Int. J. Anal. Appl. (2026), 24:371

International Journal of Analysis and Applications

# Novel Fixed Point Theorems in Extended Perturbed Supra Metric Spaces: Existence and Uniqueness for Fractional Epidemiological and Neural Network Models

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**Abstract.** In this paper, the authors propose extended perturbed supra metric spaces as a natural mathematical model to allow for the presence of errors and uncertainties in distances. In this general context, we prove some fixed point theorems which are extensions of the classical Banach contraction(B-cont ) principle and Kannan-type(Ka-ty ) and Chatterjea-type(Ch-ty ) contractions. The theoretical developments are supported by comprehensive examples and then applied to three fractional-order models, namely a Caputo fractional breast cancer dynamics model, a nonlinear Caputo fractional boundary value problem, and an ABC fractional 3D Hopfield neural network model. The examples illustrate the fact that in complex problems involving measurement uncertainties, solutions exist and are unique. The outcome validates that the proposed framework provides a sufficient mathematical platform to analyze the nonlinear issues related to fractional differential equations and neurocomputing.

## 1. Introduction

Fixed point(FP) theory is one of the most important and widely applied theories in nonlinear analysis, with many applications and insights that span an extensive area of mathematics. It is based on the well-known contraction principle of Banach [1] which has been the main cornerstone of uniqueness and existence of solutions in many mathematical contexts. Fixed point theory

Received: Aug. 11, 2026.

2020 *Mathematics Subject Classification.* 54E50; 47H09; 47H10.

*Key words and phrases.* perturbed metric space; perturbed supra metric space; extended perturbed supra metric space; fractional calculus; fixed point.

https://doi.org/10.28924/2291-8639-24-2026-371
ISSN: 2291-8639

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has been successfully applied to prove local solvability of evolution equations, solve integral and matrix equations, and tackle various other mathematical problems [2–5]. Two principal research programs have shaped the generalization of Banach's classical result. The first program introduces mappings satisfying generalized contractive conditions that differ substantially from the usual Lipschitz inequality. Some of the most significant developments in this direction are Boyd-Wong contractions [6], extensions of Boyd-Wong contractions such as the Ćirić contraction [7], Ka-ty contraction [8] and Ch-ty contraction [9] and other novel formulations [10–15]. The second research direction broadens the geometric framework by considering contraction mappings defined on spaces endowed with generalized distance structures, including b-metric spaces [16], partial metric spaces [17], and generalized metric spaces [18].

More recent contributions to this field include the generalized Wardowski-type quasi-contraction in b-metric spaces proposed by Iqbal et al. [19], the novel interpolative convex contraction introduced by Hussain [20] within F-complete F-metric spaces for solving fractional differential equations, and the strong b-metric-like spaces developed by Rasham [21] for analyzing systems of fractional differential equations and nonlinear Volterra-type integral equations.

The concept of supra metric spaces was first introduced by Berzig [22], who investigated fundamental topological properties and established fixed point results. Karapınar and Cvetković [23] demonstrated that continuity and convergence are preserved between supra metric spaces and their associated metric counterparts, and they further proved Cauchy sequence and fixed point theorems. Within the framework of supra metric spaces, Seher Sultan [24] established existence and uniqueness results for fixed points of $\psi$-$\xi$-weak contractions. Panda, Agarwal, and Karapınar [25] introduced extended supra metric spaces and derived corresponding fixed point theorems. In a related development, Berzig [26] proposed generalized supra metric spaces and proved fixed point results within this broader setting.

Jleli and Samet [27] were the first to study perturbed metric spaces and fixed point theorems. Moussaoui and Taş [29] introduced perturbed parametric metric spaces and obtained fixed point theorem. Moussaoui [30] formulated perturbed extended b-metric and perturbed b-metric spaces, proving Banach-type fixed point theorems. Within a perturbed metric space, Alsahli, Shahi, and Karapınar [31] introduced a new contraction principle in the form of $S_\tau$-contractions and derived some corresponding fixed point theorems. The concept of $C^*$-algebra valued perturbed metric spaces was introduced by Tijjani Ahmad et al. [32] and fixed point results were established in this context.

Suppose that $(\mathcal{W}, p)$ is a complete supra metric space and that $S : \mathcal{W} \to \mathcal{W}$ is a self-mapping such that

$$p(S\delta, S\nu) \leq \hbar\, p(\delta, \nu), \quad \delta, \nu \in \mathcal{W}, \tag{1.1}$$

with $\hbar \in (0, 1)$, the mapping $S$ admits a unique fixed point(Ufp). However, in real-life scenarios, the distance measurements are typically noisy or disturbed. This leaves us with a natural question:

What if the condition (1.1) is replaced by the perturbed condition,

$$A(S\delta, S\nu) \leq \hbar\, A(\delta, \nu), \quad \delta, \nu \in \mathcal{W},$$

where $A(\delta, \nu)$ denotes an experimentally observed approximation of $p(\delta, \nu)$, what conclusions can still be drawn regarding the existence of FPs? Note that $A$ may not be a supra metric on $\mathcal{W}$ (in particular, it may not be the case that the union of all of $A$ is a metric on $\mathcal{W}$). To address this key issue, we introduce ex-per-s-ms 's as an original mathematical structure. Our approach offers a unified treatment that accounts for observational errors while retaining the topological essentials required for fixed point theory. The main findings of this paper are as follows:

- A rigorous definition of ex-per-s-ms 's.
- A generalized B-cont principle within this new setting.
- Extensions of Ka-ty and Ch-ty contractions in the context of ex-per-s-ms 's.
- Illustrative examples that support and validate the theoretical findings.
- Applications to three fractional-order systems: a Caputo fractional breast cancer dynamics model, a nonlinear Caputo fractional boundary value problem, and an ABC-fractional 3D Hopfield neural network model.

This paper has the following structure: Section 2 introduces the basics of ex-per-s-ms 's. The general fixed point theorems with their proofs and examples are given in section 3. The theoretical framework is used in Section 4 to study a fractional breast cancer model introduced by Caputo. A nonlinear Caputo fractional boundary value problem is studied in section 5. In Section 6, an ABC-fractional 3D Hopfield neural network model is studied. Finally, Section 7 concludes the paper with a discussion of the findings and suggestions for future research directions.

## 2. Preliminaries

The following definition of perturbed metric spaces was introduced by Jleli and Samet [27]:

**Definition 2.1.** *[27] Suppose $A, Q : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ are two given mappings. Then $A$ is called a perturbed metric on $\mathcal{W}$ with respect to $Q$ if the difference*

$$A - Q : \mathcal{W} \times \mathcal{W} \to \mathbb{R}, \qquad (\delta, \vartheta) \mapsto A(\delta, \vartheta) - Q(\delta, \vartheta)$$

*satisfies the standard metric axioms on $\mathcal{W}$. That is, for all $\delta, \vartheta, \mathsf{x} \in \mathcal{W}$,*

*(i)* $(A - Q)(\delta, \vartheta) \geq 0$;
*(ii)* $(A - Q)(\delta, \vartheta) = 0$ *if and only if* $\delta = \vartheta$;
*(iii)* $(A - Q)(\delta, \vartheta) = (A - Q)(\vartheta, \delta)$;
*(iv)* $(A - Q)(\delta, \vartheta) \leq (A - Q)(\delta, \mathsf{x}) + (A - Q)(\mathsf{x}, \vartheta)$.

*The mapping $Q$ is referred to as a perturbed map, $p = A - Q$ is the associated metric, and the triple $(\mathcal{W}, A, Q)$ is called a perturbed metric space.*

**Definition 2.2.** *[28] Given two mappings $A, Q : \mathcal{W} \times \mathcal{W} \to [0, \infty)$, the mapping $A$ is a perturbed supra metric on $\mathcal{W}$ relative to $Q$ provided that the difference*

$$A - Q : \mathcal{W} \times \mathcal{W} \to \mathbb{R}, \qquad (\delta, \vartheta) \mapsto A(\delta, \vartheta) - Q(\delta, \vartheta)$$

*meets the supra metric requirements on $\mathcal{W}$. Specifically, for any $\delta, \vartheta, \mathsf{x} \in \mathcal{W}$,*

*(i)* $(A - Q)(\delta, \vartheta) \geq 0$;
*(ii)* $(A - Q)(\delta, \vartheta) = 0$ *precisely when* $\delta = \vartheta$;
*(iii)* $(A - Q)(\delta, \vartheta) = (A - Q)(\vartheta, \delta)$;
*(iv)* $(A - Q)(\delta, \vartheta) \leq (A - Q)(\delta, \mathsf{x}) + (A - Q)(\mathsf{x}, \vartheta) + \rho(A - Q)(\delta, \mathsf{x})(A - Q)(\mathsf{x}, \vartheta)$ *for some* $\rho \in [0, \infty)$.

*Here, $Q$ is designated as a perturbed map, $p = A - Q$ as a supra metric, and $(\mathcal{W}, A, Q)$ as a perturbed supra metric space.*

We introduce the extended perturbed supra metric space (ex-per-s-ms ) as follows:

**Definition 2.3.** *Let $A, Q : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ be two arbitrary mappings. Then $A$ is called an extended perturbed supra metric on $\mathcal{W}$ with respect to $Q$ provided that*

$$A - Q : \mathcal{W} \times \mathcal{W} \to \mathbb{R}, \qquad (\delta, \vartheta) \mapsto A(\delta, \vartheta) - Q(\delta, \vartheta)$$

*is an ex-su-metr on $\mathcal{W}$. Equivalently, for all $\delta, \vartheta, \mathsf{x} \in \mathcal{W}$,*

*(i)* $(A - Q)(\delta, \vartheta) \geq 0$;
*(ii)* $(A - Q)(\delta, \vartheta) = 0 \iff \delta = \vartheta$;
*(iii)* $(A - Q)(\delta, \vartheta) = (A - Q)(\vartheta, \delta)$;
*(iv) there exists a function $\eta : \mathcal{W} \times \mathcal{W} \to [1, +\infty)$ such that*

$$(A - Q)(\delta, \vartheta) \leq (A - Q)(\delta, \mathsf{x}) + (A - Q)(\mathsf{x}, \vartheta) + \eta(\delta, \vartheta)(A - Q)(\delta, \mathsf{x})(A - Q)(\mathsf{x}, \vartheta).$$

*We call $Q$ a perturbed map, $p = A - Q$ an ex-su-metr , and $(\mathcal{W}, A, Q)$ an extended perturbed supra metric space (ex-per-s-ms ).*

**Remark 2.1.** *Note that if $\eta(\delta, \vartheta) \equiv \rho$ (a constant) for all $\delta, \vartheta \in \mathcal{W}$, then the ex-per-s-ms reduces to the standard perturbed supra metric space with parameter $\rho$. Thus, our framework generalizes the earlier notion [28].*

Note that an extended perturbed supra metric on $\mathcal{W}$ is not guaranteed to be an ex-su-metr on $\mathcal{W}$. We now present some examples that clarify this observation.

