_\varepsilon \subset A$ and $m(A \setminus K_\varepsilon) < \varepsilon$.

(ii) outer regular if for all $A \in B(T)$ and for all $\varepsilon > 0$, there exists an open subset $O_\varepsilon$ of $T$ such that $O_\varepsilon \supset A$ and $m(O_\varepsilon \setminus A) < \varepsilon$.

(iii) regular if it is inner regular and outer regular.

(2) A signed measure $\mu : B(T) \to \mathbb{R}$ is regular if and only if its total variation $v(\mu)$ is regular. Note that $v(\mu) : B(T) \to \mathbb{R}^+$ $(A \mapsto v(\mu)(A) = \sup\{\sum_i |\mu(A_i)|; (A_i)$ finite partition of $A, A_i \in B(T))$.

**Definition 2.2.** A map $M : B(T) \to cc(E)$ is a set-valued measure if $M(A \cup B) = M(A) + M(B)$ for every pair of disjoint sets $A, B$ in $B(T)$, $M(\emptyset) = \{0\}$ and $M(\bigcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} M(A_n)$ for every sequence $(A_n)$ of mutually disjoint elements of $B(T)$; which amounts to saying that for all $y \in E'$ the map $\delta^*(y|M(\cdot)) : B(T) \to \mathbb{R}(A \mapsto \delta^*(y|M(A)))$ is a countably additive measure ([21], corollary p. 2-25). We say that a set-valued measure $M$ is:

(i) positive if for all $A \in B(T), 0 \in M(A)$

(ii) regular if for all $y \in E'$, the measure $\delta^*(y|M(\cdot))$ is regular.

Let $\varphi \in C_0(T)$, let $K$ be a compact subset of $T$. We denote by $p_\varphi$ and $p_K$ the semi-norms on $C(T)$ defined by $p_\varphi(f) = \sup\{|f(t)|\varphi(t)|; t \in T\}$ and $p_K(f) = \sup\{|f(t)|; t \in K\}$ for every $f \in C(T)$.

**Definition 2.3.** The topology determined by the set of semi-norms $\{p_\varphi; \varphi \in C_0(T)\}$ (resp. $\{p_K; K$ belongs to the family of compact subsets of $T$\}) is called the strict (resp. the compact convergence) topology. We say that a map defined on $C(T)$ is strictly continuous if it is continuous for this topology.

**Definition 2.4.** A map $L : C_+(T) \to cc(E)$ is:

(i) additive set-valued map if for all $f, g \in C_+(T) L(f + g) = L(f) + L(g)$

(ii) positively homogeneous if for $f \in C_+(T)$ and for $\lambda \geq 0$ $L(\lambda f) = \lambda L(f)$

(iii) positive if for every $f \in C_+(T), 0 \in L(f)$.

**Definition 2.5.** ([24], p. 04) Let $m$ be a bounded linear functional on $C(T)$, and let $B(0,1)$ be the unit ball of $C(T)$. We say that $m$ is tight if its restriction to $B(0,1)$ is continuous for the topology of compact convergence.
3. Main result

Lemma 3.1. Let $m$ be a bounded linear functional on $C(T)$. If $m$ is tight then for all $\varepsilon > 0$ there is a compact subset $K_\varepsilon$ of $T$ such that for all $f \in C(T)$ and $|f| \leq 1_{T\setminus K_\varepsilon}$, we have $|m(f)| < \varepsilon$.

Proof. Assume that $m$ is tight. Then for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon$ of $T$ and there is $\eta > 0$ such that for all $f \in B(0, 1)$ and $p_{K_\varepsilon}(f) = \sup \{|f(t)|; t \in K_\varepsilon\} < \eta$. We have $|m(f)| < \varepsilon$. In particular for all $f \in B(0, 1)$ such that $|f| \leq 1_{T\setminus K_\varepsilon}$, one has $|m(f)| < \varepsilon$. □

Lemma 3.2. Let $M : B(T) \to cc(E)$ be a positive, regular set-valued measure. Then the real-valued measure $\delta^*(y|M(.))$ are uniformly tight with respect to $y \in E'$, $\|y\| \leq 1$ and for every $A \in B(T)$ and for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon$ of $T$ such that $K_\varepsilon \subset A$ and $\sup \{\delta^*(y|M(A\setminus K_\varepsilon)); y \in E', \|y\| \leq 1\} \leq \varepsilon$.

