A NEW RESULT ON GENERALIZED ABSOLUTE CESÁRO SUMMABILITY

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ABSTRACT. In [4], a main theorem dealing with an application of almost increasing sequences, has been proved. In this paper, we have extended that theorem by using a general class of quasi power increasing sequences, which is a wider class of sequences, instead of an almost increasing sequence. This theorem also includes some new and known results.

1. Introduction

A positive sequence \((b_n)\) is said to be an almost increasing sequence if there exists a positive increasing sequence \((c_n)\) and two positive constants \(M\) and \(N\) such that \(Mc_n \leq b_n \leq Nc_n\) (see [1]). A sequence \((d_n)\) is said to be \(\delta\)-quasi monotone, if \(d_n \to 0\), \(d_n > 0\) ultimately, and \(\Delta d_n \geq -\delta_n\), where \(\Delta d_n = d_n - d_{n+1}\) and \(\delta = (\delta_n)\) is a sequence of positive numbers (see [2]). A positive sequence \(X = (X_n)\) is said to be a quasi-\(p\)-power increasing sequence if there exists a constant \(K = K(X, f) \geq 1\) such that \(K f_n X_n \geq f_m X_m\) for all \(n \geq m \geq 1\), where \(f = \{f_n(\sigma, \gamma)\} = \{n^\sigma (\log n)^\gamma, \gamma \geq 0, 0 < \sigma < 1\}\) (see [11]). If we take \(\gamma = 0\), then we get a quasi-\(\sigma\)-power increasing sequence. Every almost increasing sequence is a quasi-\(\sigma\)-power increasing sequence for any non-negative \(\sigma\), but the converse is not true for \(\sigma > 0\) (see [9]). Let \(\sum a_n\) be a given infinite series. We denote by \(t_n^{\alpha, \beta}\) the \(n\)th Cesàro mean of order \((\alpha, \beta)\), with \(\alpha + \beta > -1\), of the sequence \((na_n)\), that is (see [6])

\[
t_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha} t_v^{\alpha, \beta},
\]

where

\[
A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for} \quad n > 0.
\]

Let \((\theta_n^{\alpha, \beta})\) be a sequence defined by (see [3])

\[
\theta_n^{\alpha, \beta} = \begin{cases} 
|t_n^{\alpha, \beta}|, & \alpha = 1, \beta > -1 \\
\max_{1 \leq v \leq n} |t_v^{\alpha, \beta}|, & 0 < \alpha < 1, \beta > -1.
\end{cases}
\]

The series \(\sum a_n\) is said to be summable \(|C, \alpha, \beta|_k\), \(k \geq 1\), if (see [7])

\[
\sum_{n=1}^{\infty} \frac{1}{n} |t_n^{\alpha, \beta}|^k < \infty.
\]

If we take \(\beta = 0\), then \(|C, \alpha, \beta|_k\) summability reduces to \(|C, \alpha|_k\) summability (see [8]).

The first author has proved the following main theorem.

**Theorem A** ([4]). Let \((\theta_n^{\alpha, \beta})\) be a sequence defined as in (3). Let \((X_n)\) be an almost increasing sequence such that \(|\Delta X_n| = O(X_n/n)\) and let \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((A_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum \delta_n X_n < \infty\), \(\sum A_n X_n\) is convergent, and \(|

\[
\Delta \lambda_n | \leq |A_n| \quad \text{for all} \quad n.
\]

If the condition

\[
\sum_{n=1}^{\infty} \frac{(\theta_n^{\alpha, \beta})^k}{n} = O(X_m) \quad \text{as} \quad m \to \infty
\]

satisfies, then the series \(\sum a_n X_n\) is summable \(|C, \alpha, \beta|_k\), \(0 < \alpha \leq 1\), \(\alpha + \beta > 0\), and \(k \geq 1\).

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2. The main result.

The aim of this paper is to extend Theorem A by using a quasi-f-power increasing sequence, which is a general class of quasi power increasing sequences, instead of an almost increasing sequence. We shall prove the following theorem.

**Theorem.** Let \( (\theta_{n}^{\alpha, \beta}) \) be a sequence defined as in (3). Let \((X_{n})\) be a quasi-f-power increasing sequence and let \( \lambda_{n} \to 0 \) as \( n \to \infty \). Suppose that there exists a sequence of numbers \((A_{n})\) such that it is \( \delta \)-quasi-monotone with \( \Delta A_{n} \leq \delta_{n}, \sum \delta_{n}X_{n} < \infty, \sum A_{n}X_{n} \) is convergent, and \| \Delta \lambda_{n} \| \leq | A_{n} | \) for all \( n \). If the condition (5) is satisfied, then the series \( \sum a_{n} \lambda_{n} \) is summable \( | C, \alpha, \beta |_{k}, 0 < \alpha \leq 1, \alpha + \beta > 0, \) and \( k \geq 1 \).

