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On Quasi-ideals and Bi-ideals in AG-Rings

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Abstract. In this paper we study some properties of quasi-ideals and bi-ideals in AG-ring and study some interesting properties of these ideals.

1. Introduction

M.A. Kazim and MD. Naseeruddin [2] have introduced the concept of an AG-groupoid.

Definition 1.1. *A groupoid G is called a* left almost semigroup *(abbreviated as a LA-*semigroup*) if its elements satisfy the left invertive law:*

$$(ab)c = (cb)a \text{ for all } a, b, c \in G.$$

It is also called an Abel-Grassmann's groupoid *(abbreviated as* AG-groupoid*).*

Moreover every AG-groupoids G have a *medial law* hold

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall\, a, b, c, d \in G.$$

Q. Mushtaq and M. Khan [4, p.322] asserted that, in every AG-groupoids G with left identity

$$(a \cdot b) \cdot (c \cdot d) = (d \cdot c) \cdot (b \cdot a), \quad \forall\, a, b, c, d \in G.$$

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Further M. Khan, Faisal, and V. Amjid [3], asserted that, if a AG-groupoid G with left identity the following law holds

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall\, a, b, c \in G.$$

M. Sarwar (Kamran) [5, p.112] defined AG-group as the following.

Definition 1.2. *A groupoid G is called an Abel-Grassmann's group, abbreviated as AG-group, if*

(1) *there exists $e \in G$ such that $ea = a$ for all $a \in G$,*
(2) *for every $a \in G$ there exists $a' \in G$ such that, $a'a = e$,*
(3) *$(ab)c = (cb)a$ for every $a, b, c \in G$.*

S.M. Yusuf in [11, p.211] introduces the concept of an Abel-Grassmann's ring (AG-ring).

Definition 1.3. *An algebraic system $\langle R, +, \cdot\rangle$ is called a* Abel-Grassmann's ring *(AG-ring) if*

(1) *$\langle R, +\rangle$ is an AG-group,*
(2) *$\langle R, \cdot\rangle$ is an AG-groupoid,*
(3) *$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$, for all $a, b, c \in R$.*

Lemma 1.1. *In an AG-ring R,*

$$(ab)(cd) = (ac)(bd) \tag{1.1}$$

for all $a, b, c, d \in R$.

Equation (1.1) is called a *medial law* in the AG-ring R.

Lemma 1.2. *If an AG-ring R has a left identity 1, then*

$$a(bc) = b(ac)$$

for all $a, b, c \in R$.

Lemma 1.3. *If an AG-ring R has a left identity 1, then*

$$(ab)(cd) = (dc)(ba) \tag{1.2}$$

for all $a, b, c, d \in R$.

Equation (1.2) is called a *paramedial law* in the AG-ring R. Now we have the following property. T. Shah and I. Rehman [11, p.211] asserted that a commutative ring $\langle R, +, \cdot\rangle$, we can always obtain an AG-ring $\langle R, \oplus, \cdot\rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an AG-ring.

Definition 1.4. *Let $\langle R, +, \cdot\rangle$ be an LA-ring and S be a non-empty subset of R and S is itself and AG-ring under the binary operation induced by R, the S is called an* AG-subring *of R, then S is called an LA-subring of $\langle R, +, \cdot\rangle$.*

Definition 1.5. *If S is an AG-subring of an LA-ring $\langle R, +, \cdot \rangle$, then S is called a left ideal of R if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.*

Lemma 1.4. *If an AG-ring R has a left identity* 1, *then every right ideal is a left ideal.*

Proof. Let R be an AG-ring with left identity 1 and A is a right ideal of R. Then for $a \in A, r \in R$, we have

$$ra = (1r)a = (ar)1 \in (AR)R \subseteq AR \subseteq A,$$

where 1 is a left identity, that is $ra \in A$. Therefore A is left ideal of R. □

2. Main Results

Definition 2.1. *Let R be an AG-ring and Q be a non-empty subset of R. Then Q is said to be a* quasi-ideal *of R if Q is a AG-subgroup of $(R, +)$ such that $RQ \cap QR \subseteq Q$.*