**Example 2.1.** *Let $\mathcal{W} = \mathbb{N}_0 = \{0, 1, 2, \dots\}$. Define $A, Q : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ by*

$$A(\delta, \vartheta) = (\delta - \vartheta)^2 + \delta\vartheta^3, \qquad Q(\delta, \vartheta) = \delta\vartheta^3, \qquad \delta, \vartheta \in \mathcal{W}.$$

*Then $A$ is an extended perturbed supra metric on $\mathcal{W}$ under the perturbation $Q$. Indeed,*

$$p(\delta, \vartheta) := (A - Q)(\delta, \vartheta) = (\delta - \vartheta)^2$$

*satisfies axioms (i)–(iii) of Definition 2.3 trivially. For axiom (iv), fix $\delta, \vartheta, y \in \mathcal{W}$ and set $a = \delta - y,\ b = y - \vartheta$, so that $a, b \in \mathbb{Z}$ and $p(\delta, \vartheta) = (a+b)^2 = a^2 + b^2 + 2ab$.*

*If $ab \leq 0$ the inequality below is immediate. If $ab > 0$ then $|a|, |b| \geq 1$ (as $a, b$ are nonzero integers), so $|a| \leq a^2$ and $|b| \leq b^2$, giving $2ab \leq 2a^2b^2$; moreover $ab > 0$ forces $\delta \neq \vartheta$, hence $\delta + \vartheta \geq 1$ and $e^{\delta+\vartheta} \geq e > 2$. In either case*

$$2ab \ \leq\ e^{\delta+\vartheta} a^2 b^2 = \eta(\delta, \vartheta)\, p(\delta, y)\, p(y, \vartheta),$$

*so that*

$$p(\delta, \vartheta) \ \leq\ p(\delta, y) + p(y, \vartheta) + \eta(\delta, \vartheta)\, p(\delta, y)\, p(y, \vartheta), \qquad \eta(\delta, \vartheta) = e^{\delta+\vartheta} \in [1, \infty).$$

*Hence $p$ is an ex-su-metr on $\mathcal{W}$ with control function $\eta(\delta, \vartheta) = e^{\delta+\vartheta}$. However, $A$ itself is* not *an ex-su-metr on $\mathcal{W}$, since*

$$A(1,1) = (1-1)^2 + 1 \cdot 1^3 = 1 \neq 0,$$

*so the identity axiom fails for $A$. Thus $(\mathcal{W}, A, Q)$ is a genuine example of an ex-per-s-ms that is not itself an ex-su-metr space.*

We now turn our attention to several topological concepts associated with ex-per-s-ms .

**Definition 2.4.** *Let $(\mathcal{W}, A, Q)$ be an ex-per-s-ms , $\{\delta_r\}$ a sequence in $\mathcal{W}$, and $S : \mathcal{W} \to \mathcal{W}$ a self-mapping.*

- *(i) The sequence $\{\delta_r\}$ is termed perturbed convergent in $(\mathcal{W}, A, Q)$ if it converges in the ex-su-metr space $(\mathcal{W}, p)$, where $p = A - Q$ denotes the underlying ex-su-metr .*
- *(ii) A sequence $\{\delta_r\}$ in $(\mathcal{W}, A, Q)$ is called a perturbed Cauchy sequence(per-Cau-seq ) if it constitutes a Cauchy sequence with respect to the ex-su-metr $p$.*
- *(iii) The ex-per-s-ms $(\mathcal{W}, A, Q)$ is said to be complete if $(\mathcal{W}, p)$ is complete as an ex-su-metr space. Equivalently, completeness holds if every per-Cau-seq in $(\mathcal{W}, A, Q)$ converges to a limit within $(\mathcal{W}, A, Q)$.*
- *(iv) The mapping $S$ is referred to as perturbed continuous(per-cont ) if it is continuous with respect to the ex-su-metr $p$.*

## 3. Main Results

This section develops an extension of B-cont theorem applicable to ex-per-s-ms .

**Theorem 3.1.** *Let $(\mathcal{W}, A, Q)$ be a complete ex-per-s-ms and $S : \mathcal{W} \to \mathcal{W}$ be a given mapping. Suppose that:*

- *(i) $S$ is per-cont ;*
- *(ii) there exists $\hbar \in (0, 1)$ such that*

$$A(S\delta, S\vartheta) \leq \hbar A(\delta, \vartheta) \tag{3.1}$$

*for all $\delta, \vartheta \in \mathcal{W}$.*

*Then $S$ has exactly one fixed point.*

*Proof.* Let fix an element $\delta_0 \in \mathcal{W}$. Define the Picard sequence $\{\delta_r\} \subset \mathcal{W}$ by

$$\delta_{r+1} = S\delta_r, \quad r \in \mathbb{N}.$$

Taking $(\delta, \vartheta) = (\delta_0, \delta_1)$, we obtain

$$A(S\delta_0, S\delta_1) \leq \hbar A(\delta_0, \delta_1),$$

that is,

$$A(\delta_1, \delta_2) \leq \hbar A(\delta_0, \delta_1).$$

Similarly, taking $(\delta, \vartheta) = (\delta_1, \delta_2)$, we obtain

$$A(\delta_2, \delta_3) \leq \hbar A(\delta_1, \delta_2),$$

which implies that

$$A(\delta_2, \delta_3) \leq \hbar^2 A(\delta_0, \delta_1).$$

Therefore, we obtain

$$A(\delta_r, \delta_{r+1}) \leq \hbar^r \tau, \quad r \in \mathbb{N},$$

where $\tau = A(\delta_0, \delta_1)$. Let $p = A - Q$ be the ex-su-metr . Therefore, we deduce that

$$p(\delta_r, \delta_{r+1}) + Q(\delta_r, \delta_{r+1}) \leq \hbar^r \tau, \quad r \in \mathbb{N}.$$

Since $p(\delta_r, \delta_{r+1}) \leq p(\delta_r, \delta_{r+1}) + Q(\delta_r, \delta_{r+1})$, it holds that

$$p(\delta_r, \delta_{r+1}) \leq \hbar^r \tau, \quad r \in \mathbb{N}.$$

For any $m > r$,

$$\begin{aligned} p(\delta_r, \delta_m) &\leq p(\delta_r, \delta_{r+1}) + p(\delta_{r+1}, \delta_m) + \eta(\delta_r, \delta_m)\, p(\delta_r, \delta_{r+1})\, p(\delta_{r+1}, \delta_m) \\ &\leq \hbar^r \tau + p(\delta_{r+1}, \delta_m) + \eta(\delta_r, \delta_m)\hbar^r \tau\, p(\delta_{r+1}, \delta_m) \\ &= \hbar^r \tau + (1 + \eta(\delta_r, \delta_m)\hbar^r \tau)\, p(\delta_{r+1}, \delta_m). \end{aligned}$$

Consider,

$$\begin{aligned} p(\delta_{r+1}, \delta_m) &\leq p(\delta_{r+1}, \delta_{r+2}) + p(\delta_{r+2}, \delta_m) + \eta(\delta_{r+1}, \delta_m)\, p(\delta_{r+1}, \delta_{r+2})\, p(\delta_{r+2}, \delta_m) \\ &\leq \hbar^{r+1} \tau + p(\delta_{r+2}, \delta_m) + \eta(\delta_{r+1}, \delta_m)\hbar^{r+1} \tau\, p(\delta_{r+2}, \delta_m) \\ &= \hbar^{r+1} \tau + (1 + \eta(\delta_{r+1}, \delta_m)\hbar^{r+1} \tau)\, p(\delta_{r+2}, \delta_m). \end{aligned}$$

Therefore,

$$\begin{aligned} p(\delta_r, \delta_m) &\leq \hbar^r \tau + (1 + \eta(\delta_r, \delta_m)\hbar^r \tau) \left[\hbar^{r+1} \tau + (1 + \eta(\delta_{r+1}, \delta_m)\hbar^{r+1} \tau)\, p(\delta_{r+2}, \delta_m)\right] \\ &= \hbar^r \tau + (1 + \eta(\delta_r, \delta_m)\hbar^r \tau)\hbar^{r+1} \tau \\ &+ (1 + \eta(\delta_r, \delta_m)\hbar^r \tau)(1 + \eta(\delta_{r+1}, \delta_m)\hbar^{r+1} \tau)\, p(\delta_{r+2}, \delta_m). \end{aligned}$$

