COMMON FIXED POINT THEOREMS FOR $G$–CONTRACTION IN 
$C^*$–ALGEBRA–VALUED METRIC SPACES

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ABSTRACT. In this paper we prove the common fixed point theorems for two mappings in complete 
$C^*$–valued metric space endowed with the graph $G = (V,E)$, which satisfies $G$–contractive condition. 
Also, we provide an example in support of our main result.

1. INTRODUCTION AND PRELIMINARIES

The Banach contraction principle [5] plays an important role in solving non linear problems. The 
Banach contraction principle says that: if $(X,d)$ be a complete metric space and $f$ is a self mapping 
on $X$ with the condition that there exists $\lambda \in (0,1)$ such that

$$d(fx,fy) \leq \lambda d(x,y) \quad \text{for all} \quad x,y \in X,$$

then $f$ has a unique fixed point in $X$. Since then a lot of publications are devoted to the study 
and solutions of many practical and theoretical problems by using this condition. Due to a numerous 
applications of the fixed point theory, from the last few decades this theory is a central topic of research. 
In this theory one of the approach is the common fixed point theorems. The concept of the common fixed point theorems was investigated by Jungck [1]. Many authors studied the fixed and common fixed point theorems for different spaces, like in cone metric spaces [8], non-commutative Banach spaces [22], fuzzy metric spaces [14] and uniform metric spaces [21]. For more information about this topic see ([1, 6, 7, 9, 17, 18, 23]).

On the other hand the concept of $C^*$–algebra is well developed. Here we recall some basic definitions, 
notations and results of $C^*$–algebra that may be found in [13]. A $*$-algebra $A$ is a complex algebra 
with linear involution $*$ such that $x^{**} = x$ and $(xy)^* = y^*x^*$, for any $x, y \in A$. If $*$-algebra together 
with complete sub multiplicative norm satisfying $||x^*|| = ||x||$ for all $x \in A$, then $*$-algebra is said to 
be a Banach $*$-algebra. A $C^*$–algebra is a Banach $*$-algebra such that $||x^*x|| = ||x||^2$ for all $x \in A$. 
An element of $A$ is called positive element, if $A_+ = \{x^* = x | x \in A\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the 
spectrum of an element $x \in A$, i.e., $\sigma(x) = \{\lambda \in \mathbb{C} : \lambda I - x \text{ is not invertible}\}$. There is a natural 
partial ordering on $A_+$ given by $x \preceq y$ if and only if $x - y \in A_+$. In [12] Z. Ma et al., introduced 
the notion of $C^*$–algebra valued metric space and proved fixed point theorems for $C^*$–algebra valued 
contractive mapping.

Many researchers tried to obtain some fixed point theorems of Banach type contraction endowed 
with the graph $G$, we recommend [2, 3, 4, 15, 16, 20]. Recently, T. Kamran et al., in [19] extended 
the results of Ma et al., which was given in [12], by using $C^*$–valued metric spaces and $G$-contraction 
principles.

Now we give some definitions of graph theory which is found in any text on graph theory, for example 
[11]. Following Jachymski [10], let $\Delta$ denote the diagonal of the $X \times X$ in a metric space $(X,d)$, and 
consider a directed graph $G = (V(G),E(G)) = (V,E)$ the set in which $V$ of its vertices and $E$ of its 
edges, and $\Delta \subseteq E$. Assume that $G$ has no parallel edges. We may treat $G$ as a weighted graph by 
assigning to each edge the distance between its vertices.

In this paper we will continue to study common fixed points in the $C^*$–valued metric space endowed 
with the graph $G$ under $G$–contractive condition.

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Definition 1.1. Let $X$ be a nonempty set, and the mapping $d : X \times X \to A$ endowed with the graph $G = (V, E)$, if it satisfies the following conditions:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a $C^*\text{-valued}$ metric on $X$, and $(X, d, A)$ is called $C^*\text{-valued}$ metric space.

Definition 1.2. Suppose that $(X, d, A)$ is a $C^*\text{-valued}$ metric space. Let $x \in (X, d, A)$ and $\{x_n\}$ be a sequence in $X$. The sequence $\{x_n\}$ is said to be convergent, if for any $\epsilon > 0$ there exists a positive integer $N$ such that

$$||d(x_n, x)|| \leq \epsilon \quad \text{for all } n \geq N.$$ 

The sequence $\{x_n\}$ is said to be Cauchy, if for any $\epsilon > 0$ there exists a positive integer $N$ such that

$$||d(x_n, x_m)|| \leq \epsilon \quad \text{for all } n, m \geq N.$$ 

If every Cauchy sequence is convergent in $(X, d, A)$, then $(X, d, A)$ is said to be complete $C^*\text{-valued}$ metric space.

Example 1.3. Let $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$. Define $d : X \times X \to A$ such that

$$d(x, y) = \begin{pmatrix} |x - y| & 0 \\ 0 & \alpha|x - y| \end{pmatrix} \quad \text{for all } x, y \in \mathbb{R} \text{ and } \alpha \geq 0.$$ 

It is easy to verify that $d$ is a $C^*\text{-algebra valued}$ metric space and $(X, d, M_2(\mathbb{R}))$ is a complete $C^*\text{-algebra valued}$ metric space.