Theorem 2.1. *Every one-sided ideal or two-sided ideal of an-AG-ring R is a quasi-ideal of R.*

Proof. Let L be a left ideal of an AG-ring R. Then

$$LR \cap RL \subseteq LL \subseteq L.$$

Thus L is a quasi-ideal of an AG-ring R. Similarly let I be a right ideal of R then

$$IR \cap RI \subseteq II \subseteq I.$$

Thus I is a quasi-ideal of an AG-ring R. □

Theorem 2.2. *Let R be an AG-ring. Then the intersection of left ideal L and a right ideal of I of R is a quasi-ideal of R.*

Proof. Let L be a left ideal and I be a right ideal of R. Then $L \cap I$ is a AG-subgroup of $(R, +)$. Thus

$$R(L \cap I) \cap (L \cap I)R \subseteq RL \cap IR \subseteq L \cap I.$$

Therefore the intersection of left ideal L and a right ideal of I of R is a quasi-ideal of R. □

Theorem 2.3. *Arbitrary intersection of quasi-ideal of an AG-ring R is a quasi-ideal of R.*

Proof. Let $T := \bigcap_{i \in \Delta}\{Q_i \mid Q_i \text{ is a quasi-ideal of } R\}$, where Δ denotes any indexing set, be a nonempty set. Then T is a AG-subgroup of $(R, +)$. Now

$$RT \cap TR = R\left(\bigcap_{i \in \Delta} Q_1\right) \cap \left(\bigcap_{i \in \Delta} Q_1\right) R \subseteq RQ_i \cap Q_i R \subseteq Q_i,$$

for all $i \in \Delta$. So we see that $RT \cap TR \subseteq \bigcap_{i \in \Delta} Q_i = T$. This proof complete. □

Definition 2.2. *An element e of an AG-ring R is a said* idempotent element *if $e^2 = ee = e$.*

Theorem 2.4. *Let R be an AG-ring in which every quasi-ideal is idempotent. Then for left ideal L and right ideal I such that $IL = I \cap L \subseteq LI$ is true.*

Proof. Let P and Q be two quasi-ideal in R then $P \cap Q$ is also a quasi-ideal. By the idempotent of $P \cap Q$ we have

$$P \cap Q = (P \cap Q)(P \cap Q)(PQ) \cap (QP)$$

on other hand

$$(PQ) \cap (QP) \subseteq (PR) \cap (RP) \subseteq P.$$

Similarly $(PQ) \cap (QP) \subseteq Q$ and so $P \cap Q = (PQ) \cap (QP)$.

Since left and right ideal are always AG-subgroup we have $I \cap L = (IL) \cap (LI)$ but $(IL) \subseteq (R \cap L)$ and so $IL = I \cap L \subseteq LI$. This proof complete. □

Intersection of a quasi-ideal and AG-subring of R is a quasi-ideal of an AG-subring of R. We can prove this in the following theorem.

Theorem 2.5. *Let R be an AG-ring. If Q is a quasi-ideal and T is an AG-subring of R, then $Q \cap T$ is a quasi-ideal of T.*

Proof. Let Q is a quasi-ideal and T is an AG-subring of R. Then $Q \cap T$ is a AG-subgroup of $(R, +)$. Since $Q \cap T \subseteq T$ we have $Q \cap T$ is a AG-subgroup of $(T, +)$. Then

$$T(Q \cap T) \cap (Q \cap T)T \subseteq TQ \cap QT \subseteq RQ \cap QR \subseteq Q$$

and

$$T(Q \cap T) \cap (Q \cap T)T \subseteq TT \cap TT \subseteq T \cap T = T.$$

It follows that $T(Q \cap T) \cap (Q \cap T)T \subseteq Q \cap T$. Hence $Q \cap T$ is a quasi-ideal of T. □

Definition 2.3. *Let R be an AG-ring. An additive AG-subgroup B of R is called a* bi-ideal *of R if $(BR)B \subseteq B$.*