Continue this process,

$$\begin{aligned} p(\delta_{\mathrm{r}}, \delta_m) \leq & \hbar^{\mathrm{r}}\tau + (1+\eta(\delta_{\mathrm{r}}, \delta_m)\hbar^{\mathrm{r}}\tau)\hbar^{\mathrm{r}+1}\tau \\ & + (1+\eta(\delta_{\mathrm{r}}, \delta_m)\hbar^{\mathrm{r}}\tau)(1+\eta(\delta_{\mathrm{r}+1}, \delta_m)\hbar^{\mathrm{r}+1}\tau)\hbar^{\mathrm{r}+2}\tau \\ & + \cdots + (1+\eta(\delta_{\mathrm{r}}, \delta_m)\hbar^{\mathrm{r}}\tau)(1+\eta(\delta_{\mathrm{r}+1}, \delta_m)\hbar^{\mathrm{r}+1}\tau) \\ & \cdots (1+\eta(\delta_{m-2}, \delta_m)\hbar^{m-2}\tau)\hbar^{m-1}\tau + (1+\eta(\delta_{\mathrm{r}}, \delta_m)\hbar^{\mathrm{r}}\tau)(1+\eta(\delta_{\mathrm{r}+1}, \delta_m)\hbar^{\mathrm{r}+1}\tau) \\ & \cdots (1+\eta(\delta_{m-1}, \delta_m)\hbar^{m-1}\tau)\, p(\delta_m, \delta_m). \end{aligned}$$

Since $p(\delta_m, \delta_m) = 0$, we get

$$p(\delta_{\mathrm{r}}, \delta_m) \leq \sum_{\mathsf{k}=\mathrm{r}}^{m-1}\left[\left(\prod_{i=\mathrm{r}}^{\mathsf{k}-1}(1+\eta(\delta_i, \delta_m)\hbar^i\tau)\right)\hbar^{\mathsf{k}}\tau\right].$$

Since $\hbar \in (0,1)$, the infinite product $\prod_{i=\mathrm{r}}^{\infty}(1+\eta(\delta_i, \delta_m)\hbar^i\tau)$ converges, and the series $\sum_{\mathsf{k}=\mathrm{r}}^{\infty}\hbar^{\mathsf{k}}\tau$ converges. Therefore, $\{\delta_{\mathrm{r}}\}$ is a Cauchy sequence in $(\mathcal{W}, p)$, that is, $\{\delta_{\mathrm{r}}\}$ is a per-Cau-seq in $(\mathcal{W}, A, Q)$. By the completeness of $(\mathcal{W}, A, Q)$, we deduce that there exists $\delta \in \mathcal{W}$ such that

$$\lim_{\mathrm{r}\to\infty} p(\delta_{\mathrm{r}}, \delta) = 0.$$

We now show that $\delta$ is a fixed point of $S$. Since $S$ is a per-cont map,

$$\lim_{\mathrm{r}\to\infty} p(S\delta_{\mathrm{r}}, S\delta) = 0,$$

that is,

$$\lim_{\mathrm{r}\to\infty} p(\delta_{\mathrm{r}+1}, S\delta) = 0.$$

Hence, we obtain $\delta = S\delta$. Suppose that $\delta, \vartheta \in \mathcal{W}$ are two different fixed points of $S$. By (3.1), we get

$$A(\delta, \vartheta) = A(S\delta, S\vartheta) \leq \hbar A(\delta, \vartheta),$$

which yields

$$p(\delta, \vartheta) + Q(\delta, \vartheta) \leq \hbar(p(\delta, \vartheta) + Q(\delta, \vartheta)).$$

Since $\delta \neq \vartheta$, then $p(\delta, \vartheta) + Q(\delta, \vartheta) \neq 0$, which contradicts the condition $\hbar \in (0,1)$. Therefore, $\delta$ serves as the Ufpof $S$. □

Theorem 3.1 reduces to Banach's classical fixed point theorem under suitable conditions.

**Corollary 3.1.** *Let $(\mathcal{W}, p)$ be a complete ex-su-metr space and $S : \mathcal{W} \to \mathcal{W}$ be a given mapping. If there exists $\hbar \in (0,1)$ such that*

$$p(S\delta, S\vartheta) \leq \hbar p(\delta, \vartheta) \tag{3.2}$$

*for all $\delta, \vartheta \in \mathcal{W}$, then $S$ admits exactly one fixed point.*

*Proof.* Taking $A = p$ and $Q \equiv 0$, we obtain an ex-per-s-ms $(\mathcal{W}, A, Q)$. The inequality (3.2) implies that $S$ is continuous relative to $p$ and satisfies (3.1). Hence, Theorem 3.1 guarantees a Ufp. □

**Example 3.1.** *Let $\mathcal{W} = [0, \infty)$ and define*

$$Q(\delta, \vartheta) = \delta\vartheta, \qquad A(\delta, \vartheta) = e^{|\delta - \vartheta|} - 1 + \delta\vartheta, \qquad \delta, \vartheta \in [0, \infty),$$

*with $\eta(\delta, \vartheta) = e^{\delta+\vartheta}$. Then $(\mathcal{W}, A, Q)$ is a complete ex-per-s-ms .*

*Define $S : [0, \infty) \to [0, \infty)$ by*

$$S(\delta) = c\delta, \qquad 0 < c < 1.$$

*Then for all $\delta, \vartheta \in [0, \infty)$,*

$$\begin{aligned} A(S\delta, S\vartheta) &= e^{|c\delta - c\vartheta|} - 1 + c^2\delta\vartheta \\ &\leq c\left(e^{|\delta-\vartheta|} - 1 + \delta\vartheta\right) \\ &= cA(\delta, \vartheta) \end{aligned}$$

*Therefore,*

$$A(S\delta, S\vartheta) \leq c\, A(\delta, \vartheta), \qquad \forall\, \delta, \vartheta \in [0, \infty).$$

*Hence $\hbar = c \in (0, 1)$ and $S$ is per-cont . Since all the assumptions of Theorem 3.1 are fulfilled, it follows that $S$ admits a Ufp.*

Kannan [8] proposed the class of Ka-ty contractions and proved FP theorems for such mappings. In the same spirit, we establish the following result.

**Theorem 3.2.** *Let $(\mathcal{W}, A, Q)$ be a complete ex-per-s-ms and $S : \mathcal{W} \to \mathcal{W}$ be an arbitrary mapping. If the following hypotheses are fulfilled:*

*(i) $S$ is per-cont ;*

*(ii) there exists $\hbar \in (0, 1/2)$ satisfying*

$$A(S\delta, S\vartheta) \leq \hbar(A(\delta, S\delta) + A(\vartheta, S\vartheta)) \tag{3.3}$$

*for every $\delta, \vartheta \in \mathcal{W}$.*

*Then $S$ admits a Ufp.*

*Proof.* Let $\delta_0 \in \mathcal{W}$ be arbitrary. Define the sequence $\{\delta_r\}$ by $\delta_{r+1} = S\delta_r$ for $r = 0, 1, 2, \ldots$. Let $d_r = A(\delta_r, \delta_{r+1})$. Applying (3.3) with $\delta = \delta_r$, $\vartheta = \delta_{r-1}$, we get

$$\begin{aligned} A(\delta_{r+1}, \delta_r) &= A(S\delta_r, S\delta_{r-1}) \\ &\leq \hbar\left(A(\delta_r, S\delta_r) + A(\delta_{r-1}, S\delta_{r-1})\right) \\ &= \hbar\left(A(\delta_r, \delta_{r+1}) + A(\delta_{r-1}, \delta_r)\right) \\ &= \hbar(d_r + d_{r-1}). \end{aligned}$$

Thus,

$$d_r \leq \hbar(d_r + d_{r-1}) \quad \Rightarrow \quad (1 - \hbar)d_r \leq \hbar d_{r-1} \quad \Rightarrow \quad d_r \leq \frac{\hbar}{1 - \hbar} d_{r-1}.$$

Let $\mathsf{k} = \frac{\hbar}{1-\hbar}$. Since $\hbar \in (0, 1/2)$, we have $\mathsf{k} \in (0, 1)$. By induction,

$$d_r \leq \mathsf{k}^r d_0.$$

Let $p = A - Q$ be the ex-su-metr . Since $Q(\delta_{\mathrm{r}}, \delta_{\mathrm{r}+1}) \geq 0$, we have

$$p(\delta_{\mathrm{r}}, \delta_{\mathrm{r}+1}) \leq A(\delta_{\mathrm{r}}, \delta_{\mathrm{r}+1}) \leq \mathsf{k}^{\mathrm{r}} d_0.$$

Now, for $m > \mathrm{r}$,

$$\begin{aligned} p(\delta_{\mathrm{r}}, \delta_m) &\leq p(\delta_{\mathrm{r}}, \delta_{\mathrm{r}+1}) + p(\delta_{\mathrm{r}+1}, \delta_m) + \eta(\delta_{\mathrm{r}}, \delta_m)\, p(\delta_{\mathrm{r}}, \delta_{\mathrm{r}+1})\, p(\delta_{\mathrm{r}+1}, \delta_m) \\ &\leq \mathsf{k}^{\mathrm{r}} d_0 + p(\delta_{\mathrm{r}+1}, \delta_m) + \eta(\delta_{\mathrm{r}}, \delta_m) \mathsf{k}^{\mathrm{r}} d_0\, p(\delta_{\mathrm{r}+1}, \delta_m) \\ &= \mathsf{k}^{\mathrm{r}} d_0 + (1 + \eta(\delta_{\mathrm{r}}, \delta_m) \mathsf{k}^{\mathrm{r}} d_0)\, p(\delta_{\mathrm{r}+1}, \delta_m). \end{aligned}$$

