APPROXIMATION THEOREMS FOR $q$– ANALOUGE OF A LINEAR POSITIVE OPERATOR BY A. LUPAS

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ABSTRACT. The purpose of the present paper is to introduce $q$– analouge of a sequence of linear and positive operators which was introduced by A. Lupas [1]. First, we estimate moments of the operators and then prove a basic convergence theorem. Next, a local direct approximation theorem is established. Further, we study the rate of convergence and point-wise estimate using the Lipschitz type maximal function.

1. Introduction

At the International Dortmund Meeting held in Written (Germany, March, 1995), A. Lupas [1] introduced the following Linear positive operators:

\[ L_n(f; x) = (1 - a)\sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f \left( \frac{k}{n} \right), \quad x \geq 0. \]

with $f : [0, \infty] \to \mathbb{R}$. If we impose that $L_ne_1 = e_1$ we find that $a = 1/2$. Therefore operator (1) becomes

\[ L_n(f; x) = 2^{1-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2k!} f \left( \frac{k}{n} \right), \quad x \geq 0, \]

where

\[ (\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha + 1)...(\alpha + k - 1), k \geq 1. \]

The $q$– analouge of the above operators is defined as:

\[ L_{n,q}(f; x) = 2^{-[nx]} \sum_{k=0}^{\infty} \frac{([nx]_q)_k}{2k![k]_q!} f \left( \frac{[k]_q}{[n]_q} \right), \quad x \geq 0, \]

We denote $C_B[0, \infty)$ the space of real valued bounded continuous function $f$ on the interval $[0, \infty)$, the norm on the space is defined as

\[ \| f \| = \sup_{0 \leq x < \infty} |f(x)|. \]

Let $W^2 = \{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \}$. The Peetre’s $K$– functional is defined as

\[ K_2(f, \delta) = \inf_{g \in W^2} \{ \| f - g \| + \delta \| g'' \| \}, \]

where $\delta > 0$.

For $f \in C_B[0, \infty)$ a usual modulus of continuity is given by

\[ \omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{0 \leq x < \infty} |f(x + h) - f(x)|. \]

The second order modulus of smoothness is given by

\[ \omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{0 \leq x < \infty} |f(x + 2h) - 2f(x + h) + f(x)|. \]

2010 Mathematics Subject Classification. 41A36, 41A99, 41A25.

Key words and phrases. linear positive operators; $q$–operators; rate of convergence.
By [[3], p.177, Theorem 2.4] there exists an absolute constant $C > 0$ such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}).$$

In recent years, many results about the generalization of linear positive operators have been obtained by several mathematicians ([6]-[17]).

2. Moment estimates

**Lemma 1.** The following relations hold:

$$L_{n,q}(1; x) = 1, L_{n,q}(t; x) = x$$

and

$$L_{n,q}(t^2; x) = qx^2 + \frac{1 + q}{|n|}x.$$

**Proof.** We have

$$L_{n,q}(1; x) = 2^{-|n|}x \sum_{k=0}^{\infty} \frac{\left[|n|q^x\right]_k}{2^k[k]_q} \frac{[k]_q}{|n|} = 1.$$

Now,

$$L_{n,q}(t; x) = 2^{-|n|}x \sum_{k=0}^{\infty} \frac{\left[|n|q^x\right]_k}{2^k[k]_q} \frac{[k]_q}{|n|}$$

$$= 2^{-|n|}x \sum_{k=0}^{\infty} \frac{\left[|n|q^x\right]_k}{2^k[k-1]_q} \frac{1}{|n|}$$

$$= 2^{-|n|}x \sum_{k=1}^{\infty} \frac{|n|q^x([n]q^x + 1)_{k-1}}{2^{k-1}[k-1]_q}$$

$$= 2^{-|n|}x \sum_{k=1}^{\infty} \frac{|n|q^x + 1)_{k-1}}{2^{k-1}[k-1]_q} = x.$$