If we take \((X_{n})\) as an almost increasing sequence such that \| \Delta X_{n} \| = O(X_{n}/n), then we get Theorem A, in this case condition \( \Delta A_{n} \leq \delta_{n} \) is not needed.

We need the following lemmas for the proof of our theorem.

**Lemma 1 ([3]).** If \( 0 < \alpha \leq 1, \beta > -1, \) and \( 1 \leq v \leq n \), then

\[
| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} a_{p} | \leq \max_{1 \leq m \leq v} | \sum_{p=0}^{m} A_{n-p}^{\alpha-1} A_{p}^{\beta} a_{p} |.
\]

**Lemma 2 ([5]).** Let \((X_{n})\) be a quasi-f-power increasing sequence. If \((A_{n})\) is a \( \delta \)-quasi-monotone sequence with \( \Delta A_{n} \leq \delta_{n} \) and \( \sum n\delta_{n}X_{n} < \infty \), then

\[
\sum_{n=1}^{\infty} nX_{n} | \Delta A_{n} | < \infty,
\]

\[
nA_{n}X_{n} = O(1) \text{ as } n \to \infty.
\]

3. Proof of the theorem

Let \((T_{n}^{\alpha, \beta})\) be the \( n \)th \((C, \alpha, \beta)\) mean of the sequence \((na_{n}\lambda_{n})\). Then, by (1), we have

\[
T_{n}^{\alpha, \beta} = \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} va_{v} \lambda_{v}.
\]

Applying Abel’s transformation first and then using Lemma 1, we obtain that

\[
T_{n}^{\alpha, \beta} = \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta \lambda_{v} \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} pa_{p} + \frac{\lambda_{n}}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} va_{v},
\]

\[
| T_{n}^{\alpha, \beta} | \leq \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} | \Delta \lambda_{v} | \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} pa_{p} | + \frac{\lambda_{n}}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} va_{v} | \leq \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} A_{v}^{(\alpha+\beta)} \theta_{v}^{\alpha, \beta} | \Delta \lambda_{v} | + \frac{\lambda_{n}}{A_{n}^{\alpha+\beta}} T_{n,1}^{\alpha, \beta} + T_{n,2}^{\alpha, \beta}.
\]

To complete the proof of the theorem, by Minkowski’s inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \frac{1}{n} | T_{n,r}^{\alpha, \beta} |^{r} < \infty, \text{ for } r = 1, 2.
\]
When \( k > 1 \), we can apply Hölder’s inequality with indices \( k \) and \( k' \), where \( \frac{1}{k} + \frac{1}{k'} = 1 \), we get that

\[
\sum_{n=2}^{m+1} \frac{1}{n} | T_{n,1}^{\alpha,\beta} |^k \leq \sum_{n=2}^{m+1} \frac{1}{n} A_n^{(\alpha+\beta)} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \Delta \lambda_v |^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+\alpha+\beta}} \left\{ \sum_{v=1}^{n-1} v^{(\alpha+\beta)+1} | A_v | (\theta_v^{\alpha,\beta})^k \right\} \times \left\{ \sum_{v=1}^{n-1} | A_v | \right\}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)+1} | A_v | (\theta_v^{\alpha,\beta})^k \int_0^\infty \frac{dx}{x^{1+\alpha+\beta}} = O(1) \sum_{v=1}^{m} v | A_v | (\theta_v^{\alpha,\beta})^k
\]

\[
= O(1) \sum_{v=1}^{m} |(v+1)\Delta A_v| + O(1) \sum_{v=1}^{m} | A_v | X_v + O(1)m | A_m | X_m
\]

\[
= O(1) \text{ as } m \to \infty,
\]

in view of hypotheses of the theorem and Lemma 2. Similarly, we have that

\[
\sum_{n=1}^{m} \frac{1}{n} | T_{n,2}^{\alpha,\beta} |^k = O(1) \sum_{n=1}^{m} \frac{\lambda_n}{n} (\theta_n^{\alpha,\beta})^k = O(1) \sum_{n=1}^{m} (\theta_n^{\alpha,\beta})^k \sum_{v=n}^{\infty} | \Delta \lambda_v |
\]

\[
= O(1) \sum_{v=1}^{\infty} | \Delta \lambda_v | (\theta_v^{\alpha,\beta})^k = O(1) \sum_{v=1}^{\infty} | \Delta \lambda_v | X_v
\]

\[
= O(1) \sum_{v=1}^{\infty} | A_v | X_v < \infty.
\]

This completes the proof of the theorem. If we take \( \beta = 0 \), then we get a new result concerning the \(| C, \alpha |_k\) summability factors. If we set \( \beta = 0, \alpha = 1 \), and \( X_n = \log n \), then we obtain the result of Mazhar dealing with \(| C, 1 |_k\) summability factors (see [10]). Finally, if we take \( \gamma = 0 \), then we get a new result dealing with an application of quasi-\( \sigma \)-power increasing sequences.

References


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