Proof. Let us consider the set $\{\delta^*(y|M(.)), y \in E', \|y\| \leq 1\}$ of countably additive real-valued measures. It is uniformly countable additive (see [9], theorem 10, p. 88–89; [28], lemma 3.1, p. 275). According to ([10], p. 443, Theorem 10.7) there is a sequence $(c_n)$ of real numbers and there is a sequence $(\delta^*(y_n|M(.))), [y_n] \leq 1$ of measures such that $\mu(A) = \sum_{n=1}^{+\infty} c_n \delta^*(y_n|M(A))$ exists for each $A \in B(T)$ and such that the series $\sum |c_n| \delta^*(y_n|M(A))$ is uniformly convergent for $A \in B(T)$; moreover the countable additive measure $\nu : B(T) \to \mathbb{R}(A \mapsto \nu(A) = \sum_{n=1}^{+\infty} c_n \delta^*(y_n|M(A)))$ verifies the following relation: $\lim_{\nu(A) \to 0} \sup \{\delta^*(y|M(A)); y \in E', \|y\| \leq 1\} = 0$.

(*) We deduce from the uniform convergence of the series $\sum |c_n| \delta^*(y_n|M(A))$ for $A \in B(T)$, that $\nu$ is regular. Indeed, given $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that

$$\sup_{A \in B(T)} \left| \nu(A) - \sum_{k=1}^{n_0} c_k \delta^*(y_k|M(A)) \right| < \varepsilon/2.$$  

For $A \in B(T)$, choose a compact subset $K$ of $T$ such that $K \subset A$ and for every $k \in \{1, 2, ..., n_0\}$ $\delta^*(y_k|M(A\setminus K)) \leq \frac{\varepsilon}{2(n_0+1)r_0}$ with $r_0 = \sup \{|c_k|; k \in \{1, 2, ..., n_0\}\}$ then $\sum_{k=1}^{n_0} |c_k| \delta^*(y_k|M(A\setminus K)) \leq \varepsilon/2$, therefore $\nu(A\setminus K) \leq \varepsilon$.

The relation (*) and the inner regularity of $\nu$ show that for each $\varepsilon > 0$ and each $A \in B(T)$ there exists a compact subset $K$ of $T$ such that $K \subset A$ and $\sup \{\delta^*(y|M(A\setminus K)); y \in E', \|y\| \leq 1\} \leq \varepsilon$. □

Let $M$ be a positive set-valued measure defined on $B(T)$. For the construction of the integral $\int fM$, with $f \in C_+(T)$ we refer to ([23], p. 17).

Lemma 3.3. Let $M : B(T) \to cc(E)$ be a positive regular set-valued measure. Then the set-valued map $L : C_+(T) \to cc(E)(f \mapsto L(f) = \int fM)$ is additive, positively homogeneous, positive and strictly continuous.

Proof. We only prove the strict continuity. The other properties follow from the construction of the integral $\int fM, f \in C_+(T)$. For each $n \in \mathbb{N}^*$ there exists a compact subset $K_n$ of $T$ such that $\sup \{\delta^*(y|M(T\setminus K_n)); y \in E', \|y\| \leq 1\} \leq 2^{-2n}$ (Lemma 3.2). We then have a sequence $(K_n)$ of compact subsets of $T$ that we may assume monotone increasing. We repeat here the proof of R. Giles ([13], p. 471,
Lemma 4.2. Consider $\varphi = \sum_{n=1}^{+\infty} 2^{-n}K_n$, we have $2^{-n-1} \leq \varphi(x) \leq 2^{-n}$ for all $x \in K_{n+1} \setminus K_n$. The function $1/\varphi$ is measurable and is $\delta^*(y|M(.))$-integrable for each $y \in E', \|y\| \leq 1$. We have $\int 1/\varphi \delta^*(y|M(.)) = \int_{\cup_{n=1}^{+\infty}(K_{n+1} \setminus K_n)} 1/\varphi \delta^*(y|M(.)) = \sum_{n=1}^{+\infty} \int_{K_{n+1} \setminus K_n} 1/\varphi \delta^*(y|M(.))$