Similarly,

$$\begin{aligned} p(\delta_{\mathrm{r}+1}, \delta_m) &\leq p(\delta_{\mathrm{r}+1}, \delta_{\mathrm{r}+2}) + p(\delta_{\mathrm{r}+2}, \delta_m) + \eta(\delta_{\mathrm{r}+1}, \delta_m)\, p(\delta_{\mathrm{r}+1}, \delta_{\mathrm{r}+2})\, p(\delta_{\mathrm{r}+2}, \delta_m) \\ &\leq \mathsf{k}^{\mathrm{r}+1} d_0 + p(\delta_{\mathrm{r}+2}, \delta_m) + \eta(\delta_{\mathrm{r}+1}, \delta_m) \mathsf{k}^{\mathrm{r}+1} d_0\, p(\delta_{\mathrm{r}+2}, \delta_m) \\ &= \mathsf{k}^{\mathrm{r}+1} d_0 + (1 + \eta(\delta_{\mathrm{r}+1}, \delta_m) \mathsf{k}^{\mathrm{r}+1} d_0)\, p(\delta_{\mathrm{r}+2}, \delta_m). \end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned} p(\delta_{\mathrm{r}}, \delta_m) \leq & \mathsf{k}^{\mathrm{r}} d_0 + (1 + \eta(\delta_{\mathrm{r}}, \delta_m) \mathsf{k}^{\mathrm{r}} d_0) \mathsf{k}^{\mathrm{r}+1} d_0 \\ & + (1 + \eta(\delta_{\mathrm{r}}, \delta_m) \mathsf{k}^{\mathrm{r}} d_0)(1 + \eta(\delta_{\mathrm{r}+1}, \delta_m) \mathsf{k}^{\mathrm{r}+1} d_0) \mathsf{k}^{\mathrm{r}+2} d_0 \\ & + \cdots \\ & + (1 + \eta(\delta_{\mathrm{r}}, \delta_m) \mathsf{k}^{\mathrm{r}} d_0)(1 + \eta(\delta_{\mathrm{r}+1}, \delta_m) \mathsf{k}^{\mathrm{r}+1} d_0) \\ & \cdots (1 + \eta(\delta_{m-2}, \delta_m) \mathsf{k}^{m-2} d_0) \mathsf{k}^{m-1} d_0 \\ & + (1 + \eta(\delta_{\mathrm{r}}, \delta_m) \mathsf{k}^{\mathrm{r}} d_0)(1 + \eta(\delta_{\mathrm{r}+1}, \delta_m) \mathsf{k}^{\mathrm{r}+1} d_0) \\ & \cdots (1 + \eta(\delta_{m-1}, \delta_m) \mathsf{k}^{m-1} d_0)\, p(\delta_m, \delta_m). \end{aligned}$$

Since $p(\delta_m, \delta_m) = 0$, we have

$$p(\delta_{\mathrm{r}}, \delta_m) \leq \sum_{j=\mathrm{r}}^{m-1} \left[ \left( \prod_{i=\mathrm{r}}^{j-1} (1 + \eta(\delta_i, \delta_m) \mathsf{k}^i d_0) \right) \mathsf{k}^j d_0 \right].$$

Since $\mathsf{k} \in (0,1)$, the infinite product $\prod_{i=\mathrm{r}}^{\infty} (1 + \eta(\delta_i, \delta_m) \mathsf{k}^i d_0)$ converges, and the series $\sum_{j=\mathrm{r}}^{\infty} \mathsf{k}^j d_0$ converges. Therefore, $p(\delta_{\mathrm{r}}, \delta_m) \to 0$ as $\mathrm{r}, m \to \infty$, proving that $\{\delta_{\mathrm{r}}\}$ is a Cauchy sequence in $(\mathcal{W}, p)$. By the completeness of $(\mathcal{W}, A, Q)$, there exists $\delta^* \in \mathcal{W}$ such that

$$\lim_{\mathrm{r} \to \infty} p(\delta_{\mathrm{r}}, \delta^*) = 0.$$

By the per-cont of $S$, we have

$$S(\delta^*) = S\left( \lim_{\mathrm{r} \to \infty} \delta_{\mathrm{r}} \right) = \lim_{\mathrm{r} \to \infty} S(\delta_{\mathrm{r}}) = \lim_{\mathrm{r} \to \infty} \delta_{\mathrm{r}+1} = \delta^*.$$

So, $\delta^*$ is a fixed point of $S$. To show uniqueness, suppose $\vartheta^*$ is another fixed point. Then

$$\begin{aligned} A(\delta^*, \vartheta^*) &= A(S\delta^*, S\vartheta^*) \\ &\leq \hbar \left( A(\delta^*, S\delta^*) + A(\vartheta^*, S\vartheta^*) \right) \\ &= \hbar \left( A(\delta^*, \delta^*) + A(\vartheta^*, \vartheta^*) \right) = 0. \end{aligned}$$

Hence, $A(\delta^*, \vartheta^*) = 0$, and since $A$ is an extended perturbed supra metric, it follows that $\delta^* = \vartheta^*$. Therefore, $S$ has a Ufp. □

**Example 3.2.** *Let $\mathcal{W} = [0, \infty)$ and define*

$$Q(\delta, \vartheta) = \delta\vartheta, \qquad A(\delta, \vartheta) = e^{|\delta - \vartheta|} - 1 + \delta\vartheta, \qquad \delta, \vartheta \in [0, \infty).$$

*Then $p = A - Q = e^{|\delta-\vartheta|} - 1$ satisfies the ex-su-metr space axioms with $\eta(\delta, \vartheta) = e^{\delta+\vartheta}$, so $([0, \infty), A, Q)$ is a complete ex-per-s-ms . Define the map $S : [0, \infty) \to [0, \infty)$ by*

$$S(\delta) = \frac{\delta}{4}, \qquad \delta \in [0, \infty).$$

*For any $\delta, \vartheta \in [0, \infty)$, we have*

$$A(S\delta, S\vartheta) = e^{|\frac{\delta}{4} - \frac{\vartheta}{4}|} - 1 + \frac{\delta\vartheta}{16} = e^{\frac{|\delta-\vartheta|}{4}} - 1 + \frac{\delta\vartheta}{16}.$$

*Furthermore,*

$$A(\delta, S\delta) = e^{|\delta - \frac{\delta}{4}|} - 1 + \delta \cdot \frac{\delta}{4} = e^{\frac{3\delta}{4}} - 1 + \frac{\delta^2}{4}, \qquad A(\vartheta, S\vartheta) = e^{\frac{3\vartheta}{4}} - 1 + \frac{\vartheta^2}{4}.$$

*Without loss of generality assume $\delta \geq \vartheta \geq 0$, so $|\delta - \vartheta| \leq \delta$ and hence*

$$e^{\frac{|\delta-\vartheta|}{4}} - 1 \leq e^{\frac{\delta}{4}} - 1.$$

*Applying the inequality $e^{cy} - 1 \leq c(e^y - 1)$ with $c = \frac{1}{3}$ and $y = \frac{3\delta}{4}$ (so that $cy = \frac{\delta}{4}$),*

$$e^{\frac{\delta}{4}} - 1 \leq \tfrac{1}{3}\left(e^{\frac{3\delta}{4}} - 1\right) \leq \tfrac{1}{3}\left[\left(e^{\frac{3\delta}{4}} - 1\right) + \left(e^{\frac{3\vartheta}{4}} - 1\right)\right].$$

*By the AM–GM inequality, $\delta\vartheta \leq \frac{1}{2}(\delta^2 + \vartheta^2)$, so*

$$\frac{\delta\vartheta}{16} \leq \tfrac{1}{8}\left(\frac{\delta^2}{4} + \frac{\vartheta^2}{4}\right) \leq \tfrac{1}{3}\left(\frac{\delta^2}{4} + \frac{\vartheta^2}{4}\right).$$

*Adding the two bounds,*

$$A(S\delta, S\vartheta) \leq \tfrac{1}{3}\left[\left(e^{\frac{3\delta}{4}} - 1 + \frac{\delta^2}{4}\right) + \left(e^{\frac{3\vartheta}{4}} - 1 + \frac{\vartheta^2}{4}\right)\right] = \tfrac{1}{3}\Big(A(\delta, S\delta) + A(\vartheta, S\vartheta)\Big).$$

*Hence there exists $\hbar = \frac{1}{3} < \frac{1}{2}$ such that*

$$A(S\delta, S\vartheta) \leq \hbar\Big(A(\delta, S\delta) + A(\vartheta, S\vartheta)\Big), \qquad \forall\, \delta, \vartheta \in [0, \infty).$$

*Since all the hypotheses of Theorem 3.2 are fulfilled, it follows that S possesses a Ufp.*

Chatterjea [9] pioneered the notion of Chatterjea contraction and established corresponding FP results. Inspired by this foundational work, we present the following theorem.

**Theorem 3.3.** *Let $(\mathcal{W}, A, Q)$ be a complete ex-per-s-ms and $S : \mathcal{W} \to \mathcal{W}$ be an arbitrary mapping. If the following hypotheses hold:*

*(i) $S$ is per-cont ;*

*(ii) there exists $\mathsf{a} \in (0, \frac{1}{2})$ satisfying*

$$A(S\delta, S\vartheta) \leq \mathsf{a}(A(\delta, S\vartheta) + A(\vartheta, S\delta)) \tag{3.4}$$

*for every $\delta, \vartheta \in \mathcal{W}$.*

*Then $S$ admits a Ufp.*

*Proof.* Let $\delta_0 \in \mathcal{W}$ be arbitrary. Define the sequence $\{\delta_r\}$ by $\delta_{r+1} = S\delta_r$ for $r = 0, 1, 2, \ldots$. Let $d_r = A(\delta_r, \delta_{r+1})$. Applying (3.4) with $\delta = \delta_r$, $\vartheta = \delta_{r-1}$, we get

$$\begin{aligned} A(\delta_{r+1}, \delta_r) &= A(S\delta_r, S\delta_{r-1}) \\ &\leq \mathsf{a}\,(A(\delta_r, S\delta_{r-1}) + A(\delta_{r-1}, S\delta_r)) \\ &= \mathsf{a}\,(A(\delta_r, \delta_r) + A(\delta_{r-1}, \delta_{r+1})) \\ &= \mathsf{a} \cdot A(\delta_{r-1}, \delta_{r+1}) \\ &\leq \mathsf{a}\,(A(\delta_{r-1}, \delta_r) + A(\delta_r, \delta_{r+1})) \\ &= \mathsf{a}(d_{r-1} + d_r). \end{aligned}$$

Thus,

$$d_r \leq \mathsf{a}(d_{r-1} + d_r) \quad \Rightarrow \quad (1 - \mathsf{a})d_r \leq \mathsf{a}d_{r-1} \quad \Rightarrow \quad d_r \leq \frac{\mathsf{a}}{1 - \mathsf{a}} d_{r-1}.$$

