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Pollution Identification in an Age-Structured Reaction-Diffusion Model with Regulated Renewal Conditions

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Abstract. This paper deals with the identification of pollution terms in an age-structured reaction-diffusion model with nonlinear renewal condition and incomplete initial data. The model describes the joint dynamics of two fish species cohabiting in a lake represented by a bounded domain. The pollution sources are represented by unknown parameters acting on each population. Moreover, the initial data are subject to perturbations. To estimate the pollution effects independently of missing data, we introduce the notion of simultaneous sentinels. We show that the existence of such sentinels is equivalent to a simultaneous null-controllability problem for an associated adjoint system. The existence of the control relies on an observability inequality derived from a global Carleman estimate. By transforming the coupled system into an equivalent cascade-type system, we establish the existence of an optimal control of minimal norm ensuring the required insensibility condition. This control allows us to derive estimates for the unknown pollution parameters.

1. Introduction

The study of spatio-temporal systems has generated numerous applications in fields such as ecology, environmental pollution, and population dynamics. One important issue arising in such systems is the treatment of incomplete data. For distributed or dissipative systems defined on a domain Ω, many results are available in the literature; see, for example [1, 4, 6–9, 11] and the references therein. In [4] and [11], the authors focused on simultaneous sentinels for a coupled

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nonlinear heat equations system and for a nonlinear two-stroke equations system, respectively. In both works, they showed that the existence of simultaneous sentinels is equivalent to the simultaneous null controllability of the corresponding coupled system, established by means of suitable Carleman inequalities. In [11], the boundary condition is unknown on Γ_i, $i = 1;2$, one part of the boundary Γ of the domain Ω and is such that $\Gamma_1 \cup \Gamma_2 = \Gamma$. In this work, we consider two species of fish (or two cohorts), denoted by y_1 and y_2 cohabiting in a lake (domain) represented by Ω. Their dynamics are described by the following model: for each $i \in \{1;2\}$ and two strictly positive real numbers:

$$\begin{cases} \dfrac{\partial y_i}{\partial t} + \dfrac{\partial y_i}{\partial a} - \Delta y_i + \mu_i y_i &= g_i + \lambda_i \widehat{g_i} \text{ in } Q; \\ y_i(0,a,x) &= y_i^0 + \tau_i \hat{y}_i^0 \text{ in } Q_A; \\ y_i(t,0,x) &= F_i\left(\displaystyle\int_0^A \beta_i y_i da\right) \text{ in } Q_T; \\ y_i &= 0 \text{ on } \Sigma \end{cases} \tag{1.1}$$

Here, Ω is a bounded open set of $\mathbb{R}^N$, with $N \in \{1,2,3\}$, and boundary Γ of class C^∞. And $U = [0,T] \times [0,A]$, $Q = U \times \Omega$, $Q_A = [0,A] \times \Omega$, $Q_T = [0,T] \times \Omega$, $\Sigma = [0,T] \times [0,A] \times \Gamma$.

The terms of system (1.1) have the following biological interpretation. The operator $\frac{\partial y_i}{\partial t} + \frac{\partial y_i}{\partial a}$ describes evolution of the population with respect to time and age. The spatial diffusion, $-\Delta y_i$ describes the movement of fish in the lake. The functions $\beta_i(t,a,x)$ and $\mu_i(t,a,x)$ denote the fertility and mortality rates, respectively, of fish of age a at location x at time t. The initial condition $y^0 + \tau_i \widehat{y_i^0}$ indicates the distribution by age and space at the beginning of the observation and is uncertain. In other words, we do not know exactly the number of fish of age a, we have at the beginning at the location x. The birth condition $a = 0$, describes the number of newborns at time t at location x that depends on the adult population through the fertility rates and the nonlinear function F_i. This function accounts for regulatory effects such that competition for breeding sites, environmental variability, or density dependence. Thus, the term $y_i(t,0,x)$ expresses that the contribution of an adult individual to population dynamics depends not only on their age but also on local ecological conditions. We can therefore say that this expresses the impact over time of an adult individual of age a, through their regulated (non-linear) reproductive capacity, on the value of a newborn. The homogeneous Dirichlet boundary condition $y_i = 0$, $i = 1,2$ models an absorbing boundary (or lethal boundary). When an individual (fish) reaches of the domain (lake), it disappears from the system or can neither return nor reproduce. This represents total mortality linked to this boundary. Biologically, this means that any fish reaching the boundary of the habitat disappears from the system and can neither return nor reproduce. Such a condition may represent highly polluted areas, zones of desiccation, or impassable ecological barriers where survival is impossible.

In the source terms $g_i + \lambda_i \widehat{g_i}$, the quantities $\lambda_i \widehat{g_i}$ are unknown and represent the pollution that affect an age class of the fish population while g_i and $\widehat{g_i}$ are known. The main objective of this work is to estimate the pollution terms $\lambda_i \widehat{g_i}$ starting from observations distributed. One of the best tools for detecting pollution is the sentinel method introduced by J. Lions [5]. To do it, we will first

focus on the existence of simultaneous sentinels in proving such of existence is equivalent to the simultaneous null-controllability for two stroke equations studied in [11].