Definition 1.4. Let $(X, d, A)$ be a $C^*\text{-valued}$ metric space. A mapping $f : X \to X$ is said to be a $C^*\text{-algebra-valued}$ contraction mapping on $X$ if there exists an $a \in A$ with $||a|| < 1$ such that

$$(1.1) \quad d(fx, fy) \leq a^*d(x, y)\alpha, \quad \text{for all } x, y \in X.$$ 

Theorem 1.5. [12] Let $(X, d, A)$ be a complete $C^*\text{-algebra-valued}$ metric space and $f$ satisfies (1.1), then $f$ has a unique fixed point in $X$.

Property 1.6. [12]

1. For any $\{x_n\} \in X$ such that $x_n$ converges to $x$ with $(x_{n+1}, x_n) \in E$ for all $n \geq 1$ there exists a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that $(x, x_{m_k}) \in E$.
2. For any $\{f^n x\} \in X$ such that $f^n x$ converges to $x \in X$ with $(f^{n+1} x, f^n x) \in E$ there exists a subsequence $\{f^n x\}$ and $n_0 \in \mathbb{N}$ such that $(x, f^{n_k} x) \in E$ for all $k \geq n_0$.

2. Main Result

In this section, we prove common fixed point theorems for two mappings satisfying $G\text{-contractive}$ condition in a complete $C^*\text{-valued}$ metric space endowed with the graph $G = (V, E)$.

Definition 2.1. Let $(X, d, A)$ be a $C^*\text{-valued}$ metric space endowed with the graph $G = (V, E)$. The mappings $f, g : X \to X$ are said to be $C^*\text{-valued}$ $G\text{-contractive}$ on $X$, if there exists an $a \in A$ with $||a|| < 1$ such that

$$(2.1) \quad d(fx, gy) \leq a^*d(x, y)\alpha, \quad \text{for all } (x, y) \in E.$$ 

Theorem 2.2. Let $(X, d, A)$ is a complete $C^*\text{-valued}$ metric space endowed with the graph $G = (V, E)$. Suppose that the mappings $f, g : X \to X$ are $C^*\text{-valued}$ $G\text{-contractive}$ mappings on $X$ satisfying the Property 1.6 (2) and the following conditions

1. if $(x, y) \in E$ then $(fx, gy) \in E$,
2. there exists $z_0 \in X$ such that $(z_0, fz_0), (z_0, gz_0) \in E$.

Then $f$ and $g$ has a unique common fixed point in $X$. 
Proof. Let \( z_1 \in X \), and construct sequence \( \{z_n\} \in X \), such that \( z_{2n+1} = fz_{2n}, z_{2n+2} = gz_{2n+1} \), and \( (z_{2n-1}, z_{2n}) \in E \) for all \( n \in \mathbb{N} \). We have
\[
d(z_{2n+1}, z_{2n+2}) = d(gz_{2n+1}, fz_{2n}) \\
\leq a^*d(z_{2n+1}, z_{2n})a \\
\leq (a^*)^2d(z_{2n}, z_{2n-1})(a)^2 \\
\leq (a^*)^{2n+1}d(z_1, z_0)(a)^{2n+1}.
\]
Similarly,
\[
d(z_{2n+1}, z_{2n}) = d(fz_{2n}, gz_{2n-1}) \\
\leq a^*d(z_{2n}, z_{2n-1})a \\
\leq (a^*)^{2n}d(z_1, z_0)(a)^{2n} \\
= (a^*)^{2n}Q(a)^{2n}.
\]
Let us denote \( d(z_1, z_0) \) by \( Q \in A \). Then for any \( n \in \mathbb{N} \)
\[
d(z_{n+1}, z_n) = (a^*)^n d(z_1, z_0)(a)^n \\
= (a^*)^n Q(a)^n,
\]
then for any \( q \in \mathbb{N} \) and applying the triangular inequality (3) for the \( C^* \)-valued metric spaces,
\[
d(z_{n+q}, z_n) = \sum_{n+q-1}^{n} (a^*)^j d(z_1, z_0)(a)^j \\
\leq \sum_{j=n}^{n+q-1} (a^*)^j Q(a)^j \\
= \sum_{j=n}^{n+q-1} (a^*)^j Q^2(a^j) \\
= \sum_{j=n}^{n+q-1} (Q^2a^j)^* (Q^2a^j) \\
= \sum_{j=n}^{n+q-1} |Q^2a^j|^2 \\
\leq \sum_{j=n}^{n+q-1} ||Q^2a^j||^2 ||I \\
= ||Q^2||^2 \sum_{j=n}^{n+q-1} ||a^j||.
\]
Since \( ||a|| < 1 \), thus \( d(z_{n+q}, z_n) \to 0 \) as \( n \to \infty \). Thus we conclude that the sequence \( \{z_n\} \) is a Cauchy sequence, with respect to \( A \). Using the completeness of \( X \), there exists an element \( z_0 \in X = V \), such that \( z_n \to z_0 \) as \( n \to \infty \).
On the other hand, using the triangular inequality, we get
\[
d(z_0, f z_0) = d(z_0, z_{2n+1}) + d(z_{2n+1}, f z_0) \\
= d(z_0, z_{2n+1}) + d(g z_{2n}, f z_0) \\
\leq d(z_0, z_{2n+1}) + a^n d(z_{2n}, z_0) a.
\]
Thus if \( n \to \infty \), then \( d(z_0, f z_0) \to 0 \) i.e. \( f z_0 = z_0 \). Similarly we can prove that \( g z_0 = z_0 \). Now we will show the uniqueness of common fixed points in \( X \). For this we assume that there is another point \( z^* \in X = V \), such that \((z_0, z^*) \in E \). Consider
\[
d(z_0, z^*) = d(f z_0, g z_0) \leq a^* d(z_0, z^*) a.
\]
Since \( ||a|| < 1 \), then the above inequality yields that
\[
0 \leq ||d(z_0, z^*)|| \leq ||a||^2 ||d(z_0, z^*)|| < ||d(z_0, z^*)||.
\]
Which is a contradiction. Thus, \( ||d(z_0, z^*)|| = 0 \) which implies that \( d(z_0, z^*) = 0 \) i.e. \( z_0 = z^* \). Thus the proof is complete.

