Int. J. Anal. Appl. (2026), 24:373

International Journal of Analysis and Applications

Bondage Numbers in Power Dominator Coloring of Graphs

L. Jacquline[1], G. Navamani[1,*], M. R. Raghavendra[2], L. Panneerselvam[3], R. Sankar[4]

[1]*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Tandalam, Chennai - 602 105, Tamil Nadu, India*

[2]*Department of Mathematics, Dayananda Sagar Academy of Technology and Management, Kanakapura road, Bengaluru - 560 082, Karnataka, India*

[3]*Department of Mathematics, Faculty of Science and Humanities, SRM Institute of Science and Technology, Ramapuram Campus, Chennai - 600 089, Tamil Nadu, India*

[4]*Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai - 600 062, Tamil Nadu, India*

[*]*Corresponding author:* navamaniprakash69@gmail.com

Abstract. Power dominator coloring is a strengthened form of graph coloring in which every vertex must be assigned a color and must also be able to power dominate at least one color class. The minimum number of colors required for such a coloring is the power dominator chromatic number, denoted by $\chi_{pd}(G)$. In this paper, we introduce two stability parameters for $\chi_{pd}(G)$: the upper bondage number $b^+(G, \chi_{pd})$ and the lower bondage number $b^-(G, \chi_{pd})$. These values represent the minimum number of edges whose removal increases or decreases $\chi_{pd}(G)$, respectively.

We compute these bondage numbers for fundamental graph families such as paths P_n, cycles C_n, complete graphs K_n, complete bipartite graphs $K_{m,n}$, wheel graph W_n and helm graph H_n. Our results show that the power dominator chromatic number is highly sensitive to small structural modifications, especially in dense graphs like K_n where the removal of a single edge already reduces the chromatic requirement. For graphs where $\chi_{pd} = 2$, such as paths and even cycles, the lower bondage number is infinite, since it is impossible to reduce the number of colors below two.

This work establishes the first bondage-type analysis for power dominator coloring and provides a foundation for further studies on robustness of domination-based colorings.

1. Introduction

Graph coloring is a fundamental topic in graph theory and many of its extensions arise from incorporating additional structural constraints into classical coloring models. A *proper vertex*

Received: May 13, 2026.

2020 *Mathematics Subject Classification.* 05C15; 05C69.

Key words and phrases. lower bondage number; upper bondage number; power dominator coloring; structural modification; domination.

https://doi.org/10.28924/2291-8639-24-2026-373
ISSN: 2291-8639

© 2026 the author(s).

coloring of a graph $G = (V, E)$ is a mapping

$$c : V \to \{1, 2, \ldots, k\}$$

such that $c(u) \neq c(v)$ whenever $uv \in E$. The smallest integer k for which such a coloring exists is the *chromatic number* $\chi(G)$ [5].

Domination theory provides a complementary framework for studying monitoring and control in networks. A set $S \subseteq V(G)$ is said to be a dominating set of a graph G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex of S. Equivalently, the closed neighborhood of S satisfies $N[S] = V(G)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G. Any dominating set of G with cardinality $\gamma(G)$ is referred to as a γ-set of G [2, 6, 8]. Variants of domination have been developed to model monitoring processes in communication networks, electrical systems, and sensor grids.

Combining domination with coloring leads to the concept of *dominator coloring*, where c is a proper coloring and additionally, each vertex dominates at least one color class [5, 10].

A dynamic extension of domination is *power domination*, introduced to model the placement of Phasor Measurement Units (PMUs) in electric power networks.

Let $G = (V, E)$ be a graph and let $D \subseteq V$ be a set of vertices. The set of vertices monitored by D, denoted by $\mathcal{M}(D) \subseteq V$, is defined recursively as follows:

(i) $\mathcal{M}(D) \leftarrow D \cup N(D)$,

(ii) if there exists a vertex $u \in \mathcal{M}(D)$ such that $N(u) \cap \Big(V \setminus \mathcal{M}(D)\Big) = \{x\}$, then set $\mathcal{M}(D) \leftarrow \mathcal{M}(D) \cup \{x\}$.