In the rest of paper, we will give some assumptions in the section 2 to ensure well-posedness of the model. In the section 3, we will define the sentinel. Section 4 will be devoted to proving the equivalence between the existence of simultaneous sentinels and the simultaneous null-controllability. In Section 5, we will recall an observability inequality that allows us to characterize the simultaneous sentinels. In Section 6, we will estimate the pollution parameters. The next Section will conclude the paper.

2. Assumptions for Well-Posed Problem

We adopt the following hypothesis of [11].

$$(H_1)\ \begin{cases} (\mu_i,\ \nabla\mu_i) \in (L^\infty(Q))^{N+1} & \text{for all } i \in \{1;2\},\ N \in \mathbb{N}^*, \\ \mu_i \geq 0 \text{ in } Q & \text{for all } i \in \{1;2\}, \\ \mu_1 \neq \mu_2 \text{ in } Q_\omega. \end{cases}$$

$$(H_2)\ \begin{cases} \beta_i \in C^2(\overline{Q}) & \text{for all } i \in \{1;2\}, \\ \beta_i \geq 0 \text{ in } \overline{Q} & \text{for all } i \in \{1;2\}. \end{cases}$$

(H_3) There exists positive constants nonzero a_0 and a_1 with $a_0 < a_1 < A$ such that for each $i \in \{1;\ 2\}$, $\beta_i(t,a,x) = 0$ a.e $(t,a,x) \in (0,T) \times ([0,a_0] \cup [a_1,A]) \times \Omega$.

(H_4) $\forall i \in \{1,2\}, F_i : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-linear C^2 function such that F_i' belong to $L^\infty(\mathbb{R})$ is globally lipschitz and $F_i'(0) = 0$.

Furthermore, suppose that:

(i) for i=1;2 , $\widehat{g_i} \in L^2(Q)$ and $\|\widehat{g_i}\|_{L^2(Q)} \leq 1$;
(ii) for i=1;2 , $\widehat{y_i^0} \in L^2(Q_A)$ and $\|\widehat{y_i^0}\|_{L^2(Q_A)} \leq 1$;
(iii) for $i = 1;2$, the reals τ_i and λ_i are unknown and small enough.

Remark 2.1. *Under the assumptions $(H_1) - (H_4)$ and (i)-(iii), for all $i \in \{1,2\}$, system (1.1) admits an unique solution $y_i := y_i(t,a,x)$ in $L^2(U, H_0^1(\Omega))$ such that $\frac{\partial y_i}{\partial t} + \frac{\partial y_i}{\partial a} \in L^2(U; H^{-1}(\Omega))$ where $H^{-1}(\Omega)$ is the dual of the Hilbert space $H_0^1(\Omega)$. Moreover for all $i \in \{1,2\}, y_i$ belong to $C((0,T); L^2(Q_A)) \cap C((0,A); L^2(Q_T)) \cap L^2(U, H_0^1(\Omega))$ (see Lemma 0 in [3]). Such a solution will be denoted by $y_i(\lambda_i, \tau_i)$ to signify that it depends on parameters λ_i, τ_i for all $i \in \{1,2\}$.*

3. Notion of the Simultaneous Sentinels

Suppose that the pollution $\lambda_i \widehat{g_i}$ affects fish in certain places of lake. Let O, the region defined by this places, so-called observatory. We wish to determine the pollution using, on the one hand, measurements

$$y_{iobs} = m_i \text{ on } O \text{ for } i \in \{1,2\}, \tag{3.1}$$

weighted by a function h and, on the other hand, a control function w defined on ω, an open and non-empty subset of Ω such that $\omega \cap O \neq \emptyset$. Now, consider the functional defined by:

$$S(\lambda,\tau) = \int_{Q_O} h\left(y_1(\lambda_1,\tau_1) + y_2\left(\lambda_2,\tau_2\right)\right) dQ + \int_{Q_\omega} w\left(y_1(\lambda_1,\tau_1) + y_2\left(\lambda_2,\tau_2\right)\right) dQ \tag{3.2}$$

with $\lambda = (\lambda_1,\lambda_2),\ \tau = (\tau_1,\tau_2) \in \mathbb{R}^2$. We will use this function only to obtain informations on the pollution $\lambda_i\widehat{g_i}$ that is not affected by the missing data (also called perturbations) $\tau_i\widehat{y_i^0}$ for $i \in \{1,2\}$. This reduces to the following definition.

Definition 3.1. *The function S is called simultaneous sentinel if it verifies two conditions that are:*

- ***Insensibility to the missing data at the first order**:*

$$\frac{\partial S}{\partial \tau_i}\left(0_{\mathbb{R}^4}\right) = 0,\ i = \overline{1,2}\,; \tag{3.3}$$

- *and there exists the function w such that*

$$\|w\|_{L^2(Q_\omega)} \text{ is minimal norm.} \tag{3.4}$$

Remark 3.1. *As proved in Proposition 3 of [11], for each $i = 1;2$, the functions $\lambda_i \longmapsto y_i(\lambda_i,\tau_i)$ and $\tau_i \longmapsto y_i(\lambda_i,\tau_i)$ are differentiable at the point 0.*

Then the following Proposition holds.