**Corollary 2.3.** Suppose that \((X, d, \mathcal{A})\) is a \( C^* \)–valued metric space endowed with the graph \( G \), and suppose that the mappings \( f, g : X \to X \) are \( G \)–contractive, satisfying
\[
||d(f x, g y)|| \leq ||a|| ||d(x, y)||, \quad \text{for all } (x, y) \in E,
\]
where \( a \in \mathcal{A} \) with \( ||a|| < 1 \). Then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Corollary 2.4.** Let \((X, d, \mathcal{A})\) be a \( C^* \)–valued metric space endowed with the graph \( G \), and suppose that the mapping \( f : X \to X \) is \( G \)–contractive, satisfying
\[
||d(f^m x, f^n y)|| \leq a^* d(x, y) a, \quad \text{for all } (x, y) \in E,
\]
where \( a \in \mathcal{A} \) with \( ||a|| < 1 \) and \( m, n \) are positive integers. Then \( f \) has a unique fixed point in \( X \).

**Remark 2.5.** In Theorem 2.2, if \( f = g \), then we have
\[
(2.2) \quad d(f x, f y) \leq a^* d(x, y) a, \quad \text{for all } (x, y) \in E.
\]
In this case we have the following corollary, which can also be found in [12].

**Corollary 2.6.** Let \((X, d, \mathcal{A})\) be a complete \( C^* \)–valued metric space, and consider the mapping \( f : X \to X \) such that it satisfies \((2.2)\), then \( f \) has a unique fixed point in \( X \).

**Example 2.7.** Consider, \( \mathcal{A} = M_{2 \times 2}(\mathbb{R}) \), of all \( 2 \times 2 \) matrices with the usual operation of addition, scalar multiplication, and matrix multiplication. Thus \( \mathcal{A} \) becomes \( C^* \)–algebra. Let us define \( d : \mathbb{R} \times \mathbb{R} \to \mathcal{A} \) by
\[
d(x, y) = \left( \begin{array}{cc}
|x - y| & 0 \\
0 & |x - y|
\end{array} \right).
\]
It is easy to check that \( d \) satisfies all the conditions of Definition 1.1. Therefore \((\mathbb{R}, \mathcal{A}, d)\) is \( C^* \)–valued metric space. Define \( f, g : \mathbb{R} \to \mathbb{R} \) by
\[
f(x) = \frac{x^2}{4} \quad \text{and} \quad g(x) = \frac{x^2}{3},
\]
and consider the graph \( G = (V, E) \), where \( V = \mathbb{R} \) and
\[
E = \left\{ \left( \frac{1}{4m}, \frac{1}{32m+1} \right); m = 1, 2, \ldots \right\} \cup \left\{ \left( \frac{1}{4m}, 0 \right); m = 1, 2, \ldots \right\} \cup \{(x, x); x \in \mathbb{R}\}.
\]
Note that, for each \( m \in \mathbb{N} \),
\[
\left(f\left(\frac{1}{4m}\right), g\left(\frac{1}{32m+1}\right)\right) = \left(\frac{1}{42m+1}, \frac{1}{34m+1}\right) \in E,
\]
and
\[
\left(f\left(\frac{1}{4m}\right), g(0)\right) = \left(\frac{1}{32m+1}, 0\right) \in E.
\]
Also, $(fx, gx) = \left(\frac{x^2}{4}, \frac{x^2}{4}\right)$, for each $x \in \mathbb{R}$, which is again in $E$. Moreover, by taking $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$, we have $||A|| < 1$, so all the conditions of Theorem 2.2 are satisfied and thus the common fixed point of $f$ and $g$ is $0$.

References


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