ON QUASI-POWER INCREASING SEQUENCES AND THEIR SOME APPLICATIONS

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Abstract. In [6], we proved a main theorem dealing with $|\tilde{N}, p_n, \theta_n |_k$ summability factors using a new general class of power increasing sequences instead of a quasi-$\sigma$-power increasing sequence. In this paper, we prove that theorem under weaker conditions. This theorem also includes some new results.

1. Introduction

A positive sequence $X = (X_n)$ is said to be a quasi-$f$-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $K f_n X_n \geq f_m X_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^{\eta}(\log n)^{\eta}, \eta \geq 0, 0 < \sigma < 1\}$ (see [13]). If we set $\eta = 0$, then we get a quasi-$\sigma$-power increasing sequence (see [10]). We write $BV = BV \cap C_O$, where $C_O = \{ x = (x_k) \in \Omega : \lim_k |x_k| = 0 \}$, $BV = \{ x = (x_k) \in \Omega : \sum_k |x_k - x_{k+1}| < \infty \}$ and $\Omega$ being the space of all real-valued sequences. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $(s_n)$. We denote by $u_n^\alpha$ the $n$th Cesàro mean of order $\alpha$, with $\alpha > -1$, of the sequence $(s_n)$, that is (see [7]),

\[ u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \]

where

\[ A_n^\alpha = \frac{(\alpha + 1)(\alpha + 2)\ldots(\alpha + n)}{n!} = O(n^\alpha), \quad A_{n}^{\alpha-n} = 0 \text{ for } n > 0. \]

A series $\sum a_n$ is said to be summable $| C, \alpha |_k$, $k \geq 1$, if (see [8])

\[ \sum_{n=1}^{\infty} n^{\alpha-1} \left| u_n^\alpha - u_{n-1}^\alpha \right| < \infty. \]

If we take $\alpha = 1$, then we get the $| C, 1 |_k$ summability. Let $(p_n)$ be a sequence of positive real numbers such that

\[ P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = P_{-i} = 0, \ i \geq 1). \]

The sequence-to-sequence transformation

\[ v_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \]

defines the sequence $(v_n)$ of the Riesz mean or simply the $(\tilde{N}, p_n)$ mean of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$ (see [9]). Let $(\theta_n)$ be any sequence of positive constants. The series $\sum a_n$ is said to be summable $| \tilde{N}, p_n, \theta_n |_k$, $k \geq 1$, if (see [12])

\[ \sum_{n=1}^{\infty} n^{\alpha-1} \left| v_n - v_{n-1} \right|^k < \infty. \]

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If we take $\theta_n = \frac{p_n}{p_{n+1}}$, then $|\tilde{N}, p_n, \theta_n | k$ summability reduces to $|\tilde{N}, p_n | k$ summability (see [1]). Also, if we take $\theta_n = \frac{p_n}{p_{n+1}}$ and $p_n = 1$ for all values of $n$, then we get $| C, 1 | k$ summability. Furthermore, if we take $\theta_n = n$, then $|\tilde{N}, p_n, \theta_n | k$ summability reduces to $| R, p_n | k$ summability (see [2]).

2. Known Results. The following theorems are known:

**Theorem A** ([4]). Let $\left(\frac{\theta_n}{p_{n+1}}\right)$ be a non-increasing sequence. Let $(\lambda_n) \in BV_O$ and let $(X_n)$ be a quasi-$\sigma$-power increasing sequence for some $\sigma (0 < \sigma < 1)$. Suppose also that there exist sequences $(\beta_n)$ and $(\lambda_n)$ such that
\[
|\Delta \lambda_n| \leq \beta_n,
\]
(7) $\beta_n \to 0$ as $n \to \infty$,
(8) $\sum_{n=1}^{\infty} n |\Delta \beta_n | X_n < \infty$,
(9) $|\lambda_n | X_n = O(1)$.
If
(10) $\sum_{v=1}^{n} \theta_v^{k-1} \frac{|s_v|^k}{v^k} = O(X_n) \text{ as } n \to \infty$,
and $(p_n)$ is a sequence such that
(12) $P_n = O(np_n),
(13) P_n \Delta p_n = O(p_n p_{n+1}),
then the series $\sum_{n=1}^{\infty} a_n \frac{p_n \lambda_n}{np_n}$ is summable $|\tilde{N}, p_n, \theta_n | k$, $k \geq 1$.

**Remark.** We can take $(\lambda_n) \in BV$ instead of $(\lambda_n) \in BV_O$ and it is sufficient to prove Theorem A.

**Theorem B** ([6]). Let $\left(\frac{\theta_n}{p_{n+1}}\right)$ be a non-increasing sequence. Let $(\lambda_n) \in BV$ and let $(X_n)$ be a quasi-f-power increasing sequence for some $\sigma (0 < \sigma < 1)$ and $\eta \geq 0$. If the conditions (7)-(10), (12)-(13), and
(11) $\sum_{n=1}^{\infty} a_n \frac{p_n \lambda_n}{np_n}$ is summable $|\tilde{N}, p_n, \theta_n | k$, $k \geq 1$.
It should be noted that if we take $\eta = 0$, then we obtain Theorem A.