Proposition 3.1. *The functional S is differentiable at the point $0_{\mathbb{R}^4} = (0,0,0,0)$ and*

$$\frac{\partial S}{\partial \tau_i}\left(0_{\mathbb{R}^4}\right) = \int_Q \left(h\chi_O + w\chi_\omega\right) y_i^{\tau_i} dQ \text{ for } i \in \{1,2\}\,; \tag{3.5}$$

$$\frac{\partial S}{\partial \lambda_i}\left(0_{\mathbb{R}^4}\right) = \int_Q \left(h\chi_O + w\chi_\omega\right) y_i^{\lambda_i} dQ \text{ for } \ i \in \{1,2\}. \tag{3.6}$$

where for each fixed $i \in \{1,2\}$, $y_i^{\tau_i}$ solves the system:

$$\begin{cases} \dfrac{\partial y_i^{\tau_i}}{\partial t} + \dfrac{\partial y_i^{\tau_i}}{\partial a} - \Delta y_i^{\tau_i} + \mu_i y_i^{\tau_i} = 0 & \text{in} \quad Q, \\ y_i^{\tau_i}(0,a,x) = \widehat{y_i}^0(a,x) & \text{in} \quad Q_A, \\ y_i^{\tau_i}(t,0,x) = F_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right) \int_0^A \beta y_i^{\tau_i} da & \text{in} \quad Q_T, \\ y_i^{\tau_i} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{3.7}$$

and $y_i^{\lambda_i}$ solves the system

$$\begin{cases} \dfrac{\partial y_i^{\lambda_i}}{\partial t} + \dfrac{\partial y_i^{\lambda_i}}{\partial a} - \Delta y_i^{\lambda_i} + \mu_i y_i^{\lambda_i} = 0 & \text{in} \quad Q, \\ y_i^{\lambda_i}(0,a,x) = 0 & \text{in} \quad Q_A, \\ y_i^{\lambda_i}(t,0,x) = F_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right) \int_0^A \beta_i y_i^{\lambda_i} da & \text{in} \quad Q_T, \\ y_i^{\lambda_i} = 0 \text{ on } \Sigma \end{cases} \tag{3.8}$$

Moreover, for each fixed $i \in \{1,2\}$,

$$y_i^{\tau_i},\ y_i^{\lambda_i} \in C((0,T);L^2(Q_A)) \cap C((0,A);L^2(Q_T)) \cap L^2(U,H_0^1(\Omega)). \tag{3.9}$$

Remark 3.2. *A complete proof of Proposition 3.1 can be found in [11] replacing for fixed $i \in \{1,2\}$, S_i, y_{τ_i} and y_{λ_i} with S, $y_i^{\tau_i}$ and $y_i^{\lambda_i}$ respectively. Notice that for $i \in \{1,2\}$:*

$$y_i^{\tau_i} = \lim_{\tau_i \to 0} \frac{y_i(\lambda_i = 0, \tau_i) - y_i(0,0)}{\tau_i} \ \text{ and } \ y_i^{\lambda_i} = \lim_{\lambda_i \to 0} \frac{y_i(\lambda_i, \tau_i = 0) - y_i(0,0)}{\lambda_i}. \tag{3.10}$$

In the sequel, we will use y_{0i} to denote $y_i(0,0)$ for all $i \in \{1,2\}$.

4. Equivalence with Simultaneous Null-Controllability

We shall show in this section that the study of the existence of a sentinel reduces to the study of simultaneous null-controllability problem. We begin by transforming the insensibility condition (3.3). For fixed $i \in \{1,2\}$ the derivatives of the system (1.1) with respect to the parameters τ_i and λ_i near $(\lambda_i = 0; \tau_i = 0)$ is given by (3.7) and (3.8) respectively. Then, the insensibility condition (3.3) holds if and only if

$$\int_Q (h\chi_O + w\chi_\omega) y_i^{\tau_i} dQ = 0. \tag{4.1}$$

The adjoint system associated to (3.7) for a fixed $i \in \{1,2\}$, is defined by the following system:

$$\begin{cases} -\dfrac{\partial q_i}{\partial t} - \dfrac{\partial q_i}{\partial a} - \Delta q_i + \mu_i q_i &= \beta_i F_i' \left(\displaystyle\int_0^A \beta_i y_{0i} da \right) q_i(t,0,x) + h\chi_O + w\chi_\omega \quad \text{in} \quad Q, \\ q_i(T,a,x) &= 0 \quad \text{in} \quad Q_A, \\ q_i(t,A,x) &= 0 \quad \text{in} \quad Q_T, \\ q_i &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{4.2}$$

As for the problem (3.7), the problem (4.2) has a unique solution q_i (under mild assumptions on $F_i'(\cdot)$). This function q_i depends on the control w that we shall determine. Then, for a fixed $i \in \{1,2\}$, multiply the first equation in (4.2) by $y_i^{\tau_i}$, and integrate by parts over Q, we obtain:

$$\int_{Q_A} \widehat{y}_i^0(a,x) q_i(0,a,x) dQ_A = \int_Q (h\chi_O + w\chi_\omega)\, y_i^{\tau_i} dQ. \tag{4.3}$$

It follows, firstly by summation, that

$$\int_{Q_A} \left(\widehat{y}_1^0(a,x) q_1(0,a,x) + \widehat{y}_2^0(a,x) q_2(0,a,x) \right) dadx = \int_Q (h\chi_O + w\chi_\omega) \left(y_1^{\tau_1} + y_2^{\tau_2} \right) dQ, \tag{4.4}$$

and secondly by identification, that the condition (4.1) is equivalent to

$$q_i(0,a,x) = 0 \text{ in } Q_A, \forall i \in \{1,2\}. \tag{4.5}$$

This leads to simultaneous null-controllability problem of the system (4.2). The following definition holds.

