SOME FIXED POINT RESULTS FOR CARISTI TYPE MAPPINGS IN MODULAR METRIC SPACES WITH AN APPLICATION

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**Abstract.** In this paper we give Caristi type fixed point theorem in complete modular metric spaces. Moreover we give a theorem which can be derived from Caristi type. Also an application for the bounded solution of functional equations is investigated.

1. Introduction

Fixed point theory is one of the very popular tools in various fields. Since Banach introduced this theory in 1922\[6\], it has been extended and generalized by several authors. Caristi type fixed point theorem is one of these generalizations. It is a modification of $\varepsilon-$variational principle of Ekeland\[13\]. It is crucial in nonlinear analysis, in particular, optimization, variational inequalities, differential equations and control theory.

The notion of modular space was introduced by Nakano \[20\] and was intensively developed by Koshi, Shimogaki, Yamamuro (see \[18, 21\]) and others. A lot of mathematicians are interested in fixed point of modular space. In 2008, Chistyakov introduced the notion of modular metric space generated by $F$-modular and developed the theory of this space \[8\], on the same idea was defined the notion of a modular on an arbitrary set and developed the theory of metric space generated by modular such that called the modular metric spaces in 2010 \[9\]. Afrah A. N. Abdou \[1\] studied and proved some new fixed points theorems for pointwise and asymptotic pointwise contraction mappings in modular metric spaces. Azadifer et. al. \[3\] introduced the notion of modular $G$-metric spaces. Azadifer et. al. \[5\] proved the existence and uniqueness of a common fixed point of compatible mappings of integral type in this space. Kılınc and Alaca \[14\] defined $(\varepsilon,k)-$uniformly locally contractive mappings and $\eta$-chainable concept and proved a fixed point theorem for these concepts in a complete modular metric spaces. Kılınc and Alaca \[15\] proved that two main fixed point theorems for commuting mappings in modular metric spaces. Recently, many authors \[4, 7, 10, 11, 12, 19\] studied on different fixed point results for modular metric spaces. In 2014 Khamisi and Abdou investigated Hausdorff modular metric in modular metric spaces\[16\], and proved fixed point theorem for multivalued mappings.

In this paper we investigate Caristi type fixed point theorems for multivalued mappings in modular metric spaces, which is more general than the results of Khojasteh, Karapinar and Khandani\[17\]. And we also give an application of results for functional equations.

2. Preliminaries

In this section, we will give some basic concepts and definitions about modular metric spaces.

**Definition 2.1** \[9\], Definition 2.1 Let $X$ be a nonempty set, a function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular on $X$ if satisfying, for all $x, y, z \in X$ the following condition holds:

(i) $w_\lambda (x, y) = 0$ for all $\lambda > 0 \iff x = y;$

(ii) $w_\lambda (x, y) = w_\lambda (y, x)$ for all $\lambda > 0;$

(iii) $w_{\lambda + \mu} (x, y) \leq w_\lambda (x, z) + w_\mu (z, y)$ for all $\lambda, \mu > 0.$

If instead of (i), we have only the condition

(i) $w_\lambda (x, x) = 0$ for all $\lambda > 0,$ then $w$ is said to be a (metric) pseudomodular on $X.$

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The main property of a metric modular \([9]\) \(w\) on a set \(X\) is the following: given \(x, y \in X\), the function \(0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty)\) is nonincreasing on \((0, \infty)\). In fact, if \(0 < \mu < \lambda\), then (iii), (i) and (ii) imply

\[
w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(x, y) = w_\mu(x, y).
\]

It follows that at each point \(\lambda > 0\) the right limit \(w_{\lambda+0}(x, y) = \lim_{\mu \to \lambda+0} w_\mu(x, y)\) and the left limit \(w_{\lambda-0}(x, y) = \lim_{\varepsilon \to 0^+} w_{\lambda-\varepsilon}(x, y)\) exist in \([0, \infty]\) and the following two inequalities hold:

\[
w_{\lambda+0}(x, y) \leq w_\lambda(x, y) \leq w_{\lambda-0}(x, y).
\]

Theorem 2.1 ([19]) Let \(X_w\) be a complete modular metric space and \(T\) a contraction on \(X_w\). Then, the sequence \((T^n)_{n \in \mathbb{N}}\) converges to the unique fixed point of \(T\) in \(X_w\) for any initial \(x \in X_w\).

Now we give some definitions, which are useful for our main results.