Let $\mathsf{k} = \frac{\mathsf{a}}{1-\mathsf{a}}$. Since $\mathsf{a} \in (0, \frac{1}{2})$, we have $\mathsf{k} \in (0, 1)$. By induction,

$$d_r \leq \mathsf{k}^r d_0.$$

Let $p = A - Q$ be the ex-su-metr . Since $Q(\delta_r, \delta_{r+1}) \geq 0$, we have

$$p(\delta_r, \delta_{r+1}) \leq A(\delta_r, \delta_{r+1}) \leq \mathsf{k}^r d_0.$$

Now, for $m > r$,

$$\begin{aligned} p(\delta_r, \delta_m) &\leq p(\delta_r, \delta_{r+1}) + p(\delta_{r+1}, \delta_m) + \eta(\delta_r, \delta_m)\, p(\delta_r, \delta_{r+1})\, p(\delta_{r+1}, \delta_m) \\ &\leq \mathsf{k}^r d_0 + p(\delta_{r+1}, \delta_m) + \eta(\delta_r, \delta_m)\mathsf{k}^r d_0\, p(\delta_{r+1}, \delta_m) \\ &= \mathsf{k}^r d_0 + (1 + \eta(\delta_r, \delta_m)\mathsf{k}^r d_0)\, p(\delta_{r+1}, \delta_m). \end{aligned}$$

Similarly,

$$\begin{aligned} p(\delta_{r+1}, \delta_m) &\leq p(\delta_{r+1}, \delta_{r+2}) + p(\delta_{r+2}, \delta_m) + \eta(\delta_{r+1}, \delta_m)\, p(\delta_{r+1}, \delta_{r+2})\, p(\delta_{r+2}, \delta_m) \\ &\leq \mathsf{k}^{r+1} d_0 + p(\delta_{r+2}, \delta_m) + \eta(\delta_{r+1}, \delta_m)\mathsf{k}^{r+1} d_0\, p(\delta_{r+2}, \delta_m) \\ &= \mathsf{k}^{r+1} d_0 + (1 + \eta(\delta_{r+1}, \delta_m)\mathsf{k}^{r+1} d_0)\, p(\delta_{r+2}, \delta_m). \end{aligned}$$

Continuing this process, we obtain

$$p(\delta_r, \delta_m) \leq \mathsf{k}^r d_0 + (1 + \eta(\delta_r, \delta_m)\mathsf{k}^r d_0)\mathsf{k}^{r+1} d_0$$

$$
\begin{aligned}
&+ (1+\eta(\delta_{\mathrm{r}},\delta_m)\mathsf{k}^{\mathrm{r}}d_0)(1+\eta(\delta_{\mathrm{r}+1},\delta_m)\mathsf{k}^{\mathrm{r}+1}d_0)\mathsf{k}^{\mathrm{r}+2}d_0\\
&+\cdots+(1+\eta(\delta_{\mathrm{r}},\delta_m)\mathsf{k}^{\mathrm{r}}d_0)(1+\eta(\delta_{\mathrm{r}+1},\delta_m)\mathsf{k}^{\mathrm{r}+1}d_0)\\
&\cdots(1+\eta(\delta_{m-2},\delta_m)\mathsf{k}^{m-2}d_0)\mathsf{k}^{m-1}d_0\\
&+(1+\eta(\delta_{\mathrm{r}},\delta_m)\mathsf{k}^{\mathrm{r}}d_0)(1+\eta(\delta_{\mathrm{r}+1},\delta_m)\mathsf{k}^{\mathrm{r}+1}d_0)\\
&\cdots(1+\eta(\delta_{m-1},\delta_m)\mathsf{k}^{m-1}d_0)\,p(\delta_m,\delta_m).
\end{aligned}
$$

Since $p(\delta_m,\delta_m)=0$, we have

$$
p(\delta_{\mathrm{r}},\delta_m)\le\sum_{j=\mathrm{r}}^{m-1}\left[\left(\prod_{i=\mathrm{r}}^{j-1}(1+\eta(\delta_i,\delta_m)\mathsf{k}^i d_0)\right)\mathsf{k}^j d_0\right].
$$

Since $\mathsf{k}\in(0,1)$, the infinite product $\prod_{i=\mathrm{r}}^{\infty}(1+\eta(\delta_i,\delta_m)\mathsf{k}^i d_0)$ converges, and the series $\sum_{j=\mathrm{r}}^{\infty}\mathsf{k}^j d_0$ converges. Therefore, $p(\delta_{\mathrm{r}},\delta_m)\to 0$ as $\mathrm{r},m\to\infty$, proving that $\{\delta_{\mathrm{r}}\}$ is a Cauchy sequence in $(\mathcal{W},p)$. By the completeness of $(\mathcal{W},A,Q)$, there exists $\delta^*\in\mathcal{W}$ such that

$$
\lim_{\mathrm{r}\to\infty}p(\delta_{\mathrm{r}},\delta^*)=0.
$$

By the perturbed continuity of $S$, we have

$$
S(\delta^*)=S\left(\lim_{\mathrm{r}\to\infty}\delta_{\mathrm{r}}\right)=\lim_{\mathrm{r}\to\infty}S(\delta_{\mathrm{r}})=\lim_{\mathrm{r}\to\infty}\delta_{\mathrm{r}+1}=\delta^*.
$$

So, $\delta^*$ is a fixed point of $S$. To show uniqueness, suppose $\vartheta^*$ is another fixed point. Then

$$
\begin{aligned}
A(\delta^*,\vartheta^*)&=A(S\delta^*,S\vartheta^*)\\
&\le\mathsf{a}\left(A(\delta^*,S\vartheta^*)+A(\vartheta^*,S\delta^*)\right)\\
&=\mathsf{a}\left(A(\delta^*,\vartheta^*)+A(\vartheta^*,\delta^*)\right)\\
&=2\mathsf{a}A(\delta^*,\vartheta^*).
\end{aligned}
$$

Thus, $(1-2\mathsf{a})A(\delta^*,\vartheta^*)\le 0$. Since $\mathsf{a}\in(0,\frac{1}{2})$, we have $1-2\mathsf{a}>0$, so $A(\delta^*,\vartheta^*)=0$. Therefore, $\delta^*=\vartheta^*$. Hence, $S$ has a Ufp. □

**Example 3.3.** *Let $\mathcal{W}=[0,\infty)$ and define*

$$
Q(\delta,\vartheta)=\delta\vartheta,\qquad A(\delta,\vartheta)=e^{|\delta-\vartheta|}-1+\delta\vartheta,\qquad \delta,\vartheta\in[0,\infty).
$$

*Then $p=A-Q=e^{|\delta-\vartheta|}-1$ satisfies the ex-su-metr space axioms with $\eta(\delta,\vartheta)=e^{\delta+\vartheta}$, so $([0,\infty),A,Q)$ is a complete ex-per-s-ms . Define the map $S:[0,\infty)\to[0,\infty)$ by*

$$
S(\delta)=\frac{\delta}{7},\qquad \delta\in[0,\infty).
$$

*For any $\delta,\vartheta\in[0,\infty)$, we have*

$$
A(S\delta,S\vartheta)=e^{|\frac{\delta}{7}-\frac{\vartheta}{7}|}-1+\frac{\delta\vartheta}{49}=e^{\frac{|\delta-\vartheta|}{7}}-1+\frac{\delta\vartheta}{49}.
$$

*Furthermore,*

$$
A(\delta,S\vartheta)=e^{|\delta-\frac{\vartheta}{7}|}-1+\delta\cdot\frac{\vartheta}{7}=e^{|\delta-\frac{\vartheta}{7}|}-1+\frac{\delta\vartheta}{7},
$$

$$A(\vartheta, S\delta) = e^{|\vartheta - \frac{\delta}{7}|} - 1 + \vartheta \cdot \frac{\delta}{7} = e^{|\vartheta - \frac{\delta}{7}|} - 1 + \frac{\delta\vartheta}{7}.$$

*By the inequality* $e^{x/7} - 1 \leq \frac{1}{7}(e^x - 1)$ *(with* $x = |\delta - \vartheta|$*) together with the triangle-type estimate* $e^{|\delta-\vartheta|} + 1 \leq e^{|\delta - \frac{\vartheta}{7}|} + e^{|\vartheta - \frac{\delta}{7}|}$,

$$e^{\frac{|\delta-\vartheta|}{7}} - 1 \leq \tfrac{1}{7}\Big(e^{|\delta-\vartheta|} - 1\Big) \leq \tfrac{1}{7}\Big(e^{|\delta - \frac{\vartheta}{7}|} + e^{|\vartheta - \frac{\delta}{7}|} - 2\Big).$$

*Since* $Q(\delta, S\vartheta) = Q(\vartheta, S\delta) = \dfrac{\delta\vartheta}{7}$, *we have the exact identity*

$$\frac{\delta\vartheta}{49} = \tfrac{1}{14}\left(\frac{\delta\vartheta}{7} + \frac{\delta\vartheta}{7}\right) = \tfrac{1}{14}\left(Q(\delta, S\vartheta) + Q(\vartheta, S\delta)\right) \leq \tfrac{1}{7}\left(Q(\delta, S\vartheta) + Q(\vartheta, S\delta)\right).$$

*Combining these inequalities,*

$$\begin{aligned} A(S\delta, S\vartheta) &\leq \tfrac{1}{7}\left(e^{|\delta - \frac{\vartheta}{7}|} + e^{|\vartheta - \frac{\delta}{7}|} - 2\right) + \tfrac{1}{7}\left(Q(\delta, S\vartheta) + Q(\vartheta, S\delta)\right) \\ &= \tfrac{1}{7}\left[\left(e^{|\delta - \frac{\vartheta}{7}|} - 1 + \frac{\delta\vartheta}{7}\right) + \left(e^{|\vartheta - \frac{\delta}{7}|} - 1 + \frac{\delta\vartheta}{7}\right)\right], \end{aligned}$$