3. The Main result. The purpose of this paper is to prove Theorem B under weaker conditions. Now, we shall prove the following general theorem.

**Theorem.** Let $\left(\frac{\theta_n}{p_{n+1}}\right)$ be a non-increasing sequence. Let $(X_n)$ be a quasi-f-power increasing sequence for some $\sigma (0 < \sigma < 1)$ and $\eta \geq 0$. If the conditions (7)-(10), (12)-(13), and
(14) $\sum_{v=1}^{n} \theta_v^{k-1} \frac{|s_v|^k}{v^k X_v^{k-1}} = O(X_n) \text{ as } n \to \infty$
are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{p_n \lambda_n}{np_n}$ is summable $|\tilde{N}, p_n, \theta_n | k$, $k \geq 1$.

**Remark.** It should be noted that condition (14) is reduced to the condition (11), when $k=1$. When $k > 1$, the condition (14) is weaker than the condition (11), but the converse is not true. As in [14] we can show that if (11) is satisfied, then we get that
\[
\sum_{v=1}^{n} \theta_v^{k-1} \frac{|s_v|^k}{v^k X_v^{k-1}} = O\left(\frac{1}{X^{k-1}_1}\right) \sum_{v=1}^{n} \theta_v^{k-1} \frac{|s_v|^k}{v^k} = O(X_n).
\]
If (14) is satisfied, then for $k > 1$ we obtain that
\[
\sum_{v=1}^{n} \theta_v^{k-1} \frac{|s_v|^k}{v^k} = \sum_{v=1}^{n} \theta_v^{k-1} X_v^{k-1} \frac{|s_v|^k}{v^k X_v^{k-1}} = O(X_n) \sum_{v=1}^{n} \theta_v^{k-1} \frac{|s_v|^k}{v^k} = O(X_n) \neq O(X_n).
\]
Also, it should be noted that the condition ”$(\lambda_n) \in BV$” has been removed.

We require the following lemmas for the proof of the theorem.
**Lemma 1** ([5]). Under the conditions on \((X_n), (\beta_n)\) and \((\lambda_n)\) as expressed in the statement of the theorem, we have the following:

\[ nX_n\beta_n = O(1), \]

\[ \sum_{n=1}^{\infty} \beta_n X_n < \infty. \]

**Lemma 2** ([11]). If the conditions (12) and (13) are satisfied, then we have that

\[ \Delta \left( \frac{P_n}{np_n} \right) = O \left( \frac{1}{n} \right). \]

4. **Proof of the theorem.** Let \((T_n)\) be the sequence of \((\bar{N}, p_n)\) mean of the series \(\sum_{n=1}^{\infty} \frac{a_n p_n \lambda_n}{np_n}\).

Then, by definition, we have

\[ T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{vp_v}. \]

Then, for \(n \geq 1\) we obtain that

\[ T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_{v-1} P_v a_v \lambda_v. \]

Using Abel’s transformation, we get

\[
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \Delta \left( \frac{P_{v-1} P_v \lambda_v}{vp_v} \right) + \frac{\lambda_n s_n}{n} \\
= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta \lambda_v}{(v+1)p_{v+1}} \\
+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta \left( \frac{P_v}{vp_v} \right) - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} \\
= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\]

To prove the theorem, by Minkowski’s inequality, it is sufficient to show that

\[ \sum_{n=1}^{\infty} \theta_n^{k-1} | T_{n,r} |^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \]

Firstly, by using Abel’s transformation, we have that

\[
\sum_{n=1}^{m} \theta_n^{k-1} | T_{n,1} |^k = \sum_{n=1}^{m} \theta_n^{k-1} n^{-k} | \lambda_n |^{k-1} | \lambda_n || s_n |^k \\
= O(1) \sum_{n=1}^{m} | \lambda_n | \left( \frac{1}{X_n} \right)^{k-1} \theta_n^{k-1} n^{-k} | s_n |^k \\
= O(1) \sum_{n=1}^{m} \Delta | \lambda_n | \sum_{v=1}^{n} \theta_v^{k-1} \left| \frac{s_v}{X_v^{k-1} v^k} \right| \\
+ O(1) \lambda_m \sum_{v=1}^{m} \theta_v^{k-1} \left| \frac{s_v}{X_v^{k-1} v^k} \right| \\
= O(1) \sum_{n=1}^{m} | \Delta \lambda_n | X_n + O(1) | \lambda_m | X_m \\
= O(1) \sum_{n=1}^{m} \beta_n X_n + O(1) | \lambda_m | X_m = O(1) \quad \text{as} \quad m \to \infty.
\]
by virtue of the hypotheses of the theorem and Lemma 1. Now, using (12) and applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} | T_{n,2} |^k = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}^{k-1}} \left| \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}^{k-1}} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} | s_v | p_v | \Delta \lambda_v | \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}^{k-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right)^k | s_v |^k p_v (\beta_v)^k$$