$$\leq \sum_{n=1}^{+\infty} 2^{n+1} [\delta^*(y|M(K_{n+1})) - \delta^*(y|M(K_n))] \leq \sum_{n=1}^{+\infty} 2^{n+1}2^{-2n} = 2. \text{ Let } \varepsilon > 0 \text{ and let } \psi_n \in C_0 \text{ such that } \psi_n(x) = 2^{-n} \text{ for } x \in K_n \text{ and } 0 \leq \psi_n \leq 2^{-n}1_T. \text{ Put } \psi = \sum_{n=1}^{+\infty} \psi_n. \text{ Then } \psi \in C_0 \text{ and } \varphi \leq \psi. \text{ For all } f \in \{g \in C_+(T), p_{2\psi/\varepsilon}(g) < 1\} \text{ we have } f < \varepsilon/2\varphi \text{ and } \int f \delta^*(y|M(.)) < \varepsilon \text{ for all } y \in E' \text{ with } \|y\| \leq 1. \text{ Since } \delta^*(y|\int fM) = \int f \delta^*(y|M(.)), \text{ one has } \delta(\int fM, \{0\}) < \varepsilon. \text{ Therefore the map } f \mapsto \int fM \text{ is strictly continuous at } 0. \text{ The equality } \delta^*(y|\int fM) = \int f \delta^*(y|M(.)) \text{ for each } f \in C_+(T) \text{ and each } y \in E' \text{ enable us to prove the continuity on } C_+(T). \tag*{□}

Definition 3.4. A map $S : E' \rightarrow \mathbb{R}$ is said to be sublinear if for every $y \in E'$ and $y' \in E'$ and for every $\lambda \geq 0$ one has $S(y + y') \leq S(y) + S(y')$ and $S(\lambda y) = \lambda S(y)$.

The lemme below is a particular case of L. Hörmander’s result ([16], Theorem 5, p. 182). We give here an alternative proof.

Lemma 3.5. Let $E$ be a Banach space, and let $E'$ its dual space endowed with the Mackey topology $\tau(E',E)$. Let $S : E' \rightarrow \mathbb{R}$ be a sublinear map. Then $S$ is continuous if and only if there is $C \in cc(E)$ such that $S = \delta^*(\cdot|C)$.

Proof. Assume that $S$ is continuous. Let $\nabla S = \{l : E' \rightarrow \mathbb{R}; \text{ linear and } l \leq S\}$. By the Hahn-Banach theorem ([11], theorem 10, p. 62), $S(y) = \sup\{l(y); l \in \nabla S\}$ for each $y \in E'$. Let $l \in \nabla S$; then $l$ is continuous for the Mackey topology $\tau(E',E)$. Therefore $l$ determines an element $x_l \in E$ that verifies $l(y) = y(x_l)$ for each $y \in E'$. Let $\nabla ES = \{x_l; l \in \nabla S\}$. Since $\nabla S$ is equicontinuous there is a neighborhood $V$ of 0 in $E'$ such that $\nabla ES \subset V^\circ$, where $V^\circ$ is the polar of $V$ in $E$.

By the Alaoglu-Bourbaki’s theorem ([20], p. 248), one has $V^\circ \in cc(E)$. Since $\nabla ES$ is convex, its closure is one of elements of $cc(E)$ we want. The converse is obvious. Note that if $S$ is non-negative then $0 \in \nabla ES$. \tag*{□}

Theorem 3.6. Let $T$ be a completely regular topological space and let $C_+(T)$ be the space of bounded continuous non-negative functions defined on $T$ endowed with the strict topology. Let $E$ be a Banach space and $cc(E)$ be the space of convex weakly compact non-empty compact subsets of $E$ endowed with the Hausdorff distance. Let $L : C_+(T) \rightarrow cc(E)$ be a positive, additive, positively homogeneous and strictly continuous set-valued map. Then there is a unique positive regular set-valued measure $M$ defined on $B(T)$ to $cc(E)$ such that $L(f) = \int fM$ for all $f \in C_+(T)$.