Definition 4.1. *The functional S is the simultaneous sentinel if and only if for $h \in L^2(Q_O)$, there exists a control function $w \in L^2(Q_w)$ such that the solution q_i of* (4.2) *verifies* (4.5) *and w satisfies* (3.4).

Remark 4.1. *The knowledge of the optimal control w provides informations about the pollutions terms.*

In the following section, we describe the main steps of the approach used to address the null-controllability problem (4.2).

5. Existence of the Control

We have proven in section 4 that the existence of a sentinel is equivalent to the existence of a control. The sentinel problem is reduced to the simultaneous null-controllability problem.

5.1. **Equivalence between the controllability of a nonlinear system and that of its linearized system.**

Let consider the following system:

$$\begin{cases} -\dfrac{\partial q_i}{\partial t} - \dfrac{\partial q_i}{\partial a} - \Delta q_i + \mu_i q_i &= \beta_i F_i\left(\displaystyle\int_0^A \beta_i q_i da\right) q_i(t,0,x) + h\chi_O + w\chi_\omega \quad \text{in} \quad Q, \\ q_i(T,a,x) &= 0 \quad \text{in} \quad Q_A, \\ q_i(t,A,x) &= 0 \quad \text{in} \quad Q_T, \\ q_i &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{5.1}$$

We will justify that the null controllability of system (4.2) reduces to the local controllability of system (5.1) at the point $\displaystyle\int_0^A \beta_i y_{0i} da, i = 1,2.$

Indeed, the two systems differ by their nonlinear terms $F_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)$ and $F_i\left(\displaystyle\int_0^A \beta_i q_i da\right), i = 1,2.$ For $\lambda_i = \tau_i = 0, i = 1,2$; system (1.1) describes the steady states y_{0i} of the nonlinear system (5.1). Thus, $q_i, i = 1;2$ represent perturbations around the equilibrium solution $y_{0i}, i = 1;2.$

Let consider a small perturbation around the equilibrium state y_{0i} for $i \in \{1,2\}$. Applying the Taylor formula to the functional F_i around $\displaystyle\int_0^A \beta_i y_{0i} da$, we have:

$$F_i\left(\int_0^A \beta_i (y_{0i} + q_i) da\right) \approx F_i\left(\int_0^A \beta_i y_{0i} da\right) + F_i'\left(\int_0^A \beta_i y_{0i} da\right) \cdot \int_0^A \beta_i q_i da \tag{5.2}$$

System (4.2) retains only the linear term part of this approximation, the zero-order terms $F_i\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)$ being included in the equilibrium system (1.1) when $\lambda_i = \tau_i = 0$. Which gives the term $\beta_i F_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right) q_i(\cdot,0,\cdot)$. The remainder linear differential operators are identical.

Consequently, given that the nonlinear terms of system (5.1) are differentiable, and that system (4.2) with $\lambda_i = \tau_i = 0$ possesses an equilibrium point, the implicit function theorem (or the linearization method) yields that the local simultaneous controllability of the nonlinear system is equivalent to the simultaneous controllability of its linearized counterpart [12].

Hence, the simultaneous null-controllability analysis of the linearized system (4.2) is extended to that of nonlinear (5.1). An observability inequality is the key ingredient to prove this simultaneous controllability. We will quote below Carleman's global inequality and the observability inequality

(see [10]) that we need to solve the simultaneous null-controllability problem and then, the result of the existence of the optimal control.

Remark 5.1. *Although the controllability of the linearized system* (4.2) *is more straightforward than that on the controllability of the nonlinear system* (5.1)*, as it as been thoroughly investigated in [11]. As mentioned above, the remainder of this section recalls the mathematical tools employed to aid understanding. These tools use the coupled cascade system equivalent to system* (5.1)*. Details regarding to the proofs of the Theorems and Propositions below can be found in [10,11].*