*that is,*

$$A(S\delta, S\vartheta) \leq \tfrac{1}{7}\left[A(\delta, S\vartheta) + A(\vartheta, S\delta)\right].$$

*Taking* $\mathsf{a} = \frac{1}{7} \in \left(0, \frac{1}{2}\right)$, *we have*

$$A(S\delta, S\vartheta) \leq \mathsf{a}\left(A(\delta, S\vartheta) + A(\vartheta, S\delta)\right)$$

*for all* $\delta, \vartheta \in [0, \infty)$. *The map* $S$ *is per-cont , and all assumptions of Theorem 3.3 are satisfied. Hence,* $S$ *admits a Ufp.*

## 4. Application: Breast Cancer Dynamics Model (Caputo Operator)

Alzahrani, El-Dessoky and Altaf Khan [33] formulated a compartmental model to understand the dynamics of breast cancer, prevention, diagnosis and therapy, fitted to real breast cancer data reported in Saudi Arabia. The population of cancer patients is sub-divided into the following subgroups: patients admitted at stage 1 and 2 $\mathsf{X}$, patients at stage 3 $\mathsf{B}$, patients at stage 4 $\mathsf{C}$, patients with disease-free conditions after chemotherapy $\mathsf{R}$, and patients who experience cardiotoxicity $\mathsf{E}$. We shall consider the following system of ODEs

$$\begin{aligned} \mathsf{X}' &= \Lambda_1 - (\varphi_1 + \beta_2)\mathsf{X}, \\ \mathsf{B}' &= \Lambda_2 + \beta_2\mathsf{X} + \psi_1\mathsf{R} - (\varphi_2 + \beta_1 + \tau_1 + \mu_1)\mathsf{B}, \\ \mathsf{C}' &= \Lambda_3 + \beta_1\mathsf{B} + \psi_2\mathsf{R} - (\varphi_3 + \tau_2 + \mu_2)\mathsf{C}, \\ \mathsf{R}' &= \varphi_1\mathsf{X} + \varphi_2\mathsf{B} + \varphi_3\mathsf{C} - (\psi_1 + \psi_2 + \tau_3)\mathsf{R}, \\ \mathsf{E}' &= \tau_3\mathsf{R} + \tau_2\mathsf{C} + \tau_1\mathsf{B} - \mu_3\mathsf{E}. \end{aligned}$$

We replace the classical time derivative by the Caputo fractional derivative ${}_0^{\mathsf{C}}\mathsf{D}_{\varpi}^{\zeta}$ of order $\zeta \in (0, 1]$, giving

$$\begin{aligned} {}_0^{\mathsf{C}}\mathsf{D}_{\varpi}^{\zeta}(\mathsf{X}(\varpi)) &= \Lambda_1 - (\varphi_1 + \beta_2)\mathsf{X}(\varpi), \\ {}_0^{\mathsf{C}}\mathsf{D}_{\varpi}^{\zeta}(\mathsf{B}(\varpi)) &= \Lambda_2 + \beta_2\mathsf{X}(\varpi) + \psi_1\mathsf{R}(\varpi) - (\varphi_2 + \beta_1 + \tau_1 + \mu_1)\mathsf{B}(\varpi), \end{aligned}$$

$$
\begin{aligned}
{}_0^C\mathsf{D}_\omega^\zeta(\mathsf{C}(\omega)) &= \Lambda_3 + \beta_1\mathsf{B}(\omega) + \psi_2\mathsf{R}(\omega) - (\varphi_3 + \tau_2 + \mu_2)\mathsf{C}(\omega),\\
{}_0^C\mathsf{D}_\omega^\zeta(\mathsf{R}(\omega)) &= \varphi_1\mathsf{X}(\omega) + \varphi_2\mathsf{B}(\omega) + \varphi_3\mathsf{C}(\omega) - (\psi_1 + \psi_2 + \tau_3)\mathsf{R}(\omega),\\
{}_0^C\mathsf{D}_\omega^\zeta(\mathsf{E}(\omega)) &= \tau_3\mathsf{R}(\omega) + \tau_2\mathsf{C}(\omega) + \tau_1\mathsf{B}(\omega) - \mu_3\mathsf{E}(\omega),
\end{aligned}
$$

with initial conditions $\mathsf{X}(0) = \mathsf{z}_1$, $\mathsf{B}(0) = \mathsf{z}_2$, $\mathsf{C}(0) = \mathsf{z}_3$, $\mathsf{R}(0) = \mathsf{z}_4$, $\mathsf{E}(0) = \mathsf{z}_5$.

To facilitate easier comprehension, write the right-hand sides as

$$
\begin{aligned}
\Psi_1(\mathsf{X},\omega) &= \Lambda_1 - (\varphi_1 + \beta_2)\mathsf{X}(\omega),\\
\Psi_2(\mathsf{B},\omega) &= \Lambda_2 + \beta_2\mathsf{X}(\omega) + \psi_1\mathsf{R}(\omega) - (\varphi_2 + \beta_1 + \tau_1 + \mu_1)\mathsf{B}(\omega),\\
\Psi_3(\mathsf{C},\omega) &= \Lambda_3 + \beta_1\mathsf{B}(\omega) + \psi_2\mathsf{R}(\omega) - (\varphi_3 + \tau_2 + \mu_2)\mathsf{C}(\omega),\\
\Psi_4(\mathsf{R},\omega) &= \varphi_1\mathsf{X}(\omega) + \varphi_2\mathsf{B}(\omega) + \varphi_3\mathsf{C}(\omega) - (\psi_1 + \psi_2 + \tau_3)\mathsf{R}(\omega),\\
\Psi_5(\mathsf{E},\omega) &= \tau_3\mathsf{R}(\omega) + \tau_2\mathsf{C}(\omega) + \tau_1\mathsf{B}(\omega) - \mu_3\mathsf{E}(\omega).
\end{aligned}
$$

The Caputo derivative admits a single-term fractional Volterra integral representation: if ${}_0^C\mathsf{D}_\omega^\zeta(y(\omega)) = \Psi(\omega, y(\omega))$ with $y(0) = y_0$ and $0 < \zeta \le 1$, then

$$
y(\omega) = y_0 + \frac{1}{\Gamma(\zeta)}\int_0^\omega (\omega-\kappa)^{\zeta-1}\Psi(\kappa, y(\kappa))\,d\kappa.
$$

Applying this to each compartment gives the fractional Volterra integral system

$$
\begin{aligned}
\mathsf{X}(\omega) &= \mathsf{X}(0) + \frac{1}{\Gamma(\zeta)}\int_0^\omega (\omega-\kappa)^{\zeta-1}\Psi_1(\kappa,\mathsf{X})\,d\kappa,\\
\mathsf{B}(\omega) &= \mathsf{B}(0) + \frac{1}{\Gamma(\zeta)}\int_0^\omega (\omega-\kappa)^{\zeta-1}\Psi_2(\kappa,\mathsf{B})\,d\kappa,\\
\mathsf{C}(\omega) &= \mathsf{C}(0) + \frac{1}{\Gamma(\zeta)}\int_0^\omega (\omega-\kappa)^{\zeta-1}\Psi_3(\kappa,\mathsf{C})\,d\kappa,\\
\mathsf{R}(\omega) &= \mathsf{R}(0) + \frac{1}{\Gamma(\zeta)}\int_0^\omega (\omega-\kappa)^{\zeta-1}\Psi_4(\kappa,\mathsf{R})\,d\kappa,\\
\mathsf{E}(\omega) &= \mathsf{E}(0) + \frac{1}{\Gamma(\zeta)}\int_0^\omega (\omega-\kappa)^{\zeta-1}\Psi_5(\kappa,\mathsf{E})\,d\kappa.
\end{aligned}
$$

Define the operator $S : \mathcal{W} \to \mathcal{W}$ componentwise by $(S\delta)_i(\omega) = \delta_i(0) + \frac{1}{\Gamma(\zeta)}\int_0^\omega(\omega - \kappa)^{\zeta-1}\Psi_i(\kappa,\delta_i)\,d\kappa$ for $i = 1,\dots,5$, where $\delta = (\mathsf{X},\mathsf{B},\mathsf{C},\mathsf{R},\mathsf{E})$.

Let $\mathcal{W} = \mathbb{R}^5$ and define, for $\delta = (\mathsf{X},\mathsf{B},\mathsf{C},\mathsf{R},\mathsf{E})$, $\vartheta = (\mathsf{X}',\mathsf{B}',\mathsf{C}',\mathsf{R}',\mathsf{E}') \in \mathcal{W}$,

$$
Q(\delta,\vartheta) \equiv 0, \qquad A(\delta,\vartheta) = \max\left\{e^{|\mathsf{X}-\mathsf{X}'|} - 1,\ e^{|\mathsf{B}-\mathsf{B}'|} - 1,\ e^{|\mathsf{C}-\mathsf{C}'|} - 1,\ e^{|\mathsf{R}-\mathsf{R}'|} - 1,\ e^{|\mathsf{E}-\mathsf{E}'|} - 1\right\},
$$

with $\eta(\delta,\vartheta) \equiv 1$ (constant function). Then $(\mathcal{W}, A, Q)$ is a complete ex-per-s-ms .

**Theorem 4.1.** *Suppose that*

*(i) there exist constants* $\mathsf{L}_1,\dots,\mathsf{L}_5 > 0$ *such that, for each* $i = 1,\dots,5$ *and all admissible arguments,*

$$
\left|\Psi_i(\omega,\mathsf{u}_i) - \Psi_i(\omega,\mathsf{u}_i')\right| \le \mathsf{L}_i\left|\mathsf{u}_i - \mathsf{u}_i'\right|,
$$

*where* $\Psi_1,\dots,\Psi_5$ *are defined above.*

*(ii)* $\mathsf{L}\mathsf{K} < 1$, *where* $\mathsf{L} = \max\{\mathsf{L}_1, \dots, \mathsf{L}_5\}$ *and* $\mathsf{K} = \dfrac{\omega_{\max}^{\zeta}}{\Gamma(\zeta+1)}$.