Definition 2.2 Let \(X_w\) be a modular metric space. Then following definitions exist:

1. The sequence \((x_n)_{n \in \mathbb{N}}\) in \(X_w\) is said to be convergent to \(x \in X_w\) if \(w_1(x_n, x) \to 0\), as \(n \to \infty\).
2. The sequence \((x_n)_{n \in \mathbb{N}}\) in \(X_w\) is said to be Cauchy if \(w_1(x_m, x_n) \to 0\), as \(m, n \to \infty\).
3. A subset \(C\) of \(X_w\) is said to be closed if the limit of a convergent sequence of \(C\) always belong to \(C\).
4. A subset \(C\) of \(X_w\) is said to be complete if any Cauchy sequence in \(C\) is a convergent sequence and its limit is in \(C\).
5. A subset \(C\) of \(X_w\) is said to be \(w\)-bounded if

\[
\delta_w(C) = \sup \{w_1(x, y); x, y \in C\} < \infty.
\]

6. A subset \(C\) of \(X_w\) is said to be \(w\)-compact if for any \((x_n)\) in \(C\) there exists a subset sequence \((x_{n_k})\) and \(x \in X\) such that \(w_1(x_{n_k}, x) \to 0\).

7. \(w\) is said to satisfy the Fatou property if and only if for any sequence \((x_n)_{n \in \mathbb{N}}\) in \(X_w\) \(w\)-convergent to \(x\), we have

\[
w_1(x, y) \leq \liminf_{n \to \infty} w_1(x_n, y),
\]

for any \(y \in X_w\).

Now we will give some basic properties and notions of multivalued mappings in modular metric spaces which was given in [2]. For a subset \(M\) of modular metric space \(X_w\) set

(i) \(CB(M) = \{C: C\) is nonempty \(w\) - closed and \(w\) - bounded subset of \(M\}\);
(ii) \(K(M) = \{C: C\) is nonempty \(w\) - compact subset of \(M\}\);
(iii) the Hausdorff modular metric is defined on \(CB(M)\) by

\[
H_w(A, B) = \max \left\{ \sup_{x \in A} w_1(x, B), \sup_{y \in B} w_1(y, A) \right\},
\]

where \(w_1(x, B) = \inf_{y \in B} w_1(x, y)\).

Definition 2.3 Let \(X_w\) be a complete modular metric space and \(M\) be a nonempty subset of \(X_w\). A mapping \(T: M \to CB(M)\) is called a multivalued Lipschitzian mapping, if there exists a constant \(k > 0\) such that

\[
H_w(Tx, Ty) \leq kw_1(x, y),
\]

for any \(x, y \in M\).

A point \(x \in M\) is called fixed point of \(T\) whenever \(x \in Tx\). The set of fixed points of \(T\) will be denoted by \(\text{Fix}(T)\).

It was shown in [2] that Definition 2.4 is more general than Theorem 2.1.
3. Main Results

In this section we will give a fixed point theorem for Caristi type mappings and a generalization of the theorem in modular metric spaces. This works are more general than the results of [17].

Theorem Let $X_w$ be a complete modular metric space and $T : X_w \to CB(X_w)$ be a nonexpansive mapping such that for each $x \in X_w$ and for all $y \in Tx$, there exists $z \in Ty$ such that

$$w_\lambda(x, y) \leq \frac{1}{\lambda}(\varphi_w(x, y) - \varphi_w(y, z))$$

specifically for $\lambda = 1$ we can write

$$w_1(x, y) \leq \varphi_w(x, y) - \varphi_w(y, z)$$

where $\varphi : X_w \times X_w \to [0, \infty]$ is lower semicontinuous with respect to the first variable. Then for $w_1(x_n, x_{n+1}) < \infty$, $T$ has a fixed point.

Proof. Let $x_0 \in X_w$ and $x_1 \in Tx_0$. If $x_1 = x_0$, then $x_0$ is a fixed point and theorem is satisfied. Otherwise, let $x_0 \neq x_1$. By assumption there exists $x_2 \in Tx_1$ such that

$$w_1(x_0, x_1) \leq \varphi_w(x_0, x_1) - \varphi_w(x_1, x_2)$$

Alternatively, one can choose $x_n \in Tx_{n-1}$ such that $x_n \neq x_{n-1}$ and find $x_{n+1} \in Tx_n$ such that

$$w_1(x_{n-1}, x_n) \leq \varphi_w(x_{n-1}, x_n) - \varphi_w(x_n, x_{n+1}),$$

which means that $(\varphi_w(x_{n-1}, x_n))_n$ is a nonincreasing sequence in fact