Let set

$$p_1 = q_1 + q_2 \quad ; \quad p_2 = q_1 - q_2. \tag{5.3}$$

Then simultaneous null controllability of system (5.1) is equivalent to the following: find a control v such that the couple solution (p_1, p_2) of

$$\begin{cases} -\dfrac{\partial p_1}{\partial t} - \dfrac{\partial p_1}{\partial a} - \Delta p_1 + \tilde{\mu}_1 p_1 + \tilde{\mu}_2 p_2 &= \tilde{\beta}_1 b_1(t,a,x) p_1(t,0,x) + f + v\chi_\omega \\ & \quad + \tilde{\beta}_2 b_2(t,a,x) p_2(t,0,x) \qquad \text{in } Q \\ -\dfrac{\partial p_2}{\partial t} - \dfrac{\partial p_2}{\partial a} - \Delta p_2 + \tilde{\mu}_1 p_2 + \tilde{\mu}_2 p_1 &= \tilde{\beta}_1 b_2(t,a,x) p_1(t,0,x) \\ & \quad + \tilde{\beta}_2 b_1(t,a,x) p_2(t,0,x) \qquad \text{in } Q \\ p_i &= 0 \quad \text{on} \quad \Sigma,\ i = 1,2 \\ p_i(T,a,x) &= 0 \quad \text{in} \quad Q_A,\ \ i = 1,2 \\ p_i(t,A,x) &= 0 \quad \text{in} \quad Q_T,\ \ i = 1,2 \end{cases} \tag{5.4}$$

verifies

$$p_i(0,a,x) = 0 \text{ in } Q_A, i = 1,2. \tag{5.5}$$

where:

$$\begin{cases} \tilde{\mu}_1 = \frac{1}{2}(\mu_1 + \mu_2)\ ,\ \tilde{\mu}_2 = \frac{1}{2}(\mu_1 - \mu_2)\ ; \\ \tilde{\beta}_1 = \frac{1}{2}(\beta_1 + \beta_2)\ , \quad \tilde{\beta}_2 = \frac{1}{2}(\beta_1 - \beta_2)\ ; \\ f = 2h\chi_O\ ,\ k = 2w\ ; \\ b_1(t,a,x) = \tilde{\beta}_1 F_1\left(\frac{1}{2}\displaystyle\int_0^A \beta_1(p_1+p_2)da\right) + \tilde{\beta}_2 F_2\left(\frac{1}{2}\displaystyle\int_0^A \beta_2(p_1-p_2)da\right)\ ; \\ b_2(t,a,x) = \tilde{\beta}_1 F_1\left(\frac{1}{2}\displaystyle\int_0^A \beta_1(p_1+p_2)da\right) - \tilde{\beta}_2 F_2\left(\frac{1}{2}\displaystyle\int_0^A \beta_2(p_1-p_2)da\right). \end{cases} \tag{5.6}$$

and for all $i \in \{1,2\}$, $\tilde{\mu}_i$ verifies (H_1), $\tilde{\beta}_i$ satisfies $(H_2) - (H_3)$.

5.2. **Global Carleman and observability inequalities.** For any positive parameters λ and τ, we define the positive functions:

$$\alpha(t,a,x) = \tau \frac{e^{\frac{4}{3}\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{at\,(T-t)} \quad \text{and} \quad \varphi(t,a,x) = \frac{e^{\lambda\psi(x)}}{at\,(T-t)}\ ,\ \forall\,(t,a,x) \in Q.$$

Remark 5.2. *Recall (see [2])that the function $\psi \in C^2(\overline{\Omega})$ is such that:*

$$\forall x \in \Omega\,; \psi(x) > 0\;;\;\; \forall x \in \Gamma, \psi(x) = 0 \text{ and } \;\forall x \in \overline{\Omega \setminus \omega_0}\;,\; \nabla\psi(x) \neq 0$$

where ω_0 is an open set such that $\overline{\omega}_0 \subset \omega \subset \Omega$. In the sequel:

- *C represent different positive constants,*
- *we will use the next notations*

$$\mathcal{V} = \left\{\rho \in C^{\infty}\left(\overline{Q}\right) \text{ such that } \rho_{|\Sigma} = 0\right\} \quad ; \quad \mathcal{W} = \mathcal{V} \times \mathcal{V}\,,$$

$$L\rho = -\frac{\partial \rho}{\partial t} - \frac{\partial \rho}{\partial a} - \Delta\rho \quad ; \quad L^*\rho = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta\rho$$

$$M(\rho_1, \rho_2) = L^*\rho_1 + \hat{\mu}_1\rho_1 + \hat{\mu}_2\rho_2 \quad ; \quad N(\rho_1, \rho_2) = L^*\rho_2 + \hat{\mu}_1\rho_2 + \hat{\mu}_2\rho_1\,;$$

$$\| \hat{\mu}_1, \hat{\mu}_2 \|_{\infty}^2 = \| \hat{\mu}_1 \|_{\infty}^2 + \| \hat{\mu}_2 \|_{\infty}^2 \qquad \text{and} \qquad dQ = dtdadx$$

Theorem 5.1 (see [10])**.** *There exists $\lambda_0 > 0$, $\tau_0 > 0$ and a positive constant C such that for all $\lambda \geq \lambda_0$, $\tau \geq \tau_0$ and for all $s \geq -3$, the inequality*

$$\int_Q \left(\frac{1}{\lambda} \left| \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \right|^2 + \frac{1}{\lambda} \left| \Delta\rho \right|^2 + \lambda\tau^2\varphi^2 \left| \nabla\rho \right|^2 + \lambda^4\tau^4\varphi^4 \left| \rho \right|^2 \right) \varphi^{2s-1} e^{-2\alpha} \mathrm{dQ}$$

$$\leq C \left(\tau \int_Q \left| \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \pm \Delta\rho \right|^2 \varphi^{2s} e^{-2\alpha} \mathrm{dQ} + \text{ᵕ}^4 \text{ø}^4 \int_0^{\mathrm{T}} \int_0^{\mathrm{A}} \int_! |\text{æ}|^{2\,,2s+3} \mathrm{e}^{-2\mathrm{ff}} \mathrm{dQ} \right) \tag{5.7}$$

holds for any function $\rho \in \mathcal{V}$ such that the member on the right hand side of the inequality (5.7) *is finite.*