Step (ii) is repeated until no further vertices can be added to $\mathcal{M}(D)$. If, at the termination of this process, $\mathcal{M}(D) = V(G)$, then D is called a *power dominating set* of G. The minimum cardinality of a power dominating set of G is called the *power domination number* of G and is denoted by $\gamma_p(G)$ [3, 9, 13].

Let $v \in V(G)$. The set of vertices monitored by v, denoted by $\mathcal{M}(v) \subseteq V(G)$, is defined through the following iterative procedure. Initially,

$$\mathcal{M}(v) = \{v\} \cup N(v).$$

Thereafter, a vertex $x \in V(G)$ is added to $\mathcal{M}(v)$ whenever there exists a vertex $y \in \mathcal{M}(v)$ such that $N(y) \setminus \mathcal{M}(v) = \{x\}$. This step is repeated until no further vertices can be included. The vertex v is said to power dominate all vertices in the resulting set $\mathcal{M}(v)$ [11].

A *power dominator coloring* of a graph G is a proper vertex coloring with the additional property that each vertex of G power dominates all vertices of at least one color class. The minimum number of colors required for a power dominator coloring of G is called the *power dominator chromatic number* of G and is denoted by $\chi_{pd}(G)$ [1].

To study the sensitivity of $\chi_{pd}(G)$ under structural modifications, we introduce the *upper* and *lower bondage numbers*. For an edge set $E' \subseteq E(G)$, we define

$$b^+(G, \chi_{pd}) = \min\left\{|E'| : \chi_{pd}(G - E') > \chi_{pd}(G)\right\},$$

$$b^-(G, \chi_{pd}) = \min\left\{|E'| : \chi_{pd}(G - E') < \chi_{pd}(G)\right\},$$

with the understanding that the value is ∞ if no such edge set exists. These parameters quantify the robustness of the power dominator chromatic number with respect to edge deletions and describe the vulnerability of a graph's monitoring structure under failures or perturbations.

In this paper, we compute both $b^+(G, \chi_{pd})$ and $b^-(G, \chi_{pd})$ for several standard graph families, including paths P_n, cycles C_n, complete graphs K_n, complete bipartite graphs $K_{m,n}$, and wheel graphs W_n. The results show that $\chi_{pd}(G)$ exhibits markedly different behaviors depending on the structural properties of G. Dense graphs such as complete graphs are highly sensitive to edge removals, while paths and even cycles demonstrate greater stability. These findings establish foundational stability properties for power dominator coloring and motivate further work on algorithmic and structural questions related to domination-based colorings.

2. Preliminaries

Let $G = (V, E)$ be a simple connected graph. For a vertex $v \in V$, the open neighborhood of v, denoted by $N(v)$, is the set of all vertices adjacent to v, while the closed neighborhood of v is defined as $N[v] = N(v) \cup \{v\}$. A path P_n is a simple graph on n vertices arranged in a linear order such that each vertex is adjacent to at most two other vertices. A cycle C_n is obtained by joining the first and last vertices of the path P_n, forming a closed chain in which every vertex has degree two. A complete graph K_n is a graph on n vertices in which every pair of distinct vertices is connected by an edge, so that each vertex has degree $n - 1$. A complete bipartite graph $K_{m,n}$ is a bipartite graph whose vertex set is partitioned into two disjoint sets of sizes m and n, with every vertex in one part adjacent to all vertices in the other part and no edges joining vertices within the same part. A wheel graph W_n is constructed by taking a cycle C_n on n vertices and adding one additional vertex, called the hub, which is adjacent to all vertices of the cycle; thus, the resulting graph has $n + 1$ vertices and a structure resembling a wheel with spokes.