$$0 < w_1(x_{n-1}, x_n) + \varphi_w(x_n, x_{n+1}) \leq \varphi_w(x_{n-1}, x_n)$$

Thus it is bounded below, so it converges to some $r \geq 0$. By taking the limit on both sides of (3.2) we get

$$\lim_{n \to \infty} w_1(x_{n-1}, x_n) \leq \lim_{n \to \infty} (\varphi_w(x_{n-1}, x_n) - \varphi_w(x_n, x_{n+1}))$$

$$\lim_{n \to \infty} w_1(x_{n-1}, x_n) \leq \lim_{n \to \infty} \varphi_w(x_{n-1}, x_n) - \lim_{n \to \infty} \varphi_w(x_n, x_{n+1})$$

$$\lim_{n \to \infty} w_1(x_{n-1}, x_n) = r - r$$

$$\lim_{n \to \infty} w_1(x_{n-1}, x_n) = 0$$

Now we show that $w_1(x_n, x_{m-n})$ is Cauchy sequence. For all $m, n \in \mathbb{N}$ with $m > n$,

$$w_1(x_n, x_m) \leq \sum_{i=n+1}^{m} w_{\frac{1}{m-n}}(x_{i-1}, x_i)$$

On the other hand from the main property of metric modular one can choose that

$$w_1(x_{n-1}, x_n) \leq w_{\frac{1}{m-n}}(x_{n-1}, x_n)$$

$$w_{\frac{1}{m-n}}(x_{n-1}, x_n) \leq \frac{1}{m-n} \varphi_w(t)$$

when we write this in (3.3) we get

$$w_1(x_n, x_m) \leq \frac{1}{m-n}(\varphi_w(x_{n-1}, x_n) - \cdots - \varphi_w(x_m, x_{m+1}))$$

$$w_1(x_n, x_m) \leq \frac{1}{m-n}(\varphi_w(x_{n-1}, x_n) - \varphi_w(x_m, x_{m+1}))$$
When we take the limsup on both sides of the inequalities above, we have

\[
\lim_{n \to \infty} \left( \sup \left\{ w_1(x_n, x_m) : m > n \right\} \right) \leq \lim_{n \to \infty} \left( \sup \frac{1}{(m-n)} \left( \varphi_w(x_{n-1}, x_n) - \varphi_w(x_m, x_{m+1}) \right) \right)
\]

\[
\lim_{n \to \infty} \left( \sup \left\{ w_1(x_n, x_m) : m > n \right\} \right) \leq \lim_{n \to \infty} \left( \sup \frac{1}{(m-n)} \right) (r - r)
\]

\[
\lim_{n \to \infty} \left( \sup \left\{ w_1(x_n, x_m) : m > n \right\} \right) = 0
\]

Thus \((x_n)\) is a Cauchy sequence. Since \(X_w\) is complete it converges to \(u \in X_w\). Now we show that \(u\) is a fixed point of \(T\). We have

\[
w_1(u, Tu) \leq w_\frac{1}{2}(u, x_{n+1}) + w_\frac{1}{2}(Tu, x_{n+1})
\]

\[
\leq w_\frac{1}{2}(u, x_{n+1}) + H_w(Tu, Tx_n)
\]

\[
\leq w_\frac{1}{2}(u, x_{n+1}) + w_\frac{1}{2}(u, x_n)
\]

When we take the limit on both sides of the inequalities above, we get

\[
\lim_{n \to \infty} w_1(u, Tu) \leq \lim_{n \to \infty} w_\frac{1}{2}(u, x_{n+1}) + \lim_{n \to \infty} w_\frac{1}{2}(u, x_n)
\]

\[
\lim_{n \to \infty} w_1(u, Tu) \leq w_\frac{1}{2}(u, u) + w_\frac{1}{2}(u, u)
\]

\[
\lim_{n \to \infty} w_1(u, Tu) = 0
\]

Hence we get \(u \in Tu\). Therefore \(u\) is fixed point of \(T\). \(\square\)

Now let us give the theorem which is a generalized version of the theorem above.

Theorem Let \(X_w\) be a complete modular metric space and \(T : X_w \to CB(X_w)\) be a multivalued mapping such that

\[
H_w(Tx, Ty) \leq \eta(w_1(x, y))
\]

for all \(x, y \in X_w\); where \(\eta : [0, \infty) \to [0, \infty]\) is a lower semicontinuous map such that \(\eta(t) < t\) for all \(t \in [0, \infty]\), \(\frac{\eta(t)}{t}\) is nondecreasing. Then \(T\) has a fixed point.