*Then the Caputo-fractional breast cancer model admits a unique solution.*

*Proof.* For $\kappa \in [0, \omega]$,

$$|S\mathsf{X} - S\mathsf{X}'| = \left| \frac{1}{\Gamma(\zeta)} \int_0^{\omega} (\omega - \kappa)^{\zeta-1} \{\Psi_1(\kappa, \mathsf{X}) - \Psi_1(\kappa, \mathsf{X}')\} \, d\kappa \right|$$
$$\leq \frac{1}{\Gamma(\zeta)} \int_0^{\omega} (\omega - \kappa)^{\zeta-1} \left| \Psi_1(\kappa, \mathsf{X}) - \Psi_1(\kappa, \mathsf{X}') \right| d\kappa.$$

Using the Lipschitz condition for $\Psi_1$,

$$|S\mathsf{X} - S\mathsf{X}'| \leq \mathsf{L}_1 |\mathsf{X} - \mathsf{X}'| \cdot \frac{1}{\Gamma(\zeta)} \int_0^{\omega} (\omega - \kappa)^{\zeta-1} d\kappa = \mathsf{L}_1 |\mathsf{X} - \mathsf{X}'| \cdot \frac{\omega^{\zeta}}{\Gamma(\zeta+1)}.$$

Let $\mathsf{K} = \dfrac{\omega_{\max}^{\zeta}}{\Gamma(\zeta+1)}$. Then $|S\mathsf{X} - S\mathsf{X}'| \leq \mathsf{L}_1 \mathsf{K} |\mathsf{X} - \mathsf{X}'|$, and hence, since $e^{cx} - 1 \leq c(e^x - 1)$ for $c \in (0,1)$ and $x \geq 0$,

$$e^{|S\mathsf{X} - S\mathsf{X}'|} - 1 \leq \mathsf{L}_1 \mathsf{K} \left( e^{|\mathsf{X} - \mathsf{X}'|} - 1 \right).$$

The same argument applied to each of the remaining four components, using the Lipschitz constants $\mathsf{L}_2, \dots, \mathsf{L}_5$ for $\Psi_2, \dots, \Psi_5$ respectively, gives

$$e^{|(S\delta)_i - (S\vartheta)_i|} - 1 \leq \mathsf{L}_i \mathsf{K} \left( e^{|\delta_i - \vartheta_i|} - 1 \right), \qquad i = 1, \dots, 5.$$

Let $\mathsf{L} = \max\{\mathsf{L}_1, \dots, \mathsf{L}_5\}$ and assume $\mathsf{L}\mathsf{K} < 1$. Taking the maximum over $i = 1, \dots, 5$,

$$A(S\delta, S\vartheta) \leq \mathsf{L}\mathsf{K} A(\delta, \vartheta).$$

It follows that $S$ satisfies the contraction condition with $\hbar = \mathsf{L}\mathsf{K} < 1$. By Theorem 3.1, $S$ admits exactly one fixed point. □

## 5. Application: A Fractional Boundary Value Problem

As a second application, we consider a nonlinear Caputo fractional two-point boundary value problem (BVP)

$$\begin{aligned} {}^{C}\mathsf{D}_{0^+}^{\zeta} u(\omega) + g(\omega, u(\omega)) = 0, \qquad \omega \in [0,1], \quad 1 < \zeta \leq 2, \\ u(0) = 0, \qquad u(1) = 0, \end{aligned} \tag{5.1}$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{C}\mathsf{D}_{0^+}^{\zeta}$ denotes the Caputo fractional derivative of order $\zeta$.

It is well known that the BVP (5.1) is equivalent to the integral equation

$$u(\omega) = \int_0^1 \mathcal{G}(\omega, \mathsf{q}) \, g(\mathsf{q}, u(\mathsf{q})) \, d\mathsf{q}, \qquad \omega \in [0,1],$$

where the Green's function $\mathcal{G}$ is given by [34]

$$\mathcal{G}(\omega,\mathsf{q}) = \frac{1}{\Gamma(\zeta)}\begin{cases}[\omega(1-\mathsf{q})]^{\zeta-1} - (\omega-\mathsf{q})^{\zeta-1}, & 0 \le \mathsf{q} \le \omega \le 1,\\ [\omega(1-\mathsf{q})]^{\zeta-1}, & 0 \le \omega \le \mathsf{q} \le 1,\end{cases}$$

and satisfies $\mathcal{G}(\omega,\mathsf{q}) \ge 0$ for all $\omega,\mathsf{q} \in [0,1]$.

Define the operator $S : \mathcal{W} \to \mathcal{W}$, where $\mathcal{W} = C[0,1]$, by

$$(Su)(\omega) = \int_0^1 \mathcal{G}(\omega,\mathsf{q})\, g(\mathsf{q},u(\mathsf{q}))\, d\mathsf{q}, \qquad \omega \in [0,1].$$

A function $u^* \in \mathcal{W}$ solves the BVP (5.1) if and only if $u^*$ is a fixed point of $S$.

Let $\mathcal{W} = C[0,1]$ and, for $u,v \in \mathcal{W}$, define

$$Q(u,v) \equiv 0, \qquad A(u,v) = e^{\|u-v\|_\infty} - 1,$$

with $\eta(u,v) \equiv 1$, where $\|u-v\|_\infty = \max_{\omega\in[0,1]} |u(\omega)-v(\omega)|$. Consequently, $(\mathcal{W},A,Q)$ is a complete ex-per-s-ms .

**Theorem 5.1.** *Suppose the following conditions hold.*

*(i) There exists* $\mathsf{L} > 0$ *such that*

$$|g(\omega,u) - g(\omega,v)| \le \mathsf{L}\,|u-v|, \qquad \forall\, \omega \in [0,1],\ u,v \in \mathbb{R}.$$

*(ii)* $\mathsf{L}\,\mathsf{L}_{\mathcal{G}} < 1$, *where*

$$\mathsf{L}_{\mathcal{G}} := \max_{\omega\in[0,1]} \int_0^1 \mathcal{G}(\omega,\mathsf{q})\, d\mathsf{q} = \frac{1}{\Gamma(\zeta+1)}.$$

*Then the boundary value problem* (5.1) *admits a unique solution* $u^* \in C[0,1]$.

*Proof.* Let $u,v \in \mathcal{W}$. For any $\omega \in [0,1]$, using $\mathcal{G}(\omega,\mathsf{q}) \ge 0$ on $[0,1]\times[0,1]$ and the Lipschitz condition on $g$, we obtain

$$\begin{aligned}|(Su)(\omega) - (Sv)(\omega)| &= \left|\int_0^1 \mathcal{G}(\omega,\mathsf{q})\,[g(\mathsf{q},u(\mathsf{q})) - g(\mathsf{q},v(\mathsf{q}))]\, d\mathsf{q}\right|\\ &\le \int_0^1 \mathcal{G}(\omega,\mathsf{q})\,\big|g(\mathsf{q},u(\mathsf{q})) - g(\mathsf{q},v(\mathsf{q}))\big|\, d\mathsf{q}\\ &\le \mathsf{L}\int_0^1 \mathcal{G}(\omega,\mathsf{q})\,|u(\mathsf{q}) - v(\mathsf{q})|\, d\mathsf{q}\\ &\le \mathsf{L}\,\|u-v\|_\infty \int_0^1 \mathcal{G}(\omega,\mathsf{q})\, d\mathsf{q}\\ &\le \mathsf{L}\,\mathsf{L}_{\mathcal{G}}\,\|u-v\|_\infty.\end{aligned}$$

Taking the maximum over $\omega \in [0,1]$, we get

$$\|Su - Sv\|_\infty \le \mathsf{L}\,\mathsf{L}_{\mathcal{G}}\,\|u-v\|_\infty.$$

Set $\hbar := \mathsf{L}\,\mathsf{L}_{\mathcal{G}}$. By hypothesis (ii), $\hbar \in (0,1)$. Since $\mathsf{x} \mapsto e^{\mathsf{x}} - 1$ is convex and increasing on $[0,\infty)$ with $e^0 - 1 = 0$, for any $c \in (0,1)$ and $\mathsf{x} \geq 0$ we have $e^{c\mathsf{x}} - 1 \leq c(e^{\mathsf{x}} - 1)$. Applying this with $c = \hbar$ and $\mathsf{x} = \|u - v\|_\infty$,

$$\begin{aligned} A(Su, Sv) &= e^{\|Su - Sv\|_\infty} - 1 \\ &\leq e^{\hbar\|u-v\|_\infty} - 1 \\ &\leq \hbar\left(e^{\|u-v\|_\infty} - 1\right) \\ &= \hbar\, A(u,v). \end{aligned}$$

Thus $S$ satisfies condition of Theorem 3.1 with $\hbar = \mathsf{L}\mathsf{L}_{\mathcal{G}} \in (0,1)$. Moreover, since $g$ and $\mathcal{G}$ are continuous, $S$ is continuous with respect to $\|\cdot\|_\infty$, and hence per-cont with respect to $p$. It follows from Theorem 3.1 that $S$ has exactly one fixed point. □

**Example 5.1.** *Let $\zeta = \frac{3}{2}$ and consider $g(\omega, u) = \frac{1}{10}\sin(u) + \omega^2$. Then $g$ satisfies the Lipschitz condition with $\mathsf{L} = \frac{1}{10}$, since $|\sin(u) - \sin(v)| \leq |u - v|$. Here, $\mathsf{L}_{\mathcal{G}} = \dfrac{1}{\Gamma(\zeta+1)} = \dfrac{1}{\Gamma\left(\frac{5}{2}\right)} = \dfrac{4}{3\sqrt{\pi}} \approx 0.7523$. Hence,*

$$\mathsf{L}\,\mathsf{L}_{\mathcal{G}} \approx \frac{1}{10} \times 0.7523 = 0.07523 < 1.$$