3. Main Results

Theorem 3.1. *[12] For $n \geq 2$, $\chi_{pd}(P_n) = 2$.*

Theorem 3.2. *[12] For $n \geq 1$, $\chi_{pd}(K_n) = n$.*

Theorem 3.3. *For $n \geq 3$,*

$$b^+(P_n, \chi_{pd}) = 1, \qquad b^-(P_n, \chi_{pd}) = \infty.$$

Proof. Let $G \cong P_n$, $n \geq 3$, with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_{n-1}\}$. By Theorem 3.1, $\chi_{pd}(P_n) = 2$.

Case 1: Pendant edge removal

The removal of pendant edge leads to the disconnected graph G' whose components are G_1 and G_2, where $G_1 \cong K_1$ and $G_2 \cong P_{n-1}$. Each vertex of G' power dominates at least one color class, so by theorem 3.2 $\chi_{pd}(G_1) = 1$ and by theorem 3.1 $\chi_{pd}(G_2) = 2$. Therefore $\chi_{pd}(G') = 3$. Hence $b^+(P_n, \chi_{pd}) = 1$

Case 2: Non-pendant edge removal

Let G' be the graph obtained from G by removing a non-pendant edge $e \in E(G)$. Then G' becomes disconnected and consists of two components, say G_1 and G_2, where $G_1 \cong P_k$ and $G_2 \cong P_{n-k}$ for some $1 < k < n$.

Let $\{v_1, v_2, \ldots, v_k\}$ and $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ be the vertex set of the components P_k and P_{n-k} respectively. Since each component is a path, every vertex can power dominate all vertices in its own component. To construct a power dominator coloring for G', assign color 1 to the odd-indexed vertices of G_1 and color 2 to the odd-indexed vertices of G_2. The vertices in the component G_1 power dominates the color class C^1 and the vertices in the component G_2 power dominates the color class C^2. The remaining vertices in both components cannot be colored with these two colors. Therefore, a third color is necessary. Hence, color 3 is assigned to all even-indexed vertices of G'.

Thus, the resulting color classes are

$$C^1 = \{ v_{2i-1} : 1 \leq i \leq \left\lceil \tfrac{k}{2} \right\rceil \},$$

$$C^2 = \{ v_{2i-1} : \left\lceil \tfrac{k}{2} \right\rceil + 1 \leq i \leq \left\lceil \tfrac{k}{2} \right\rceil + \left\lceil \tfrac{n-k}{2} \right\rceil \},$$

$$C^3 = \{ v_{2i} : 1 \leq i \leq \left\lfloor \tfrac{n}{2} \right\rfloor \}.$$

Hence, $\chi_{pd}(G') = 3$. The power dominator chromatic number increases for a single edge removal. Therefore,

$$b^+(P_n, \chi_{pd}) = 1.$$

Since $\chi_{pd}(P_n) = 2$ and no connected graph with at least two vertices can have a power dominator chromatic number smaller than 2, removing edges cannot decrease $\chi_{pd}(P_n)$. Thus,

$$b^-(P_n, \chi_{pd}) = \infty.$$

□

Theorem 3.4. *[12] For $C_n, n \geq 3$,*

$$\chi_{pd}(C_n) = \begin{cases} 2, & \text{if } n \text{ is even}, \\ 3, & \text{if } n \text{ is odd} \end{cases}$$

Theorem 3.5. *Let C_n be a cycle of order $n \geq 3$. Then the lower bondage number of C_n with respect to the power dominator chromatic number is given by*

$$b^-(C_n, \chi_{pd}) = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ \infty, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Let $V(C_n) = \{k_1, k_2, \ldots, k_n\}$ be the vertex set and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ be the edge set of C_n, where $e_1 = k_1k_2$, $e_2 = k_2k_3$, …, $e_{n-1} = k_{n-1}k_n$, and $e_n = k_nk_1$.