Setting now

$$\theta = e^{\alpha} \text{ and } \;\delta = \theta^{\frac{r}{2}-1}, \tag{5.8}$$

we have the following observability inequality:

Proposition 5.1 (see [10])**.** *Under the hypothesis of the Theorem 5.1, there exists a positive constant C such that for all $\rho = (\rho_1\,, \rho_2) \in \mathcal{W}$, we have*

$$\int_{Q_T} \left(\left|\rho_1(t,0,x)\right|^2 + \left|\rho_2(t,0,x)\right|^2 \right) dxdt + \int_{Q_A} \left(\left|\rho_1(0,a,x)\right|^2 + \left|\rho_2(0,a,x)\right|^2 \right) dxda$$

$$\leq\; C \left(\int_Q \left(\left|M(\rho)\right|^2 + \left|N(\rho)\right|^2 \right) \mathrm{dQ} \;+ \int_{\mathrm{Q!}} \mathrm{ffi}^2 \left|\text{æ}_1\right|^2 \mathrm{dQ} \right) \tag{5.9}$$

5.3. **Study of the simultaneous problem.** From Proposition 5.1, we cite the following [11]:

Theorem 5.2. *Let Ω an open subset of $\mathbb{R}^N$ with boundary Γ of class C^2 and ω be a non empty subset of Ω. Assume that the hypothesis $(H_1) - (H_4)$ hold. For any $f \in L^2(\Omega)$ such that $\theta f \in L^2(\Omega)$, there exists an unique control $\tilde{k} \in L^2(Q_\omega)$, such that $(\tilde{k}, \tilde{p}_1, \tilde{p}_2)$ is solution of the simultaneous null controllability problem* (5.4)-(5.5)*. Moreover, the control $\tilde{k}$ is given by*

$$\tilde{k} = \tilde{\eta}_1 \chi_\omega \tag{5.10}$$

and verifies

$$\|\tilde{k}\|_{L^2(Q_\Omega)} \leq C\left(\|\theta f\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right) \tag{5.11}$$

where (η_1, η_2) *verifies*

$$\begin{cases} \dfrac{\partial \tilde{\eta}_1}{\partial t} + \dfrac{\partial \tilde{\eta}_1}{\partial a} - \Delta\tilde{\eta}_1 + \tilde{\mu}_1\tilde{\eta}_1 + \tilde{\mu}_2\tilde{\eta}_2 = 0 & \text{in } Q, \\ \dfrac{\partial \tilde{\eta}_2}{\partial t} + \dfrac{\partial \tilde{\eta}_2}{\partial a} - \Delta\tilde{\eta}_2 + \tilde{\mu}_1\tilde{\eta}_2 + \tilde{\mu}_2\tilde{\eta}_1 = 0 & \text{in } Q, \\ \tilde{\eta}_1(0,a,x) = \tilde{\eta}_2(0,a,x) = 0 & \text{in } Q_A, \\ \tilde{\eta}_1 = \tilde{\eta}_2 = 0 & \text{on } \Sigma, \\ \tilde{\eta}_1(t,0,x) = \displaystyle\int_0^A b_1(\tilde{\eta}_1 + \tilde{\eta}_2)da & \\ \qquad + \displaystyle\int_0^A b_2(t,a,x)(\tilde{\eta}_1 - \tilde{\eta}_2)da & \text{in } Q_T, \\ \tilde{\eta}_2(t,0,x) = \displaystyle\int_0^A b_1(\tilde{\eta}_1 + \tilde{\eta}_2)da & \\ \qquad + \displaystyle\int_0^A b_2(\tilde{\eta}_2 - \tilde{\eta}_1)da & \text{in } Q_T. \end{cases} \tag{5.12}$$

It follows that problem (5.4)-(5.5) is locally simultaneously null-controllable. And so, there exists a function control in $L^2(Q_\omega)$ such that the conditions of Definition 4.1 are satisfied.