Next,

$$L_{n,q}(t^2; x) = 2^{-|n|}x \sum_{k=0}^{\infty} \frac{\left[|n|q^x\right]_k}{2^k[k]_q} \frac{[k]_q^2}{|n|^2}$$

$$= 2^{-|n|}x \sum_{k=0}^{\infty} \frac{|n|q^x([n]q^x + 1)_{k-1}}{2^k[k]_q[k-1]_q} \frac{[k]_q^2}{|n|^2}$$

$$= 2^{-|n|}x \sum_{k=1}^{\infty} \frac{|n|q^x + 1)_{k-1}}{2^{k-1}[k-1]_q} \frac{[k]_q}{|n|}$$

$$= 2^{-|n|}x \sum_{k=1}^{\infty} \frac{|n|q^x + 1)_{k-1}}{2^{k-1}[k-1]_q} \frac{[k+1]_q}{|n|}$$

$$= 2^{-|n|}x \sum_{k=0}^{\infty} \frac{(|n]q^x + 1)_{k}(1 + q[k]_q)}{2^k[k]_q}$$

$$= 2^{-|n|}x \sum_{k=0}^{\infty} \frac{(|n]q^x + 1)_{k}}{2^k[k]_q} + 2^{-|n|}x \sum_{k=0}^{\infty} \frac{(|n]q^x + 1)_{k}q[k]_q}{2^k[k]_q}$$

$$= I_1 + I_2, \text{say.}$$
We find that $I_1 = \frac{x}{[n]_q}$.

Now,

$$I_2 = \frac{2^{-[n]_q x - 1} x}{[n]_q} \sum_{k=0}^{\infty} \frac{([n]_q x + 1)_k q[k]_q}{2^k [k]_q !}$$

$$= \frac{2^{-[n]_q x - 2} qx}{[n]_q} \sum_{k=1}^{\infty} \frac{([n]_q x + 1)([n]_q x + 2)_k}{2^{k-1} [k - 1]_q !}$$

$$= \frac{2^{-[n]_q x - 2} qx ([n]_q x + 1)}{[n]_q} \sum_{k=1}^{\infty} \frac{([n]_q x + 2)_k}{2^{k-1} [k - 1]_q !}$$

$$= \frac{2^{-[n]_q x - 2} qx ([n]_q x + 1)}{[n]_q} \sum_{k=0}^{\infty} \frac{([n]_q x + 2)_k}{2^k [k]_q !} = \frac{qx ([n]_q x + 1)}{[n]_q}.$$

Hence, on combining $I_1$ and $I_2$, we get

$$L_{n,q}(t^2; x) = \frac{(1 + q)x}{[n]_q} + qx^2.$$

Let us define $m$th order moment by $\psi_{n,m}(q; x) = L_{n,q}((t - x)^m; x)$.

**Lemma 2.** Let $0 < q < 1$, then for $x \in [0, \infty)$ we have

$$\psi_{n,1}(q; x) = 0 \text{ and } \psi_{n,2}(q; x) = \frac{x([2] - (1 - q)[n]_q x)}{[n]_q}.$$

**Proof.** We have

$$\psi_{n,1}(q; x) = L_{n,q}(t - x; x) = 0.$$

Now,

$$\psi_{n,2}(q; x) = L_{n,q}((t - x)^2; x)$$

$$= L_{n,q}(t^2 + x^2 - 2tx; x)$$

$$= \frac{(1 + q)x}{[n]_q} + (q - 1)x^2.$$


3. Basic Pointwise Convergence

The operators $L_{n,q}$ do not satisfy the conditions of the Bohman-Korovkin theorem in case $0 < q < 1$. To make this theorem applicable, we can choose a sequence $(q_n)$ in place of the number $q$ such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$. With this modification we obtain the following Korovkin type result:

**Theorem 1.** Let $f \in C_B[0, \infty)$ and $q_n$ be a real sequence in $(0, 1)$ such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$. Then, for each $x \in [0, \infty)$ we have

$$\lim_{n \to \infty} L_{n,q_n}(f; x) = f(x).$$

**Proof.** The proof is based on the well known Korovkin theorem regarding the convergence of a sequence of linear positive operators. So, it is enough to prove the conditions

$$\lim_{n \to \infty} L_{n,q_n}(t^m; x) = x^m, m = 0, 1, 2.$$

Now, using Lemma 1 we obtain

$$\lim_{n \to \infty} L_{n,q_n}(1; x) = 1,$$

$$\lim_{n \to \infty} L_{n,q_n}(t; x) = x.$$
and
\[ \lim_{n \to \infty} L_{n,q_n}(t; x) = \lim_{n \to \infty} q_n x^2 + \frac{1 + q_n}{|n|q_n} x = x^2. \]

This completes the proof. \(\square\)

4. DIRECT RESULTS

**Theorem 2.** Let \( f \in C_B[0, \infty) \) and \( q \in (0, 1) \). Then, for each \( x \in [0, \infty) \) and \( n \in \mathbb{N} \) there exists an absolute constant \( C > 0 \) such that
\[ |L_{n,q}(f; x) - f(x)| \leq C \omega_2\left( f, \sqrt{x([2] - (1 - q)[n]q)x} \right). \]