Lemma 2.1. *Every left (right) ideal of an AG-ring R is a bi-ideal of R.*

Proof. Let L be a left ideal of R. Then A is an additive AG-subgroup of R. Thus $(LR)L \subseteq (RR)L \subseteq RL \subseteq L$. This implies that L is a bi-ideal of R. Let I be a right ideal of R. Then I is an additive AG-subgroup of R. Thus $(IR)I \subseteq II \subseteq IR \subseteq I$. This implies I is a bi-ideal of R. □

Corollary 2.1. *Every ideal of a Γ-AG-ring R is a bi-ideal of R.*

Lemma 2.2. *Let B be an idempotent bi-ideal of a Γ-AG-ring R with left identity* 1*. Then B is an ideal of R.*

Proof. Let B be an idempotent bi-ideal of a Γ-AG-ring R. Then B is an additive AG-subgroup of R. Thus

$$BR = (BB)R = (RB)B = (R(BB))B.$$

By Lemma 1.2 so

$$(R(BB))B = ((BB)R)B = (BR)B \subseteq B.$$

Which implies that B is a right ideal. By Lemma 1.4 so it is left ideal of R. Hence B is an ideal of R. □

Theorem 2.6. *The product of two bi-ideals of an AG-ring R with left identity 1 is again a bi-ideal of R.*

Proof. Let H and K be two bi-ideals of R. Then H and K are additive AG-subgroup of R. Thus using medial and $RR = R$, we get

$$\begin{aligned}
[(HK)R](HK) &= [(HK)(RR)](HK), && \text{by medial} \\
&= [(HR)(KR)](HK), && \text{by medial} \\
&= [(HR)H][(KR)K], && H, K \text{ is a bi-ideal of } R \\
&\subseteq HK.
\end{aligned}$$

Hence HK is a bi-ideal of R. □

Theorem 2.7. *Let B be a bi-ideal of an AG-ring R and A be a left ideal of R with left identity 1, then BA is a bi-ideal of R.*

Proof. Since A is a left ideal of R and B is a bi-ideal of an AG-ring R, we have BA is an additive AG-subgroup of R. Thus

$$\begin{aligned}
[(BA)R](BA) &= [(RA)B](BA) = [(BA)B](RA) \\
&\subseteq [(BR)B]A, && B \text{ is a bi-ideal of } R \\
&\subseteq BA.
\end{aligned}$$

It following that BA is a bi-ideal of R. □

Theorem 2.8. *Let B be a bi-ideal of an AG-ring R and A be a right ideal of R with left identity 1. If $A \subseteq B$ and $BB \subseteq B$, then AB is a bi-ideal of R.*

Proof. Since A is a right ideal of R and B is a bi-ideal of an AG-ring R, we have AB is an additive AG-subgroup of R. Let $A \subseteq B$ and $BB \subseteq B$. Then using Lemma 1.1, we get

$$\begin{aligned}
[(AB)R](AB) &= [(RB)A](AB) = [(AB)A](RB) \\
&\subseteq [(AR)A](RB), \quad B \subseteq R \\
&\subseteq (AA)(RB), \quad A \text{ is a right ideal of } R \\
&= (AR)(AB), \quad \text{by } \Gamma\text{-medial} \\
&\subseteq A(AB), \quad A \text{ is a right ideal of } R \\
&\subseteq A(BB), \quad A \subseteq B \\
&\subseteq AB, \quad BB \subseteq B.
\end{aligned}$$

It follows that AB is a bi-ideal of R. □

Theorem 2.9. *Let R be an AG-ring and A, B be bi-ideals of an AG-ring R. Then $A \cap B$ is a bi-ideal of R.*

Proof. Since A, B is bi-ideals of an AG-ring R, we have $A \cap B$ is an additive AG-subgroup of R. Thus $[(A \cap B)R](A \cap B) \subseteq (AR)(A \cap B) = [(A \cap B)R]A \subseteq (AR)A \subseteq A$ and $[(A \cap B)R](A \cap B) \subseteq (BR)(A \cap B) = [(A \cap B)R]B \subseteq (BR)B \subseteq B$. It following that $A \cap B$ is a bi-ideal of R. □