Case 1: n is odd

By theorem 3.4, if n is odd, $\chi_{pd}(C_n) = 3$. Let e_i be an arbitrary edge of C_n, for some i with $1 \leq i \leq n$ and define $G' = C_n - e_i$. Then G' is a path graph on n vertices, that is, $G' \cong P_n$. Consequently,

$$\chi_{pd}(G') = \chi_{pd}(P_n) = 2 < 3,$$

which shows that the removal of a single edge strictly decreases the power dominator chromatic number. Therefore, $b^-(C_n, \chi_{pd}) \leq 1$, and since b^- cannot be zero, we obtain

$$b^-(C_n, \chi_{pd}) = 1 \quad \text{for odd } n.$$

Case 2: n even.

By theorem 3.4, $\chi_{pd}(C_n) = 2$ when n is even. Let $e \in E(C_n)$ be any edge such that $C_n - e$ remains connected. Then $C_n - e$ is a path on n vertices, that is, $C_n - e \cong P_n$. Since $\chi_{pd}(P_n) = 2$, the removal of a single edge does not decrease the power dominator chromatic number.

Further, consider the removal of two edges from the cycle graph C_n. If the removed edges are adjacent, say $E' = \{e_1, e_2\}$, then the resulting graph $G' = C_n - E'$ is disconnected and consists of two components G_1 and G_2, where $G_1 \cong K_1$ and $G_2 \cong P_{n-1}$. In this case, the coloring constraints follow from Case (i) of Theorem 3.3, and hence $\chi_{pd}(G') = 3$.

If the removed edges are nonadjacent, then the resulting graph G' is a disjoint union of two path graphs, say P_k and P_{n-k}, for some $1 \leq k \leq n-1$. Each component requires at least one color class to be power dominated by vertices within that component. Therefore, the coloring constraints follow from Case (ii) of Theorem 3.3, and again $\chi_{pd}(G') = 3$.

Thus, in both cases, the power dominator chromatic number does not decrease under the removal of two edges; rather, it may increase due to the loss of connectivity and structural constraints. Hence, no edge removal can reduce $\chi_{pd}(C_n)$ when n is even.

By definition,

$$b^-(C_n, \chi_{pd}) = \min\Big\{ |E'| : \chi_{pd}(C_n - E') < \chi_{pd}(C_n) \Big\},$$

with the convention that $b^-(C_n, \chi_{pd}) = \infty$ if no such edge set exists.

Therefore,

$$b^-(C_n, \chi_{pd}) = \infty \quad \text{when n is even.}$$

□

Theorem 3.6. *Let C_n be a cycle of order $n \geq 3$. Then the upper bondage number of C_n with respect to the power dominator chromatic number is given by*

$$b^+(C_n, \chi_{pd}) = \begin{cases} 3, & \text{if } n \text{ is odd}, \\ 2, & \text{if } n \text{ is even}. \end{cases}$$

Proof. The upper bondage number with respect to the power dominator chromatic number is defined as

$$b^+(C_n, \chi_{pd}) = \min\left\{|E'| : \chi_{pd}(C_n - E') > \chi_{pd}(C_n)\right\}.$$

We distinguish the proof according to the parity of n.

Case 1: n is odd. For odd n, $\chi_{pd}(C_n) = 3$.

Subcase 1.1: Adjacent edge removal. First, remove two adjacent edges $E' = \{e_1, e_2\}$. The resulting graph $G' = C_n - E'$ is disconnected and consists of two components $G_1 \cong P_{n-1}$ and $G_2 \cong K_1$. By Theorem 3.1, $\chi_{pd}(P_{n-1}) = 2$, and by Theorem 3.2, $\chi_{pd}(K_1) = 1$. Thus,

$$\chi_{pd}(G') = 3 = \chi_{pd}(C_n), \chi_{pd}(G') \not> \chi_{pd}(C_n).$$

Hence, the removal of two adjacent edges does not increase the power dominator chromatic number.