Conversely for all positive regular set-valued measure $M : B(T) \rightarrow cc(E)$, the set-valued map $\theta : C_+(T) \rightarrow cc(E) (f \mapsto \theta(f) = \int fM)$ is positive, additive, positively homogeneous and strictly continuous.

Proof. Let $y \in E'$. The map $\delta^*(y|L(.)) : C_+(T) \rightarrow \mathbb{R} (f \mapsto \delta^*(y|L(f)))$ is additive, positively homogeneous and continuous. Then it can be extended to a continuous linear functional on $C(T)$. This extension is unique. It is denoted by $\delta^*(y|\bar{L}(.))$. Let $f \in C(T)$, one has $f = f^+ - f^-$ and $\delta^*(y|\bar{L}(.))$ is defined by $\delta^*(y|\bar{L}(.))(f) =$...
\[ \delta^*(y\mid L(f^+)) - \delta^*(y\mid L(f^-)) \] Since \( \delta^*(y\mid L(.)) \) is strictly continuous it is tight ([26], p. 41). By the lemma 3.1 and ([3], Proposition 5, p.58) there exists a unique regular positive Borel measure \( \mu_y \) on \( T \) that verifies \( \delta^*(y\mid L(f)) = \int f\mu_y \) for all \( f \in C(T) \). Let 0 an open subset of \( T \) and let \( S_O \) the map defined on \( E' \) to \( \mathbb{R} \) by \( S_O(y) = \mu_y(O) \) for each \( y \in E' \). We have \( \mu_y(O) = \sup\{ \int f\mu_y; f \in C_+(T), f \leq 1_O \} = \sup\{ \delta^*(y\mid L(f)); f \in C_+(T), f \leq 1_O \} \), therefore \( S_O \) is a sublinear map. Let now \( A \in \mathcal{B}(T) \). We denote by \( S_A \) the map defined on \( E' \) to \( \mathbb{R} \) by \( S_A(y) = \mu_y(A) \) for each \( y \in E' \). Since the measure \( \mu_y \) is regular we have \( S_A(y) = \inf\{\mu_y(O); O \subset T, O \text{ open and } O \supset A \} = \inf\{S_O(y); O \subset T, O \text{ open and } O \supset A \} \). Let \( y, y' \in E' \) and let \( \varepsilon > 0 \), there exists two open subsets \( O_e \) and \( O_{e}' \) of \( T \) containing \( A \) and such that \( S_A(y) \geq \mu_y(O_e) - \varepsilon/2, S_A(y') \geq \mu_y(O_{e}') - \varepsilon/2 \). We have \( \mu_y(O_e) + \mu_y(O_{e}') \leq S_A(y) + S_A(y') + \varepsilon \), then \( \mu_y(O_e \cap O_{e}') \leq S_A(y) + S_A(y') + \varepsilon \), therefore \( \mu_y(y'+y') \leq S_A(y) + S_A(y') + \varepsilon \). We have \( \mu_y(y+y') \leq \mu_y(y'+y') \leq S_A(y) + S_A(y') + \varepsilon \). \(~\varepsilon \text{ follows from this}~ S_A(y+y') \leq S_A(y) + S_A(y') \). It is obvious that for all \( \lambda \geq 0 \) and for all \( y \in E' \), \( S_A(\lambda y) = \lambda S_A(y) \). So \( S_A \) is a non-negative sublinear map. Let us prove now that \( S_A \) is continuous for the Mackey topology \( \tau(E', E) \). We have \( S_A(y) \leq \mu_y(T) = \delta^*(y\mid L(1_T)) \). Let \( \overline{L(1_T)} \) be the closed absolutely convex cover of \( L(1_T) \), one has \( \overline{L(1_T)} \in \mathcal{C}(E) \) and \( S_A(y) \leq \delta^*(y\mid \overline{L(1_T)}) \) for each \( y \in E' \) and \( A \in \mathcal{B}(T) \). We deduce that \( S_A \) is continuous for the Mackey topology for each \( A \in \mathcal{B}(T) \). By the lemma 3.5 there is \( C_A \in \mathcal{C}(E) \) such that \( S_A(y) = \delta^*(y\mid C_A) \) for all \( y \in E' \). Let \( M : B(T) \to \mathcal{C}(E) (A \mapsto M(A) = C_A) \). We have \( \delta^*(y\mid M(A)) = \mu_y(A) \) for all \( y \in E' \), hence the map \( \delta^*(y\mid M(.)) : B(T) \to \mathbb{R} (A \mapsto \delta^*(y\mid M(A))) \) is a positive regular countably additive measure. Then \( M \) is a regular set-valued measure. Since \( S_A \) is non-negative then \( M \) is positive. Let \( f \in C_+(T) \) and let \( y \in E' \), \( \int \delta^*(y\mid M(.)) = \int M(f) \). It follows that \( L(f) = \int fM \) for all \( f \in C_+(T) \) because \( \int f\delta^*(y\mid M(.)) = \delta^*(y\mid \int fM) \). Let us prove that \( M \) is unique. Assume that there exist two regular positive set-valued measures \( M \) and \( M' \) which verify \( \int fM = L(f) = \int fM' \). Let 0 be an open subset of \( T \) and let \( y \in E' \). According to the inner regularity of \( \delta^*(y\mid M(.)) \) and ([3] Lemme 1 p. 55) we have \( \delta^*(y\mid M(O)) = \sup\{\delta^*(y\mid L(f)); f \in C_+(T), f \leq 1_O \} = \delta^*(y\mid M'(O)) \). Moreover the outer regularity of \( \delta^*(y\mid M(.)) \) shows that \( \delta^*(y\mid M(A)) = \delta^*(y\mid M'(A)) \) for all \( A \in \mathcal{B}(T) \) and \( y \in E' \), hence \( M(A) = M'(A) \) for all \( A \in \mathcal{B}(T) \). The second assertion of the theorem is justified by the lemma 3.3.