**Proof.** Let \( g \in W^2 \) and \( x, t \in [0, \infty) \). Using Taylor's expansion we can write
\[ g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - v)g''(v)dv. \]

On application of Lemma 2 we obtain
\[ L_{n,q}(g(t); x) - g(x)) = L_{n,q}\left( \int_x^t (t - v)g''(v)dv; x \right). \]

Now, we have \( \left| \int_x^t (t - v)g''(v)dv \right| \leq (t - x)^2\|g''\|. \) Therefore
\[ |L_{n,q}(g(t); x) - g(x)| \leq L_{n,q}\left( (t - x)^2; x \right) \|g''\| = x([2] - (1 - q)[n]q)x\|g''\|. \]

By Lemma 1, we have
\[ |L_{n,q}(f; x)| \leq 2^{-|n|}\sum_{k=0}^{\infty} \frac{(|n|q)^k}{2^k|k|_q^2} \|f\| \frac{|k|_q}{|n|_q}. \]

Thus
\[ |L_{n,q}(f; x) - f(x)| \leq |L_{n,q}(f - g; x) - (f - g)(x)| + |L_{n,q}(g; x) - g(x)| \leq 2\|g\| + x([2] - (1 - q)[n]q)x\|g''\|. \]

At last, taking the infimum over all \( g \in W^2 \) and on application of the inequality \( K_2(f, \delta) \leq C\omega_2(f, \delta^{1/2}), \delta > 0 \), we get the required result. This completes the proof of the theorem. \(\square\)

5. POINTWISE ESTIMATES

In this section, we obtain some pointwise estimates of the rate of convergence of the \( q \)-Baskakov-Durrmeyer operators. First, we discuss the relationship between the local smoothness of \( f \) and the local approximation.

**Theorem 3.** Let \( 0 < \alpha \leq 1 \) and \( E \) be any bounded subset of the interval \( [0, \infty) \). If \( f \in C_B[0, \infty) \cap \text{Lip}_M(\alpha) \) then we have
\[ |L_{n,q}(f; x) - f(x)| \leq M \{ \psi_{n,q}^2(q; x) + 2(d(x, E))^\alpha \}, x \in [0, \infty), \]
where \( M \) is a constant depending on \( \alpha \) and \( f \), \( d(x, E) \) is the distance between \( x \) and \( E \) defined as \( d(x, E) = \inf \{ |t - x|; t \in E \} \) and \( \psi_{n,q}(q; x) = L_{n,q}((t - x)^2; x) \).
Proof. From the property of infimum, it follows that there exists a point \( t_0 \in \tilde{E} \) such that \( d(x, E) = |t_0 - x| \).

In view of the triangle inequality we have

\[
|f(t) - f(x)| \leq |f(t) - f(t_0)| + |f(t_0) - f(x)|.
\]

Using the definition of \( \text{Lip}_M(\alpha) \), we get

\[
|L_{n,q}(f; x) - f(x)| \leq L_{n,q}(|f(t) - f(t_0)|; x) + L_{n,q}(|f(x) - f(t_0)|; x)
\]\[
\leq M\{L_{n,q}(|t - t_0|^{\alpha}; x) + |x - t_0|^{\alpha}\}
\]\[
\leq M\{L_{n,q}(|t - x|^{\alpha}; x) + 2|x - t_0|^{\alpha}\}.
\]

Choosing \( p_1 = \frac{2}{\alpha} \) and \( p_2 = \frac{2}{2 - \alpha} \), we get \( \frac{1}{p_1} + \frac{1}{p_2} = 1 \). Then, Hölder’s inequality yields

\[
|L_{n,q}(f; x) - f(x)| \leq M\{(L_{n,q}(|t - x|^{\alpha p_1}; x))^{1/p_1}(L_{n,q}(|t - x|^{\alpha p_2}; x))^{1/p_2} + 2(d(x, E))^{\alpha}\}
\]\[
\leq M\{(L_{n,q}((t - x)^2; x))^{\alpha/2} + 2(d(x, E))^{\alpha}\}
\]\[
= M\{\psi_{n,2}^{\alpha/2}(q; x) + 2(d(x, E))^{\alpha}\}.
\]

This completes the proof of the theorem. \( \square \)

Next, we obtain a local direct estimate of operators \( L_{n,q} \) using the Lipschitz-type maximal function of order \( \alpha \) introduced by Lenze [2] as

\[
\bar{\phi}_\alpha(f, x) = \sup_{t \neq x, t \in [0, \infty)} \frac{|f(t) - f(x)|}{|t - x|^{\alpha}}, \quad x \in [0, \infty) \text{ and } \alpha \in (0, 1].
\]