$$\times \left( \frac{1}{P_n-1} \sum_{v=1}^{n-1} p_v \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \left| s_v \right|^k p_v (\beta_v)^k \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \left| s_v \right|^k p_v (\beta_v)^k \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \left| s_v \right|^k (\beta_v)^k \left( \frac{p_v}{P_v} \right)^{k-1} \theta_v^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \left( \frac{1}{X_v} \right)^{k-1} | v \theta_v^{k-1} | s_v |^k$$

$$= O(1) \sum_{v=1}^{m} \Delta(v \beta_v) \sum_{r=1}^{v} \theta_r^{k-1} \left| s_r \right|^k + O(1) m \beta_m \sum_{v=1}^{m} \theta_v^{k-1} \left| s_v \right|^k$$

$$= O(1) \sum_{v=1}^{m} \Delta(v \beta_v) | X_v | + O(1) m \beta_m X_m$$

$$= O(1) \sum_{v=1}^{m} (v+1) \Delta \beta_v - \beta_v | X_v | + O(1) m \beta_m X_m$$

$$= O(1) \sum_{v=1}^{m} v | \Delta \beta_v | X_v + O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) m \beta_m X_m = O(1)$$

as $m \to \infty$, in view of the hypotheses of the theorem and Lemma 1. Again, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} | T_{n,3} |^k = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}^{k-1}} \left\{ \sum_{v=1}^{n-1} P_v | s_v | \lambda_v \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}^{k-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right)^k v^{-k} p_v | s_v |^k \lambda_v |^k$$

$$\times \left\{ \frac{1}{P_n-1} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$
\[ = O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^k v^{-k} \left| s_v \right|^k \left| \lambda_v \right|^k \left| \sum_{n=v}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right) \right|^{k-1} \frac{p_n}{P_n P_{n-1}} \]
\[ = O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^{k-1} v^{-k} \theta_{v-1}^{k-1} \left( \frac{p_v}{P_v} \right)^{k-1} \left| s_v \right|^k \left| \lambda_v \right|^k \left| \sum_{n=v}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right) \right|^{k-1} \frac{p_n}{P_n P_{n-1}} \]
\[ = O(1) \sum_{v=1}^{m} \left| \lambda_v \right| \left( \frac{1}{X_v} \right)^{k-1} \theta_{v-1}^{k-1} v^{-k} \left| s_v \right|^k \]
\[ = O(1) \sum_{v=1}^{m} \left| \lambda_v \right| \theta_{v-1}^{k-1} \left| s_v \right|^k \frac{1}{v^k X_v} = O(1) \quad \text{as} \quad m \rightarrow \infty, \]

in view of the hypotheses of the theorem, Lemma 1 and Lemma 2. Finally, using Hölder’s inequality, as in \( T_{n,1} \) we have that
\[
\sum_{n=2}^{m+1} \theta_{n-1}^{k-1} | T_{n,4} |^k = \sum_{n=2}^{m+1} \theta_{n-1}^{k-1} \left( \frac{p_n}{P_n} \right)^k \left| \frac{1}{P_n^{k-1}} \right| \left| \sum_{v=1}^{n-1} s_v P_v p_n \lambda_v \right|^k
\]
\[ = O(1) \sum_{n=2}^{m+1} \theta_{n-1}^{k-1} \left( \frac{p_n}{P_n} \right)^k \left| \frac{1}{P_n^{k-1}} \right| \left| \sum_{v=1}^{n-1} s_v P_v p_n \lambda_v \right|^k
\]
\[ = O(1) \sum_{n=2}^{m+1} \theta_{n-1}^{k-1} \left( \frac{p_n}{P_n} \right)^k \left| \frac{1}{P_n^{k-1}} \right| \left| \sum_{v=1}^{n-1} s_v P_v p_n \lambda_v \right|^k
\]
\[ \times \left( \frac{1}{P_n^{k-1}} \sum_{v=1}^{n-1} P_v \right)^{k-1}
\]
\[ = O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^k v^{-k} \left| s_v \right|^k \left| \lambda_v \right|^k \left| \frac{1}{P_v} \right| \theta_v^{k-1} \left( \frac{P_v}{P_v} \right)^{k-1} \left| s_v \right|^k \]
\[ = O(1) \sum_{v=1}^{m} \left| \lambda_v \right| \theta_v^{k-1} \left| s_v \right|^k \frac{1}{v^k X_v} = O(1) \quad \text{as} \quad m \rightarrow \infty. \]

This completes the proof of the theorem. If we set \( \eta \geq 0 \), then we obtain Theorem B under weaker conditions. If we take \( p_n = 1 \) for all values of \( n \), then we have a new result for \( | C, 1, \theta_n |_k \) summability. Furthermore, if we take \( \theta_n = n \), then we have another new result for \( | R, p_n |_k \) summability. Finally, if we take \( p_n = 1 \) for all values of \( n \) and \( \theta_n = n \), then we get a new result dealing with \( | C, 1 |_k \) summability factors.

References


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