Now remove three adjacent edges $E'' = \{e_1, e_2, e_3\}$. The resulting graph $G'' = C_n - E''$ consists of three components $H_1 \cong P_{n-2}$, $H_2 \cong K_1$, and $H_3 \cong K_1$. By Theorem 3.1, $\chi_{pd}(P_{n-2}) = 2$, and each isolated vertex requires one distinct color. Hence,

$$\chi_{pd}(G'') = \chi_{pd}(H_1 + H_2 + H_3) = 2 + 1 + 1 = 4 > \chi_{pd}(C_n).$$

Therefore,

$$b^+(C_n, \chi_{pd}) = 3 \quad \text{for odd } n \text{ under adjacent edge removal.}$$

Subcase 1.2: Nonadjacent edge removal. Let $E' = \{e_i, e_j\}$ be two nonadjacent edges of C_n. Then $G' = C_n - E'$ is a disjoint union of two path graphs P_k and P_{n-k} for some $1 \leq k \leq n-1$. By Theorem 3.3,

$$\chi_{pd}(G') = 3 = \chi_{pd}(C_n),$$

and the removal of two nonadjacent edges does not increase the power dominator chromatic number when n is odd.

Now, remove three nonadjacent edges $E'' = \{e_i, e_j, e_k\}$ from C_n. The resulting graph $G'' = C_n - E''$ is disconnected and consists of three disjoint path components. Each path component requires at least one color class to be power dominated by vertices within that component, and one additional color is required to properly color the remaining vertices of all components. Therefore,

$$\chi_{pd}(G'') = 4 > \chi_{pd}(C_n).$$

Therefore,

$$b^+(C_n, \chi_{pd}) = 3 \quad \text{for odd } n \text{ under nonadjacent edge removal.}$$

Case 2: n is even. For even n, $\chi_{pd}(C_n) = 2$.

Subcase 2.1: Adjacent edge removal. Removing a single edge from C_n yields a path P_n and does not increase χ_{pd}. Removing two adjacent edges $E' = \{e_1, e_2\}$ results in a disconnected graph $G' = C_n - E'$ with components P_{n-1} and K_1. By case (i) of theorem 3.3 $\chi_{pd}(G') = 3$. Thus,

$$\chi_{pd}(G') = 3 > \chi_{pd}(C_n),$$

which implies

$$b^+(C_n, \chi_{pd}) = 2 \quad \text{for even } n \text{ under adjacent edge removal.}$$

Subcase 2.2: Nonadjacent edge removal. Removing two nonadjacent edges $E' = \{e_i, e_j\}$ from C_n produces two path components P_k and P_{n-k}. Each component requires at least one color class to be power dominated by each vertex. By theorem 3.3 $\chi_{pd}(C_n - E') = 3$. Hence,

$$\chi_{pd}(C_n - E') = 3 > \chi_{pd}(C_n).$$

Therefore,

$$b^+(C_n, \chi_{pd}) = 2 \quad \text{for even } n \text{ under nonadjacent edge removal.}$$

Combining all subcases completes the proof. □

Theorem 3.7. *[12] For $K_n, n \geq 1$, then $\chi_{pd}(K_n) = n$*

Theorem 3.8. *For $K_n, n \geq 1$,*

$$b^-(K_n, \chi_{pd}) = 1, \qquad b^+(K_n, \chi_{pd}) = \infty.$$

Proof. Removing one edge makes two vertices nonadjacent, allowing them to share a color, reducing $\chi_{pd}(K_n)$ to $n-1$, so $b^- = 1$. Since no graph on n vertices can require more than n colors, increasing beyond n is impossible, so $b^+ = \infty$. □

Theorem 3.9. *[12] For $K_{m,n}$, $\chi_{pd}(K_{m,n}) = 2$*

Theorem 3.10. *For $K_{m,n}$,*

$$b^-(K_{m,n}, \chi_{pd}) = \infty, \qquad b^+(K_{m,n}, \chi_{pd}) = 1.$$

Proof. Since $\chi_{pd} = 2$ cannot be reduced for any nontrivial connected graph, $b^- = \infty$. Removing a single carefully chosen edge may make it impossible for a vertex to power dominate its opposite partition, forcing an additional color, so $b^+ = 1$. □

Theorem 3.11. *[4] For any $n \geq 3$, the power dominator chromatic number of the wheel graph $W_{1,n}$ is given by*

$$\chi_{pd}\big(W_{1,n}\big) = \begin{cases} 3, & \text{if } n \text{ is even,} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$$

In this paper, the wheel graph $W_{1,n}$ is denoted by W_n. Both notations refer to the same structure, namely the graph obtained by joining each vertex of the cycle graph C_n to a single central (hub) vertex.

