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The Weibull-Rayleigh Distribution: Moments and Characterizations Through Generalized Order Statistics

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Abstract. Some recurrence relations for the single and product moments of the Weibull–Rayleigh distribution (WRD) are examined within the framework of generalized order statistics (GOS). The order statistics (OS), record values, and progressively type-II censored samples, are also discussed as a special case of GOS. Furthermore, the Weibull–Rayleigh distribution is characterized based on recurrence relations.

1. Introduction

1.1. **Weibull Rayleigh Distribution.** Statistical modeling plays a crucial role in understanding complex systems, enabling prediction, informed decision-making, and the extraction of data-driven insights. In particular, lifetime data arise in various disciplines, including reliability engineering, biomedical research, actuarial science, demography, and environmental studies. Each of these fields encounters distinct sources of uncertainty that can be effectively managed through suitable statistical models. Accurate modeling of lifetime data allows researchers and practitioners to identify meaningful patterns and make well-founded decisions. To meet these analytical demands, a wide range of lifetime distributions has been proposed in the literature, one of which involves the inverse of the exponentiated Weibull hazard rate function(HRF), particularly in the context of change point analysis (CPA).

The CPA has emerged as a powerful statistical tool for detecting gaps and turning points in real-life data. This method identifies specific time points at which a distribution undergoes significant changes—whether in its mean, variance, or overall structural form. CPA has found wide-ranging applications across diverse fields, including quality control, medical diagnostics,

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performance observing, and time-series analysis in industrial or dynamic environments. The methodology typically relies on classical statistical tests, information criteria, and data-driven techniques, making it a sophisticated approach for uncovering unobserved structural shifts within data. Beyond its extensive use in applied sciences, CPA is also highly effective in examining probability distributions, particularly for identifying structural changes in their characteristics, such as parameter variations or alterations in distributional shape.

Motivated by the effectiveness of recent methodological advances, [1] introduced a new class of distributions, termed the Weibull–Rayleigh (WR) distribution. This flexible distribution effectively captures structural changes in data, making it particularly suitable for situations where distributional characteristics exhibit abrupt shifts, such as variations in treatment effects or operational conditions.

The cumulative distribution function (CDF) and the probability density function (PDF) for WR distribution are shown below

$$F_U(u) = 1 - e^{-\frac{1}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}, \quad u > 0 \tag{1.1}$$

$$f_U(u) = \frac{\lambda(1-\alpha)u^{2\lambda(1-\alpha)-1}}{\theta^2(\beta\alpha)^{2\lambda}} e^{-\frac{1}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}. \tag{1.2}$$

where λ is a shape parameter, θ and $\beta > 0$ are scale parameters, and $0 < \alpha < 1$.

The survival function $S(u)$ for WR distribution is shown by the following expression

$$S(u) = e^{-\frac{1}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}.$$

The functional relation between (1.1) and (1.2) is presented by the following expression

$$\overline{F}(u) = \frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)} u^{1-2\lambda(1-\alpha)} f(u). \tag{1.3}$$

The r^{th} moments of WRD can be expressed as

$$\mu_r = \left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{r}{2\lambda(1-\alpha)}} \Gamma\left(\frac{r}{2\lambda(1-\alpha)} + 1\right).$$

1.2. **Generalized Order Statistics.** The *GOS* model is based on ascending ordered random variables $(ORVs)$ and offers a unified framework that encompasses several existing models of *ORVs*. The concept of *GOS* was first introduced by [2]. Since its introduction, the model has gained increasing attention and interest from researchers due to its flexibility and broad applicability. The GOS is defined as follows.