Proof. Let \(x \in X_w\) and \(y \in Tx\). If \(x = y\) then \(T\) has a fixed point and the proof is complete, so we suppose that \(x \neq y\). Let define that

\[
\theta(t) = \frac{\eta(t) + t}{2}, \text{ for all } t \in [0, \infty]
\]

Then we have

\[
H_w(Tx, Ty) \leq \eta(w_1(x, y))
\]

Since \(\eta\) is nondecreasing and \(\eta(t) < t\) we get

\[
\eta(t) < t \Leftrightarrow \eta(w_1(x, y)) \leq w_1(x, y)
\]

\[
\theta(w_1(x, y)) = \frac{\eta(w_1(x, y)) + w_1(x, y)}{2} \leq w_1(x, y)
\]

\[
\theta(w_1(x, y)) = \frac{\eta(w_1(x, y)) + w_1(x, y)}{2} \geq \eta(w_1(x, y))
\]

Then we get

\[
(3.4) \quad H_w(Tx, Ty) \leq \eta(w_1(x, y)) \leq \theta(w_1(x, y)) \leq w_1(x, y)
\]

Thus there exists \(\epsilon_0 > 0\) such that \(\theta(w_1(x, y)) = H_w(Tx, Ty) + \epsilon_0\). So there exists \(z \in Ty\) such that
\[ (3.5) \quad w_1(y, z) \leq H_w(T_1, T_2) + \epsilon_0 = \theta(w_1(x, y)) \leq w_1(x, y) \]

Again suppose that \( y \neq z \). Then
\[ w_1(x, y) - \theta(w_1(x, y)) \leq w_1(x, y) - w_1(y, z) \]
or we can rewrite this inequality as
\[ (3.6) \quad w_1(x, y) \leq \frac{w_1(x, y)}{1 - \frac{\theta(w_1(x, y))}{w_1(x, y)}} - \frac{w_1(y, z)}{1 - \frac{\theta(w_1(x, y))}{w_1(x, y)}} \]

Since \( \theta(t) \) is nondecreasing and \( w_1(y, z) < w_1(x, y) \) we rewrite the (3.5) as

\[ w_1(x, y) \leq \frac{w_1(x, y)}{1 - \frac{\theta(w_1(x, y))}{w_1(x, y)}} - \frac{w_1(y, z)}{1 - \frac{\theta(w_1(y, z))}{w_1(y, z)}} \]

Define
\[ \phi_w(x, y) = \begin{cases} \frac{w_1(x, y)}{1 - \frac{\theta(w_1(x, y))}{w_1(x, y)}}, & x \neq y \\ 0, & x = y \end{cases} \]

Therefore \( T \) satisfies (3.1) of Theorem 1, so we conclude that \( T \) has a fixed point \( u \in X_w \) and the proof is complete. \( \square \)

4. AN APPLICATION TO FUNCTIONAL EQUATIONS

We use fixed point theory in many fields of mathematics. One of these fields is mathematical optimization. Dynamic programming is useful for mathematical optimization and it is related to a multistage process reduces to solving functional equation \( p(x) \). In this section we try to give an application of Theorem 2 to a functional equation defined as

\[ (4.1) \quad p(x) = \sup_{y \in T} \{ f(x, y) + \Im(x, y, p(y(x))) \}, \quad x \in Z, \]

where \( \eta : Z \times T \to Z \), \( f : Z \times T \to \mathbb{R} \), and \( \Im : Z \times T \times \mathbb{R} \to \mathbb{R} \). We assume that \( M \) and \( N \) are Banach spaces \( Z \subset M \) and \( T \subset N \).

Now we study the existence of the bounded solution of the functional equation (4.1). Let \( B(Z) \) denote the set of all bounded real-valued functions on \( Z \), and for an arbitrary \( h \in B(Z) \), define \( \| h \| = \sup_{x \in Z} | h(x) | \). Clearly, \( (B(Z), \| \cdot \|) \) endowed with the metric modular

\[ (4.2) \quad w_\lambda(h, k) = \frac{1}{1 + \lambda} \sup_{x \in Z} | h(x) - k(x) | \]

specifically for \( \lambda = 1 \) written as

\[ w_1(h, k) = \frac{1}{2} \sup_{x \in Z} | h(x) - k(x) | \]

for all \( h, k \in B(Z) \), is Banach space, so the convergence in this space according to \( w_1(h, k) \) is uniform which means a Cauchy sequence in \( B(Z) \) is uniformly convergent to a function say \( h_\ast \), that is bounded and so \( h_\ast \in B(Z) \).