Theorem 3.12. *Let W_n be a wheel graph of order $n \geq 3$. Then the lower bondage number with respect to power dominator coloring of W_n is*

$$b^-(W_n, \chi_{pd}) = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ n, & \text{if } n \text{ is even}. \end{cases}$$

Proof. To determine the lower bondage number of the wheel graph W_n, we identify the minimum number of edges whose removal decreases the power dominator chromatic number of W_n.

Let $v_1, v_2, \ldots, v_n$ be rim vertices of the wheel graph W_n, and let v_{n+1} denote the central vertex.

The edge set of the wheel graph W_n is composed of two categories of edges. The *spoke edges* are given by $e_1 = v_1v_{n+1}$, $e_2 = v_2v_{n+1}$, $\ldots$, $e_n = v_nv_{n+1}$, which connect the central vertex to each vertex of the outer cycle, while the *cycle edges* are $e_{n+1} = v_1v_2$, $e_{n+2} = v_2v_3$, $\ldots$, $e_{2n} = v_nv_1$, which together form the rim of the wheel.

We now consider the following two cases.

Case 1: n is odd.

Consider the removal of the cycle edge $e_{2n} = v_nv_1$ from W_n. Define the color class $C^1 = \{v_{n+1}\}$. The vertices $v_1, v_2, \ldots, v_n$ power dominate the color class C^1, and the central vertex v_{n+1} power dominates its own color class C^1.

The remaining vertices are properly colored as

$$C^2 = \{v_{2i+1} \mid 0 \leq i \leq \lfloor n/2 \rfloor\},$$

$$C^3 = \{v_{2i} \mid 0 \leq i \leq \lfloor n/2 \rfloor\}.$$

Hence the resulting graph admits a power dominator coloring using three colors, that is, $\chi_{pd}(W_n) = 3$. Since the removal of a single edge reduces the power dominator chromatic number, we obtain

$$b^-(W_n, \chi_{pd}) = 1.$$

Case 2: n is even.

First, remove all spoke edges $e_1, e_2, \ldots, e_n$ from W_n. The resulting graph is the cycle C_n. For even n, we have $\chi_{pd}(C_n) = 2$.

Alternatively, removing all cycle edges $e_{n+1}, e_{n+2}, \ldots, e_{2n}$ from W_n yields the star graph $K_{1,n}$, for which $\chi_{pd}(K_{1,n}) = 2$. In both cases, exactly n edges must be removed to reduce the power dominator chromatic number of W_n. Hence, $b^-(W_n, \chi_{pd}) = n$.

This completes the proof. □

Theorem 3.13. *Let W_n be a wheel graph of order $n \geq 5$. Then the upper bondage number with respect to the power dominator coloring of W_n is given by*

$$b^+(W_n, \chi_{pd}) = \begin{cases} 5, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Let the rim vertices of the wheel graph W_n be $v_1, v_2, \ldots, v_n$ and let v_{n+1} denote the central vertex.

Case 1: n is odd.

Remove the edges $e_1 = v_1v_{n+1}, \quad e_n = v_nv_{n+1}, \quad e_{n+1} = v_1v_2, \quad e_{2n} = v_1v_n, \quad e_{2n-1} = v_nv_{n-1}$.

The removal of these edges disconnects the graph and isolates the vertices v_1 and v_n. Since each isolated vertex power dominates only itself, they must receive distinct colors. Let $C^1 = \{v_1\}, \quad C^2 = \{v_n\}$. Assign a third color to the central vertex $C^3 = \{v_{n+1}\}$.

The remaining vertices $v_2, v_3, \ldots, v_{n-1}$ are adjacent to the central vertex and hence power dominate the color class C^3, while the vertex v_{n+1} power dominates its own color class.