For simplicity, let, F throughout represent the *CDF* with *PDF* f. The random variables (RV_s) $U(1, n, \widetilde{m}, k), \ldots, U(n, n, \widetilde{m}, k)$ are called *GOS* based on F if their joint *PDF* as follows

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\overline{F}_U(u_i)]^{m_i} f_U(u_i)\right)[\overline{F}_U(u_n)]^{k-1} f_U(u_n) \tag{1.4}$$

TABLE 1. The Sub models of WR Distribution

S. No.	Distribution	β	α	λ	θ	U
1	Rayleigh	-1	-1	$\frac{1}{2}$	-	-
2	Weibull	1	-	$\frac{1}{2}$	$\frac{1}{\sqrt{2\theta}}$	$(\alpha y^{\alpha})^{\frac{1}{1-\alpha}}$
3	Weibull-Exponential	-	-	$\frac{\lambda}{2}$	$\frac{1}{\sqrt{2\theta}}$	$(\alpha^2 y)^{\frac{1}{1-\alpha}}$
4	Rayleigh-Rayleigh	$2\beta^2$	-	1	1	$y^{\frac{2}{1-\alpha}}$
5	Gompertz	$\frac{1}{\beta}$	-	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$(e^{\alpha y}-1)^{\frac{1}{1-\alpha}}$
6	Exponentiated exponential	1	-	-	$\frac{1}{\sqrt{2}}$	$\left\{\alpha\left[-ln\left[1-(1-e^{-\alpha y})^{\lambda}\right]\right]^{\frac{1}{2\lambda}}\right\}^{\frac{1}{1-\alpha}}$
7	Weibull-Lomax	-	-	$\frac{\lambda}{2}$	$\frac{1}{\sqrt{2}}$	$\left[\alpha\theta^2 log(1+\alpha y)\right]^{\frac{1}{1-\alpha}}$

TABLE 2. Descriptive measures of WRD for selected parameters

$\theta=\beta=1,\ \alpha=0.25$					$\theta=\beta=1,\ \alpha=0.50$			
λ	**Mean**	**variance**	**skewness**	**Kurtosis**	**mean**	**variance**	**skewness**	**kurtosis**
0.25	4.012	197.923	13.601	437.584	96.0	635904.0	60.092	13216.7
0.50	0.473	0.409	3.121	18.987	2.0	20.0	6.619	87.72
1.00	0.226	0.023	1.072	4.390	0.5	0.25	2.0	9.0
1.50	0.190	0.008	0.481	3.001	0.358	0.059	1.072	4.390
2.00	0.177	0.004	0.168	2.729	0.313	0.027	0.631	3.245

for $F^{-1}(0+) < u_1 \leq u_2 \leq \ldots \leq u_n < F^{-1}(1)$, with parameters $n \in N$, $n \geq 2$, $k \geq 1$, $\widetilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$ and $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j > 0$ for all $1 \leq r \leq n-1$.

For $\gamma_i \neq \gamma_j$, $i \neq j$ for all $i, j \in (1,\ 2, \ldots, n)$, the *PDF* of $f_{U(r,n,\widetilde{m},k)}(u)$ is given by [3]

$$f_{U(r,n,\widetilde{m},k)}(u) = C_{r-1} \sum_{i=1}^{r} a_i(r) f_U(u) [\overline{F}_U(u)]^{\gamma_i - 1} \tag{1.5}$$

The joint pdf of $f_{U(r,s,n,\widetilde{m},k)}(u)$ and $f_{U(s,n,\widetilde{m},k)}(v)$, $1 \leq r < s \leq n$ is given by [3]

$$f_{UV(r,\ s,n,\widetilde{m},k)}(u,v) = C_{s-1} \sum_{j=r+1}^{s} a_j^{(r)}(s) \left(\frac{\bar{F}_V(v)}{\bar{F}_U(u)}\right)^{\gamma_j} \left[\sum_{i=1}^{r} a_i(r) [\bar{F}(u)]^{\gamma_i}\right] \frac{f(u)}{\bar{F}_U(u)} \frac{f(v)}{\bar{F}_V(v)} \tag{1.6}$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n,$$