The following corollary is the result of R. Giles.

**Corollary 3.7.** ([13], Theorem 4.6) For any completely regular space \( T \) the dual of \( C(T) \) under the strict topology is the space of all bounded signed Borel regular measures on \( T \).

**Proof.** Let \( L \) be a strictly continuous linear functional on \( C(T) \); \( L \) is bounded. Therefore \( L \) is the difference of two non-negative linear functional. We may assume that \( L \) is non-negative. Let \( K_0 \) be an element of \( \mathcal{C}(E) \) that contains 0 and that is subset of the unit ball of \( E \). Consider the map \( L' : C_+(T) \to \mathcal{C}(E) \) defined by \( L'(f) = L(f)K_0 = \{L(f)k; k \in K_0 \} \) for all \( f \in C_+(T) \). The map \( L' \) is positive, positively homogeneous and strictly continuous. Let us prove that \( L' \) is additive. The inclusion \( L'(f+g) \subset L'(f) + L'(g) \) for all \( f, g \in C_+(T) \) is trivial. Let \( u \in K_0 \) and each let \( v \in K_0 \), \( L(f)u + L(g)v = L(f+g) \left[ \frac{L(f)}{L(f+g)}u + \frac{L(g)}{L(f+g)}v \right] \). Since \( K_0 \) is convex and \( L \) positive, \( \frac{L(f)}{L(f+g)}u + \frac{L(g)}{L(f+g)}v \in K_0 \). Then \( L'(f) + L'(g) \subset L'(f+g) \).
By the Theorem 3.6, there is a unique positive regular set-valued measure $M : B(T) \to \text{cc}(E)$ that satisfies the condition $\int fM = L'(f)$ for all $f \in C_+(T)$.

Let $y_0 \in E'$ such that $\delta^*(y_0|L'(.)) = L$. Since $\delta^*(y_0|\int fM) = \int f\delta^*(y_0|M(.))$ for all $f \in C_+(T)$ we then have $\int f\delta^*(y_0|M(.)) = L(f)$ for all $f \in C_+(T)$ and therefore $\int f\delta^*(y_0|M(.)) = L(f)$ for all $f \in C(T)$. The uniqueness of $\delta^*(y_0|M(.))$ follows from the regularity of $M$. Taking the lemma 3.3 (for the scalar measures) into account we conclude that there is a bijection between the dual space of $(C(T), \beta)$ and the space of all bounded signed regular Borel measures on $T$. □

References


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