To properly color the remaining vertices on the rim, at most two additional colors are sufficient. Assign

$$C^4 = \{v_{2r} \mid 1 \le r \le \lfloor n/2 \rfloor\}, \quad C^5 = \{v_{2r+1} \mid 1 \le r \le \lfloor (n-2)/2 \rfloor\}.$$

Thus, the power dominator chromatic number increases only when five edges are removed. Hence,

$$b^+(W_n, \chi_{pd}) = 5 \quad \text{for odd } n.$$

Case 2: n is even.

Remove the edge $e_1 = v_1v_{n+1}$. Assign the color $C^1 = \{v_{n+1}\}$ to the central vertex. All rim vertices except v_1 are adjacent to v_{n+1} and therefore power dominate the color class C^1.

The vertex v_1 power dominates the vertex set $\{v_1, v_2, v_n\}$. Assign color 2 to the vertices v_2 and v_n:

$$C^2 = \{v_2, v_n\}.$$

Hence, the vertex v_1 power dominates the color class C^2, while the central vertex v_{n+1} power dominates its own color class C^1. By the observation of the previous theorem, the power dominator chromatic number increases with the removal of a single edge. Therefore,

$$b^+(W_n, \chi_{pd}) = 1 \quad \text{for even } n.$$

□

Theorem 3.14. *[4] For $H_n, n \ge 3$, $\chi_{pd}(H_n) = n + 1$*

Theorem 3.15. *[7] For spider $SP(b_1{}^{y_1}, b_2{}^{y_2} \ldots b_m{}^{y_m})$, $\chi_{pd}\left(SP(b_1{}^{y_1}, b_2{}^{y_2} \ldots b_m{}^{y_m})\right) =$*

$$\begin{cases} 3, & \text{if there exists at least one } b_i > 2,\ 1 \le i \le m \\ 2, & \text{if every path has length at most 1} \end{cases}$$

Theorem 3.16. *For $H_n, n \ge 3$,*

$$b^+(H_n, \chi_{pd}) = 1, \qquad b^-(H_n, \chi_{pd}) = 3.$$

Proof. Let H_n be a helm graph with vertex set consisting of the cycle vertices $\{y_1, y_2, \ldots, y_n\}$, the pendant vertices $\{h_1, h_2, \ldots, h_n\}$, and the central vertex h_0.

Let $\{e_1, e_2, \ldots, e_n\}$ denote the pendant edges, where $e_i = y_i h_i$ for $1 \leq i \leq n$, and let $\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ denote the edges joining the central vertex h_0 to the cycle vertices.

Consider the removal of any pendant edge, say $e_1 = y_1 h_1$. The resulting graph becomes disconnected and consists of two components, G_1 and G_2, where $G_1 \cong K_1$ is the isolated vertex h_1, and $G_2 = H_n - \{h_1\}$.

To power dominate the isolated vertex h_1, assign color 1 to it. In the component G_2, each pendant vertex power dominates its closed neighborhood. Hence, assign distinct colors $2, 3, \ldots, n$ to the cycle vertices $y_2, y_3, \ldots, y_n$, respectively. Assign color $n+1$ to the central vertex h_0 and to the remaining pendant vertices of G_2.

The vertex y_1 power dominates the color classes C^2 and C^n. Therefore, in order to obtain a proper power dominator coloring, assign a new color $n+2$ to y_1.

Thus, the removal of a single edge increases the power dominator chromatic number by one, implying

$$b^+(H_n, \chi_{pd}) = 1.$$

The removal of all pendant edges of the helm graph H_n increases the power dominator chromatic number. Hence, pendant edges do not contribute to the reduction of $\chi_{pd}(H_n)$.

To determine the lower power dominator bondage number, we consider the removal of the cycle edges of H_n. The deletion of all n cycle edges converts H_n into a spider graph $SP({b_1}^{y_1}, {b_2}^{y_2} \ldots {b_m}^{y_m})$ with center vertex h_0.