Thanks to equivalence between the controllability of a nonlinear system and that of its linearized system, we can say that follows. The study of the existence of the control satisfying (3.4) and such that the solution of (4.2) satisfies (4.5) is reduced to the study of the linear case in [11] using a cascade coupled system thanks to following changes. On the one hand, let consider (5.3) replacing p_i with z_i for each $i \in \{1,2\}$ and on the other hand,

$$X_{01} = y_{01} + y_{02} \text{ and } X_{02} = y_{01} - y_{02}, \tag{5.13}$$

with $y_{0i} = y_i(0,0), i \in \{1,2\}$ a solution of (1.1). Consequently (4.2) is written as follows:

$$\begin{cases} -\dfrac{\partial \tilde{z}_1}{\partial t} - \dfrac{\partial \tilde{z}_1}{\partial a} - \Delta\tilde{z}_1 + \tilde{\mu}_1\tilde{z}_1 + \tilde{\mu}_2\tilde{z}_2 = \tilde{b}_1(t,a,x)\tilde{z}_1(t,0,x) + f + v\chi_\omega & \\ \qquad + \tilde{b}_2(t,a,x)z_2(t,0,x) \quad \text{in } Q \\ -\dfrac{\partial \tilde{z}_2}{\partial t} - \dfrac{\partial \tilde{z}_2}{\partial a} - \Delta\tilde{z}_2 + \tilde{\mu}_1\tilde{z}_2 + \tilde{\mu}_2\tilde{z}_1 = \tilde{b}_2(t,a,x)\tilde{z}_1(t,0,x) & \\ \qquad + \tilde{b}_1(t,a,x)z_2(t,0,x) \quad \text{in } Q \\ \tilde{z}_i = 0 \quad \text{on} \quad \Sigma,\ i = 1,2 \\ \tilde{z}_i(T,a,x) = 0 \quad \text{in} \quad Q_A,\ i = 1,2 \\ \tilde{z}_i(t,A,x) = 0 \quad \text{in} \quad Q_T,\ i = 1,2 \end{cases} \tag{5.14}$$

The condition (4.5) is equivalent to

$$z_i(0,a,x) = 0 \text{ in } Q_A,\ i = 1,2. \tag{5.15}$$

where:

$$\begin{cases} \tilde{b}_1(t,a,x) &= \tilde{\beta}_1 F_1'\left(\frac{1}{2}\int_0^A \beta_1(X_{01}+X_{02})da\right) + \tilde{\beta}_2 F_2'\left(\frac{1}{2}\int_0^A \beta_2(X_{01}-X_{02})da\right); \\ \tilde{b}_2(t,a,x) &= \tilde{\beta}_1 F_1'\left(\frac{1}{2}\int_0^A \beta_1(X_{01}+X_{02})da\right) - \tilde{\beta}_2 F_2'\left(\frac{1}{2}\int_0^A \beta_2(X_{01}-X_{02})da\right). \end{cases} \tag{5.16}$$

We can state the following result:

Theorem 5.3. *Suppose that assumptions (H_1) to (H_4) hold. For any function $f \in L^2(Q)$ such that $\theta f \in L^2(Q)$, there exists a control $\tilde{v}$ in $L^2(Q_\omega)$ such that $(\tilde{v},(\tilde{z}_1,\tilde{z}_2))$ is solution of simultaneous null controllability problem (5.14)-(5.15). Moreover, this threesome $(\tilde{v},\tilde{z}_1,\tilde{z}_2)$ verifies*

$$\tilde{v} = \tilde{u}_1\chi_\omega \tag{5.17}$$

$$\|\tilde{z}_1\|_{L^2(U;H^1(\Omega)} \leq C\left(\|\theta f\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right) \tag{5.18}$$

$$\|\tilde{z}_2\|_{L^2(U;H^1(\Omega)} \leq C\left(\|\theta f\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right) \tag{5.19}$$

where $\tilde{u} = (\tilde{u}_1,\tilde{u}_2)$ satisfies

$$\begin{cases} \frac{\partial \tilde{u}_1}{\partial t} + \frac{\partial \tilde{u}_1}{\partial a} - \Delta\tilde{u}_1 + \tilde{\mu}_1\tilde{u}_1 + \tilde{\mu}_2\tilde{u}_2 = 0 & in \quad Q, \\ \frac{\partial \tilde{u}_2}{\partial t} + \frac{\partial \tilde{u}_2}{\partial a} - \Delta\tilde{u}_2 + \tilde{\mu}_1\tilde{u}_2 + \tilde{\mu}_2\tilde{u}_1 = 0 & in \quad Q, \\ \tilde{u}_1(0,a,x) = \tilde{u}_2(0,a,x) = 0 & in \quad Q_A, \\ \tilde{u}_1 = \tilde{u}_2 = 0 & on \quad \Sigma, \\ \tilde{u}_1(t,0,x) = b_1\int_0^A \tilde{\beta}_1(\tilde{u}_1+\tilde{u}_2)da & \\ \qquad + b_2\int_0^A \tilde{\beta}_2(\tilde{u}_1-\tilde{u}_2)da & in \quad Q_T, \\ \tilde{u}_2(t,0,x) = b_1\int_0^A \tilde{\beta}_1(\tilde{u}_1+\tilde{u}_2)da & \\ \qquad + b_2\int_0^A \tilde{\beta}_2(\tilde{u}_2-\tilde{u}_1)da & in \quad Q_T. \end{cases} \tag{5.20}$$

Remark 5.3. *The proof of this theorem had been done in two steps in [11]. The first step proved the existence and the uniqueness of the optimal control using the function defined on $L^2(Q_\omega)$ by*

$$J_\varepsilon(v) = \tfrac{1}{2}\|v\|^2_{L^2(Q_\omega)} + \tfrac{1}{2\varepsilon}\int_{Q_A}\left(z_1^2(0,a,x)+z_2^2(0,a,x)\right)dQ_A, \tag{5.21}$$

where $z=(z_1,z_2)$ is solution of (5.14). The existence and uniqueness of the optimal control are obtained through continuity, convexity and coercivity of J_ε on $L^2(Q_\omega)$. The second step proved the inequalities (5.18), (5.19) *and the system* (5.20).