Corollary 2.2. *Let R be a Γ-AG-ring and H_i is a bi-ideal of R, for all $i \in I$. Then $\bigcap_{i\in I} H_i$ is a bi-ideal of R.*

Proof. Since $0 \in H_i$ for all $i \in I$, we have $0 \in \bigcap_i H_i$. Then $\bigcap_i H_i \neq \emptyset$. Since H_i is a bi-ideal of R, we have H_i is an additive AG-subgroup of R Let $x, y \in H_i$ then $x - y \in H_i$. Thus $x - y \in \bigcap_i H_i$. Let $x, y \in \bigcap_i H_i, r \in R$. Then $(xr)y \in (H_iR)H_i \subseteq H_i$ for all $i \in I$ Thus $(xr)y \in H_i$. Hence $\bigcap_{i\in I} H_i$ is a bi-ideal of R. □

Theorem 2.10. *Let I and L be respectively right and left AG-subgroup of R. Then any AG-subgroup B of R such that $IL \subseteq B \subseteq I \cap L$ is a bi-ideal of R.*

Proof. Since B is a AG-subgroup of $(R, +)$ with $IL \subseteq B \subseteq I \cap L$ we have

$$\begin{aligned}
(BR)B &\subseteq ((I \cap L)R)(I \cap L), && \text{by } B \subseteq I \cap L \\
&\subseteq (IR)L, && \text{by } S \subseteq I \cap L \text{ and } L \subseteq I \cap L \\
&\subseteq IL, && \text{by } I \text{ is a right ideal of } R \\
&\subseteq B, && \text{by } IL \subseteq B.
\end{aligned}$$

Then B is a bi-ideal of R. □

Corollary 2.3. *Intersection of an arbitrary set of bi-ideal $B_\lambda \quad (\lambda \in \wedge)$ of an AG-ring R is again a bi-ideal of R.*

Proof. Set $B := \bigcap_{\lambda\in\wedge} B_\lambda$. Since B is an AG-subgroup of R. From the inclusion $(B_\lambda R)B_\lambda \subseteq B_\lambda$ and $B \subseteq B_\lambda$. This implies that

$$(BR)B \subseteq (B_\lambda R)B_\lambda \subseteq B_\lambda \quad (\forall\lambda \in \wedge).$$

Hence $(BR)B \subseteq B$. □

Theorem 2.11. *Every idempotent quasi-ideal is a bi-ideal.*

Proof. Let Q be an idempotent quasi-ideal of Γ-AG-ring R. Then

$$(QR)Q \subseteq (RR)Q \subseteq RQ$$

and by Lemma 1.2

$$\begin{aligned}(QR)Q &\subseteq (QR)\Gamma(QQ) = (QQ)(RQ)\\ &\subseteq Q(RQ) \subseteq Q(RR) \subseteq QR\end{aligned}$$

which implies that $(QR)Q \subseteq QR \cap RQ \subseteq Q$. □

Definition 2.4. *An AG-ring R is called a* regular AG-ring *if for any $x \in R$ there exists $y \in R$ such that $x = (xy)x$.*

Theorem 2.12. *Let R be a regular of an AG-ring and B be a bi-ideal of R. Then $(BR)B = B$.*

Proof. Since B is a bi-ideal of R we have $(BR)B \subseteq B$. Let $x \in B$ then there exist $a \in R$ such that $x = (xa)x \in (BR)B$, since R is a regular of an AG-ring. This implies that $B \subseteq (BR)B$ so $(BR)B = B$. □

Theorem 2.13. *For a quasi-ideal Q in a regular AG-ring R, then $QR \cap RQ = Q$.*

Proof. Let Q be a quasi-ideal in R then $QR \cap RQ \subseteq Q$. Let $x \in Q$ then there exist $a \in R$ such that $x = (xa)x$, since R is a regular of a AG-ring. So

$$x = (xa)x \in (QR)Q \subseteq QR$$

and

$$x = (xa)x \in (QR)Q \subseteq (RR)Q \subseteq RQ$$

Then $x \in QR \cap RQ$. Thus $Q \subseteq QR \cap RQ$. Hence $QR \cap RQ = Q$. □

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