Now we will give the theorem that gives the solution of the functional equation defined in (4.1). To give the solution let us define an operator as follows:

\[ (4.3) \quad S(h)(x) = \sup_{y \in T} \{ f(x, y) + \Im(x, y, h(y(x, y))) \} \]

for all \( h \in B(Z) \) and \( x \in Z \).

Now we give the theorem for the bounded solution of functional equation given in (4.4)
Theorem Let $S : B(\mathcal{Z}) \to B(\mathcal{Z})$ be an upper semicontinuous operator defined by (4.4) and assume that the following conditions are satisfied:

(i) $f : \mathcal{Z} \times \mathcal{T} \to \mathbb{R}$, and $\mathcal{S} : \mathcal{Z} \times \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded;

(ii) for all $h, k \in B(\mathcal{Z})$, if

\[
0 < w_1(h, k) < 1
\]

implies

\[
\frac{1}{2} \left| \mathcal{S}(x, y, h(x, y)) - \mathcal{S}(x, y, k(x, y)) \right| \leq w_1(h, k)
\]

implies

\[
\frac{1}{2} \left| \mathcal{S}(x, y, h(x, y)) - \mathcal{S}(x, y, k(x, y)) \right| \leq w_1(h, k)
\]

Then the functional equation (4.1) has a bounded solution.

Proof. Clearly $B(\mathcal{Z})$ is a complete modular metric space. Let $\mu > 0$ be arbitrary, $x \in \mathcal{Z}$, and $h_1, h_2 \in B(\mathcal{Z})$, then there exists $y_1, y_2 \in \mathcal{T}$ such that

\[
S(h_1)(x) < f(x, y) + \mathcal{S}(x, y, h_1(x, y)) + \mu
\]

\[
S(h_2)(x) < f(x, y) + \mathcal{S}(x, y, h_2(x, y)) + \mu
\]

\[
S(h_1)(x) \geq f(x, y) + \mathcal{S}(x, y, h_1(x, y))
\]

\[
S(h_2)(x) \geq f(x, y) + \mathcal{S}(x, y, h_2(x, y))
\]

Let $\varrho : [0, \infty) \to [0, \infty]$ be defined as

\[
\varrho(t) = \begin{cases} 
  t^2, & 0 < t < 1 \\
  t, & t \geq 1
\end{cases}
\]

Then we get

\[
\frac{1}{2} \left| \mathcal{S}(x, y, h(x, y)) - \mathcal{S}(x, y, k(x, y)) \right| \leq \varrho(w_1(h, k))
\]

for all $h, k \in B(\mathcal{Z})$. It is clear that $\varrho(t) < t$ for all $t > 0$ and $\frac{\varrho(t)}{t}$ is nondecreasing function. Therefore when we use the inequalities above we get

\[
\frac{1}{2} (S(h_1)(x) - S(h_2)(x)) < \frac{1}{2} \left\{ \mathcal{S}(x, y, h_1(x, y)) - \mathcal{S}(x, y, h_2(x, y)) \right\} + \mu
\]

\[
\leq \frac{1}{2} \left| \mathcal{S}(x, y, h_1(x, y)) - \mathcal{S}(x, y, h_2(x, y)) \right| + \mu
\]

\[
\leq \varrho(w_1(h_1, h_2)) + \mu
\]

Then inequality turns into

\[
\frac{1}{2} \left\{ S(h_1)(x) - S(h_2)(x) \right\} < \varrho(w_1(h_1, h_2)) + \mu
\]

Analogously we get

\[
\frac{1}{2} \left\{ S(h_2)(x) - S(h_1)(x) \right\} < \varrho(w_1(h_1, h_2)) + \mu
\]

Hence these inequalities equal to

\[
\frac{1}{2} \left| S(h_1)(x) - S(h_2)(x) \right| < \varrho(w_1(h_1, h_2)) + \mu
\]

which means that

\[
w_1(S(h_1)(x), S(h_2)(x)) < \varrho(w_1(h_1, h_2)) + \mu
\]
Since the above inequality turns into

\[ w_1(S(h_1)(x), S(h_2)(x)) \leq \varrho(w_1(h_1, h_2)), \]

hence we get that \( S \) is a \( \varrho \)-contraction. And \( S \) satisfies the conditions in Theorem 2 and \( S \) has a fixed point say \( h_* \in B(Z) \), which means is a bounded solution of the functional equation (4.1). \( \square \)

References


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