Theorem 4. Let \( 0 < \alpha \leq 1 \) and \( f \in C_B[0, \infty) \), then for all \( x \in [0, \infty) \) we have

\[
|L_{n,q}(f; x) - f(x)| \leq \bar{\phi}_\alpha(f, x)\psi_{n,2}^{\alpha/2}(q; x).
\]

Proof. In view of (2), we get

\[
|f(t) - f(x)| \leq \bar{\phi}_\alpha(f, x)|t - x|^{\alpha}
\]

and hence

\[
|L_{n,q}(f; x) - f(x)| \leq L_{n,q}(|f(t) - f(x)|; x) \leq \bar{\phi}_\alpha(f, x)L_{n,q}(|t - x|^{\alpha}; x).
\]

Now, using the Hölder’s inequality with \( p = \frac{2}{\alpha} \) and \( \frac{1}{q} = 1 - \frac{1}{p} \), we obtain

\[
|L_{n,q}(f; x) - f(x)| \leq \bar{\phi}_\alpha(f, x)(L_{n,q}(|t - x|^{2}; x))^{\alpha/2} = \bar{\phi}_\alpha(f, x)\psi_{n,2}^{\alpha/2}(x).
\]

Thus, the proof is completed. \( \square \)

6. Weighted Approximation

In this section, we discuss about the weighted approximation theorem for the operators \( L_{n,q}(f) \).

Let \( C_{x^2}[0, \infty) \) be the subspace of all functions \( f \in C_{x^2}[0, \infty) \) for which \( \lim_{x \to \infty} \frac{|f(x)|}{1 + x^2} \) is finite.

Theorem 5. Let \( q_n \) be a sequence in \((0, 1)\) such that \( q_n \to 1 \) and \( q_n^a \to 0 \), as \( n \to \infty \). For each \( C_{x^2}[0, \infty) \), we have

\[
\lim_{n \to \infty} \|L_{n,q_n}(f) - f\|_{x^2} = 0.
\]
Proof. In order to prove (3) it is sufficient to show that ([5])
\[
(4) \quad \lim_{n \to \infty} \|L_{n,q_n}(t^n; x) - x^n\|_{x^2} = 0, \quad \nu = 0, 1, 2.
\]
Since, \(L_{n,q_n}(1; x) = 1\), (4) holds true for \(\nu = 0\).

Now, by Lemma 1, we have
\[
\|L_{n,q_n}(t; x) - x\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|L_{n,q_n}(t; x) - x|}{1 + x^2}
\]
\[
\to 0, \quad \text{as} \quad n \to \infty.
\]

Therefore, (4) is true for \(\nu = 1\).

Again, by Lemma 1, we may write
\[
\|L_{n,q_n}(t^2; x) - x^2\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|L_{n,q_n}(t^2; x) - x^2|}{1 + x^2}
\]
\[
\leq \frac{1 + q_n}{|n|_{q_n}} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2}
\]
\[
+ (q_n - 1) \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2}
\]
\[
= \frac{1 + q_n}{|n|_{q_n}} + (q_n - 1).
\]

Hence, (4) follows for \(\nu = 2\). This completes the proof of the theorem. \(\square\)

Theorem 6. Let \(f \in C_{x^2}[0, \infty), q = q_n \in (0, 1)\) such that \(q_n \to 1\) and \(q_n^a \to 0\) as \(n \to \infty\) and \(\omega_{a+1}\) be its modulus of continuity on the finite interval \([0, a + 1] \subset [0, \infty), a > 0\). Then, for every \(n \geq 1\)
\[
\|L_{n,q_n}(f) - f\|_{C[0,\infty)} \leq \frac{12M_f(1+a^2)a}{|n|_{q_n}} + 2\omega_{a+1} \left( f, \sqrt{\frac{2a}{|n|_{q_n}}} \right).
\]

Proof. For \(x \in [0, a]\) and \(t > a + 1\). Since \(t - x > 1\), we have
\[
|f(t) - f(x)| \leq M_f(2 + x^2 + t^2)
\]
\[
\leq M_f(2 + 3x^2 + 2(t - x)^2)
\]
\[
\leq 3M_f(1 + x^2 + (t - x)^2)
\]
\[
\leq 6M_f(1 + x^2)(t - x)^2
\]
\[
\leq 6M_f(1 + a^2)(t - x)^2.
\]

For \(x \in [0, a]\) and \(t \leq a + 1\), we have
\[
(6) \quad |f(t) - f(x)| \leq \omega_{a+1}(f, |t - x|) \leq \left( 1 + \frac{|t - x|}{\delta} \right) \omega_{a+1}(f, \delta),
\]
where \(\delta > 0\).