6. Detection of Pollution

Having established the existence of the optimal control, we now estimate the pollution terms in this section. For $i \in \{1,2\}$, multiply the first equation in (4.2) by $y_i^{\lambda_i}$ and integrate by parts over Q. One obtains thanks to regularities of F_i that

$$\int_Q \lambda_i \widehat{g}_i y_i^{\lambda_i} dQ = \int_Q \left(h\chi_O + w\chi_\omega\right) y_i^{\lambda_i} dQ \tag{6.1}$$

Using the expansion of the sentinel functional near to $0_{\mathbb{R}^4}$ on S, one derive that

$$S(\lambda,\tau) \approx S\left(0_{\mathbb{R}^4}\right) + \lambda_1 \frac{\partial S}{\partial \lambda_1}\left(0_{\mathbb{R}^4}\right) + \lambda_2 \frac{\partial S}{\partial \lambda_2}\left(0_{\mathbb{R}^4}\right) + \tau_1 \frac{\partial S}{\partial \tau_1}\left(0_{\mathbb{R}^4}\right) + \tau_2 \frac{\partial S}{\partial \tau_2}\left(0_{\mathbb{R}^4}\right). \tag{6.2}$$

From (3.3), one obtains

$$\lambda_1 \frac{\partial S}{\partial \lambda_1}\left(0_{\mathbb{R}^4}\right) + \lambda_2 \frac{\partial S}{\partial \lambda_2}\left(0_{\mathbb{R}^4}\right) \approx S(\lambda,\tau) - S\left(0_{\mathbb{R}^4}\right) \tag{6.3}$$

It follows that

$$\lambda_1 \frac{\partial S}{\partial \lambda_1}\left(0_{\mathbb{R}^4}\right) + \lambda_2 \frac{\partial S}{\partial \lambda_2}\left(0_{\mathbb{R}^4}\right) \approx \int_Q \left(h\chi_O + w\chi_\omega\right)\left[\left(y_1^{\lambda_1} - y_{01}\right) + \left(y_2^{\lambda_2} - y_{02}\right)\right] dQ \tag{6.4}$$

Hence, from (3.6) and (6.1), we have

$$\int_Q \left(\lambda_1 \widehat{g_1} y_1^{\lambda_1} + \lambda_2 \widehat{g_2} y_2^{\lambda_2}\right) dQ \approx \int_Q \left(h\chi_O + w\chi_\omega\right)\left[\left(y_1^{\lambda_1} - y_{01}\right) + \left(y_2^{\lambda_2} - y_{02}\right)\right] dQ \tag{6.5}$$

From (3.1), one has

$$\int_{Q_{O\cap\omega}} \left(\lambda_1 \widehat{g_1} m_1 + \lambda_2 \widehat{g_2} m_2\right) dQ \approx \int_{Q_{O\cap\omega}} (h+w)\left(m_1 + m_2 - y_{01} - y_{02}\right) dQ \tag{6.6}$$

where $Q_{O\cap\omega} = U \times O \cap \omega$. From system (6.1) and (6.6), we can estimate the parameters λ_1 and λ_2 and then pollution terms $\lambda_i \widehat{g_i}$ for $i \in \{1,2\}$.

Conclusion and Perspectives

In this paper, we have investigated parameters identification in a missing-data population dynamics model with a nonlinear birth process. This model describe the joint dynamics of two fish species living in a lake represented by a bounded domain Ω, each affected by pollution. This pollution was modeled by unknown parameters $\lambda_i g_i, i \in \{1,2\}$.

To estimate pollution terms, we introduced simultaneous sentinels with prescribed sensitivity and established a link between the simultaneous sentinels problem and a simultaneous null-controllability problem with constraints on the control. This equivalence made it possible to characterize the optimal control associated with the problem and to use it to derive estimates for the unknown pollution parameters.

These results extend the sentinel approach to a coupled population dynamics setting involving two time-like variables and incomplete data. They also provide a theoretical framework for detecting pollution effects independently of unknown perturbations in the initial data.

From a biological perspective, these results imply that, even when the exact initial distribution of each fish population is not fully known, it remains possible to detect and estimate the pollution pressure exerted on each species, provided that observations are available on the suitable observatory region O and that the control is acted on the non-empty sub-region ω of Ω with non-empty intersection with the observatory O. The nonlinear renewal condition, which accounts for density-dependent effects such as competition for breeding sites, does not hinder this identification.

Several extensions are worth considering. First the analysis is conducted at the first order by linearizing the renewal term; a full treatment going beyond the neighborhood of $F'(0)$ would

enhance the robustness of the identification. Second, the estimates obtained are qualitative, quantitative refinement including errors bounds, stability estimates regarding observation noise, and numerical simulations using realistic ecological data would be necessary to make the method operational for the actual monitoring of pollution in aquatic ecosystems.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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