*All hypotheses of Theorem 5.1 are satisfied. Hence, the boundary value problem*

$${}^{\mathsf{C}}\mathsf{D}^{3/2}_{0^+} u(\omega) + \frac{1}{10}\sin(u(\omega)) + \omega^2 = 0, \qquad \omega \in [0,1], \qquad u(0) = u(1) = 0,$$

*admits a unique solution $u^* \in C[0,1]$.*

## 6. Application: A Fractional 3D Hopfield Neural Network Model

As a third application, we consider the Atangana–Baleanu–Caputo fractional 3D Hopfield neural network (HNN) model studied by Rezapour, Kumar, Erturk and Etemad [35]. Hopfield neural networks are artificial models derived from brain structures and are widely used in neurocomputing; the fractional-order formulation captures memory and hereditary effects that classical integer-order HNNs cannot describe. The model consists of three coupled neuron states $y_1, y_2, y_3$ governed by

$$\begin{smallmatrix} \mathsf{ABC} \\ 0 \end{smallmatrix}\mathsf{D}^{\zeta}_{\omega}(y_i(\omega)) = -\mathsf{c}_i\, y_i(\omega) + \sum_{j=1}^{3} \mathsf{r}_{ij} \tanh(y_j(\omega)) + \mathsf{I}_i, \qquad i = 1,2,3, \tag{6.1}$$

subject to $y_i(0) = y_{i,0}$, $i = 1,2,3$, where $\mathsf{c}_i > 0$ is the self-inhibition (decay) rate of neuron $i$, $\mathsf{r} = (\mathsf{r}_{ij})_{1\leq i,j\leq 3}$ is the synaptic weight matrix, $\tanh(\cdot)$ is the neuron activation function, and $\mathsf{I}_i$ is a constant external bias input to neuron $i$ [35].

Define, for $i = 1,2,3$,

$$\Xi_i(y_1, y_2, y_3, \omega) = -\mathsf{c}_i\, y_i(\omega) + \sum_{j=1}^{3} \mathsf{r}_{ij} \tanh(y_j(\omega)) + \mathsf{I}_i,$$

and write $\Omega = (y_1, y_2, y_3)$, $\Xi = (\Xi_1, \Xi_2, \Xi_3)$. System (6.1) is equivalent to the fractional Volterra integral equation

$$\Omega(\varpi) = \Omega(0) + \frac{1-\zeta}{\wp(\zeta)}\Xi(\Omega, \varpi) + \frac{\zeta}{\wp(\zeta)\Gamma(\zeta)} \int_0^{\varpi} (\varpi - \kappa)^{\zeta-1} \Xi(\Omega, \kappa)\, d\kappa.$$

Define the operator $S : \mathcal{W} \to \mathcal{W}$, $\mathcal{W} = \mathbb{R}^3$, by

$$S\Omega(\varpi) = \Omega(0) + \frac{1-\zeta}{\wp(\zeta)}\Xi(\Omega, \varpi) + \frac{\zeta}{\wp(\zeta)\Gamma(\zeta)} \int_0^{\varpi} (\varpi - \kappa)^{\zeta-1} \Xi(\Omega, \kappa)\, d\kappa.$$

A solution of the HNN model (6.1) corresponds to a fixed point of $S$.

Let $\mathcal{W} = \mathbb{R}^3$ and, for $\Omega = (y_1, y_2, y_3)$, $\Omega' = (y_1', y_2', y_3') \in \mathcal{W}$, define

$$Q(\Omega, \Omega') \equiv 0, \qquad A(\Omega, \Omega') = \max_{1 \le i \le 3} \left\{ e^{|y_i - y_i'|} - 1 \right\},$$

with $\eta(\Omega, \Omega') \equiv 1$. Then $(\mathcal{W}, A, Q)$ is a complete ex-per-s-ms .

**Theorem 6.1.** *Suppose*

$$\left( \frac{1-\zeta}{\wp(\zeta)} + \frac{\varpi_{\max}^{\zeta}}{\wp(\zeta)\Gamma(\zeta+1)} \right) \mathsf{L}_{\mathrm{r}} < 1, \qquad \textit{where } \mathsf{L}_{\mathrm{r}} := \max_{1 \le i \le 3} \left( \mathsf{c}_i + \sum_{j=1}^{3} |\mathrm{r}_{ij}| \right).$$

*Then the fractional 3D Hopfield neural network model* (6.1) *admits a unique solution.*

*Proof.* Since tanh is 1-Lipschitz, i.e. $|\tanh(u) - \tanh(v)| \le |u - v|$ for all $u, v \in \mathbb{R}$, for $\Omega, \Omega' \in \mathcal{W}$ and each $i = 1, 2, 3$,

$$\begin{aligned} |\Xi_i(\Omega, \varpi) - \Xi_i(\Omega', \varpi)| &\le \mathsf{c}_i\, |y_i - y_i'| + \sum_{j=1}^{3} |\mathrm{r}_{ij}|\, |\tanh(y_j) - \tanh(y_j')| \\ &\le \mathsf{c}_i\, |y_i - y_i'| + \sum_{j=1}^{3} |\mathrm{r}_{ij}|\, |y_j - y_j'| \\ &\le \left( \mathsf{c}_i + \sum_{j=1}^{3} |\mathrm{r}_{ij}| \right) \|\Omega - \Omega'\|_\infty \le \mathsf{L}_{\mathrm{r}}\, \|\Omega - \Omega'\|_\infty. \end{aligned}$$

Thus, $\|\Xi(\Omega, \varpi) - \Xi(\Omega', \varpi)\|_\infty \le \mathsf{L}_{\mathrm{r}} \|\Omega - \Omega'\|_\infty$ for all $\varpi \in [0, \varpi_{\max}]$. Hence,

$$\|S\Omega - S\Omega'\|_\infty \le \mathsf{L}_{\mathrm{r}}\, \mathsf{K}\, \|\Omega - \Omega'\|_\infty, \qquad \mathsf{K} = \frac{1-\zeta}{\wp(\zeta)} + \frac{\varpi_{\max}^{\zeta}}{\wp(\zeta)\Gamma(\zeta+1)},$$

and, using $e^{cy} - 1 \le c(e^y - 1)$ for $c = \mathsf{L}_{\mathrm{r}}\mathsf{K} \in (0, 1)$ and $y \ge 0$ componentwise,

$$A(S\Omega, S\Omega') \le \mathsf{L}_{\mathrm{r}}\mathsf{K}\, A(\Omega, \Omega').$$

Thus $S$ satisfies condition (3.1) of Theorem 3.1 with $\hbar = \mathsf{L}_{\mathrm{r}}\mathsf{K} \in (0, 1)$, and $S$ is continuous since $\Xi$ is continuous (as tanh is continuous). It follows from Theorem 3.1 that $S$ has exactly one fixed point. □

**Example 6.1.** *Let* $\zeta = \frac{4}{5}$, $\varpi_{\max} = 1$, *and take the self-inhibition rates* $\mathsf{c}_1 = \mathsf{c}_2 = \mathsf{c}_3 = 1$ *together with the illustrative synaptic weight matrix*

$$\mathrm{r} = \begin{pmatrix} 0.10 & -0.05 & 0.02 \\ 0.04 & 0.10 & -0.03 \\ -0.02 & 0.03 & 0.10 \end{pmatrix}.$$

*Then* $\mathsf{L}_{\mathrm{r}} = \max_i \left(\mathsf{c}_i + \sum_{j=1}^{3} |\mathrm{r}_{ij}|\right) = 1 + 0.17 = 1.17$. *With* $\wp(\zeta) \equiv 1$ *(normalization* $\wp(0) = \wp(1) = 1$*), we get* $\mathsf{K} = \frac{1-\zeta}{\wp(\zeta)} + \frac{\varpi_{\max}^{\zeta}}{\wp(\zeta)\Gamma(\zeta+1)} = 0.2 + \frac{1}{\Gamma(1.8)} \approx 0.2 + 1.0737 = 1.2737$. *Hence* $\mathsf{L}_{\mathrm{r}}\mathsf{K} \approx 1.17 \times 1.2737 \approx 1.490$, *which exceeds* 1; *rescaling the weights (or shrinking* $\varpi_{\max}$*) so that* $\mathsf{L}_{\mathrm{r}} < 1/\mathsf{K} \approx 0.785$ *restores condition of Theorem 6.1 and guarantees a unique solution on* $[0, \varpi_{\max}]$.

## 7. Conclusion

In this paper, we have introduced and thoroughly examined ex-per-s-ms 's as a natural mathematical framework for handling errors in distance measurements. In this general context, we obtained a number of important fixed point results such as extensions of the classical B-cont principle, Ka-ty and Ch-ty contractions. The proposed findings are supported by comprehensive examples which also validate the results and show their wide applicability.

The practical utility of our presented framework is corroborated by the three fractional-order models: the Caputo fractional breast cancer dynamics model, the nonlinear Caputo fractional boundary value problem and the ABC-fractional 3D Hopfield neural network model. The applications show that the proposed approach is able to solve complex systems with measurement uncertainties and offers a solid mathematical theory for the analysis of fractional-order problems in applied sciences.

The findings reported here are suggestive of a number of interesting avenues for future research. There are other classes of contraction mappings that are not covered here which are worthy of further research, as are common fixed point theorems for families of mappings. The framework can be extended to other fields of mathematical biology, physics or engineering. Furthermore the relationships with other generalized metric spaces between ex-per-s-ms should be investigated further.

**Funding:** The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/36775).

**Author Contribution:** Conceptualisation - KHK, RR, GM Methodology - RR, SP, SSR, GM, Analysis -KHK, AKSH, Software -SSR, AKSH, Supervision - RR, KHK, Project Management - KHK, RR, GM, Writing Orignisal Draft - SP, SSR, KHK, AKSH, Writing -Review and Editing - KHK, SP, SSR, RR, AKSH, GM. All authors have read and agreed to the present version of the manuscript.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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