From (5) and (6), we can write
\[
(7) \quad |f(t) - f(x)| \leq 6M_f(1 + a^2)(t - x)^2 + \left( 1 + \frac{|t - x|}{\delta} \right) \omega_{a+1}(f, \delta)
\]

For \(x \in [0, a]\) and \(t \geq 0\) and applying Schwarz inequality, we obtain
\[
|L_{n,q}(f;x) - f(x)| \leq L_{n,q}(|f(t) - f(x)|; x)
\]
\[
\leq 6M_f(1 + a^2)L_{n,q}((t - x)^2; x)
\]
\[
+ \omega_{a+1}(f, \delta) \left( 1 + \frac{1}{\delta} L_{n,q}((t - x)^2; x)^{\frac{1}{2}} \right).
\]
Hence, using Lemma 2, for every $q \in (0,1)$ and $x \in [0,a]$

$$|L_{n,q}(f; x) - f(x)| \leq 6Mf(1 + a^2)\frac{x([2] - (1 - q)[n]q)}{|n|_q} + C\omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\frac{x([2] - (1 - q)[n]q)}{|n|_q}}\right) \leq 12Mf(1 + a^2)\frac{\alpha}{|n|_q} + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\frac{2\alpha}{|n|_q}}\right).$$

Taking $\delta = \sqrt{\frac{2\alpha}{|n|_q}}$, we get the required result.

This completes the proof of Theorem.

Now, we prove a theorem to approximate all functions in $C_x[0, \infty)$. Such type of results are given in [4] for locally integrable functions.

**Theorem 7.** Let $q = q_n \in (0, 1)$ such that $q_n \to 1$ and $q_n^\alpha \to 0$, as $n \to \infty$. For each $f \in C_x^*[0, \infty)$, and $\alpha > 1$, we have

$$\lim_{n \to \infty} \sup_{x \in [0, \infty)} \frac{|L_{n,q_n}(f; x) - f(x)|}{(1 + x^2)\alpha} = 0.$$

**Proof.** For any fixed $x_0 > 0$,

$$\sup_{x \in [0, \infty)} \frac{|L_{n,q_n}(f; x) - f(x)|}{(1 + x^2)\alpha} \leq \sup_{x \leq x_0} \frac{|L_{n,q_n}(f; x) - f(x)|}{(1 + x^2)\alpha} + \sup_{x > x_0} \frac{|L_{n,q_n}(f; x) - f(x)|}{(1 + x^2)\alpha} \leq \|L_{n,q_n}(f) - f\|_{C[0, x_0]} + \|f\|_x \sup_{x \geq x_0} \frac{|L_{n,q_n}(1 + t^2; x)|}{(1 + x^2)^\alpha} + \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^\alpha}.
\tag{8}$$

Since, $|f(x)| \leq Mf(1 + x^2)$, we have

$$\frac{\sup_{x \geq x_0} |f(x)|}{(1 + x^2)^\alpha} \leq \frac{\sup_{x \geq x_0} Mf}{(1 + x^2)^{\alpha - 1}} \leq \frac{Mf}{(1 + x_0^2)^{\alpha - 1}}.
\tag{9}$$

Let $\epsilon > 0$ be arbitrary. We can choose $x_0$ to be large that

$$\frac{Mf}{(1 + x_0^2)^{\alpha - 1}} < \frac{\epsilon}{3}$$

and in view of Lemma 1, we obtain

$$\|f\|_x \sup_{n \to \infty} \frac{|L_{n,q_n}(1 + t^2; x)|}{(1 + x^2)\alpha} = \frac{1 + x^2}{(1 + x^2)^\alpha} \|f\|_x \leq \frac{\|f\|_x^2}{(1 + x^2)^{\alpha - 1}} \leq \frac{\|f\|_x^2}{(1 + x_0^2)^{\alpha - 1}} < \frac{\epsilon}{3}.
\tag{10}$$

Using Theorem 6 we can see that the first term of the inequality (8) implies that

$$\|L_{n,q_n}(f; .) - f\|_{C[0, x_0]} < \frac{\epsilon}{3}, \text{ as } n \to \infty.
\tag{11}$$

Combining (8)-(11), we get the desired result.
ACKNOWLEDGEMENT

The research work of the third author Deepmala is supported by the Science and Engineering Research Board (SERB), Government of India under SERB NPDF scheme, File Number: PDF/2015/000799.

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