By the previously established theorem, the power dominator chromatic number of a spider graph satisfies

$$\chi_{pd}\left(SP({b_1}^{y_1}, {b_2}^{y_2} \ldots {b_m}^{y_m})\right) = 3$$

Thus, the removal of the n cycle edges reduces the power dominator chromatic number of H_n to 3. Since fewer than n edge deletions do not yield such a reduction, the minimum number of edges whose removal decreases the power dominator chromatic number is n. Therefore,

$$b^-(H_n, \chi_{pd}) = n.$$

□

4. Conclusion

This paper introduced the upper and lower bondage numbers for the power dominator chromatic number and computed these values for several fundamental graph families. Our study shows that $\chi_{pd}(G)$ is highly sensitive to minor structural changes, especially in dense graphs where even a single edge removal impacts the coloring.

Future research may include the study of trees, grids, hypercubes, and algorithmic complexity questions related to computing bondage numbers for power dominator coloring.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. Bala Samuvel, R. Stella Maragatham, K. Prem, A. Divya Jebaseeli, Impact of Vertex Duplication on Total Power Dominator Coloring in Graphs, IAENG Int. J. Comput. Sci. 52 (2025), 3991–4015.

[2] M.L. Caay, Equitable Rings Domination in Graphs, J. Algebr. Syst. 13 (2025), 157–168. https://doi.org/10.22044/jas.2023.12812.1693.

[3] L. Girish, K. Somasundaram, k-Fault Tolerant Power Domination Number for Block Graphs, J. Combin. Math. Combin. Comput. 130 (2026), 3–18. https://doi.org/10.61091/jcmcc130-01.

[4] A. Uma Maheswari, J. Bala Samuvel, Power Dominator Chromatic Number for Some Special Graphs, Int. J. Innov. Technol. Explor. Eng. 8 (2019), 3957–3960. https://doi.org/10.35940/ijitee.l3466.1081219.

[5] S. Madhumitha, S. Naduvath, Rainbow Dominator Coloring of Graphs, Creat. Math. Inform. 34 (2025), 317–327. https://doi.org/10.37193/cmi.2025.02.14.

[6] M. Miotk, On a Class of Graphs with Equal Domination and Certified Domination Numbers, Discret. Math. Lett. 16 (2025), 29–34. https://doi.org/10.47443/dml.2025.085.

[7] G. Navamani, L. Jacquline, V. Sumathi, A. Jyothi Bala, Comparative Analysis of Power Dominator and Power Dominator Equitable Coloring for Specific Tree Structures, IAENG Int. J. Appl. Math. 56 (2026), 988–993.

[8] I. Rios-Villamar, A. Cabrera-Martínez, G. Reyna Hernández, J.M. Sigarreta, Double Total Domination Number of Some Graph Operators, Bull. Malays. Math. Sci. Soc. 49 (2026), 18. https://doi.org/10.1007/s40840-025-02016-y.

[9] T.N.P. Saibavani, N. Parvathi, Power Domination in Different Graphs with Applications, Math. Model. Eng. Probl. 10 (2023), 546–550. https://doi.org/10.18280/mmep.100222.

[10] H. Nouri Samani, S. Alikhani, N. Ghanbari, Global Dominator Chromatic Number of Certain Graphs, Math. Interdiscip. Res. 10 (2025), 183–198. https://doi.org/10.22052/mir.2025.255889.1487.

[11] H. Samuel, S.K. Krishnamurthy, Power Dominator Coloring of Line Graph of Certain Graphs, Appl. Math. E-Notes 25 (2025), 497–503.

[12] K. Sathish Kumar, N. Gnanamalar David, K.G. Subramanian, Graphs and Power Dominator Colorings, Ann. Pure Appl. Math. 11 (2016), 67–71.

[13] S. Varghese, S. Varghese, A. Vijayakumar, Power Domination in Mycielskian of Spiders, AKCE Int. J. Graphs Comb. 19 (2022), 154–158. https://doi.org/10.1080/09728600.2022.2082900.