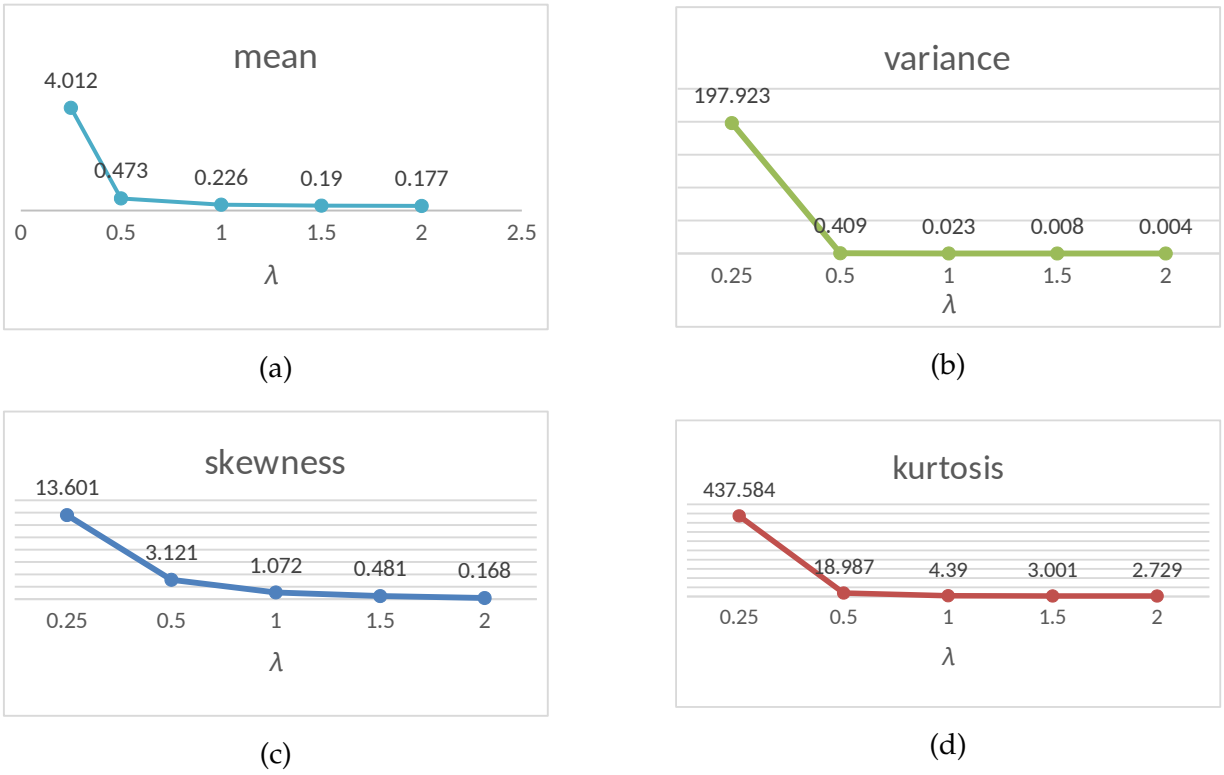


FIGURE 1. Plots of descriptive measures for different parameters values $\theta = \beta = 1$, $\alpha = 0.25$ and $\lambda = 0.25 - 2.0$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j\neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \leq i \leq s \leq n.$$

It is worth noting that for $m_1 = m_2 = \ldots = m_{n-1} = m \neq 1$

$$a_i(r) = \frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!}$$

and

$$a_j^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}}(-1)^{s-j}\frac{1}{(j-i-1)!(s-j)!}$$

Therefore the *PDF* given in (1.5) reduces to

$$f_{U(r,n,m,k)}(u) = \frac{C_{r-1}}{(r-1)!}[\overline{F}_U(u)]^{\gamma_r - 1} f_U(u) g_m^{r-1}[F_U(u)] \tag{1.7}$$

and the joint *PDF* given in (1.6) reduces to

$$\begin{aligned} f_{UV(r,s,\ n,m,k)}(u,v) = {} & \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\overline{F}_U(u)]^m f_U(u) g_m^{r-1}(F_U(u)) \\ & \times [h_m(F_V(v)) - h_m(F_U(u)]^{s-r-1}[\overline{F}_V(v)]^{\gamma_s - 1} f_V(v), \quad u < v \end{aligned} \tag{1.8}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

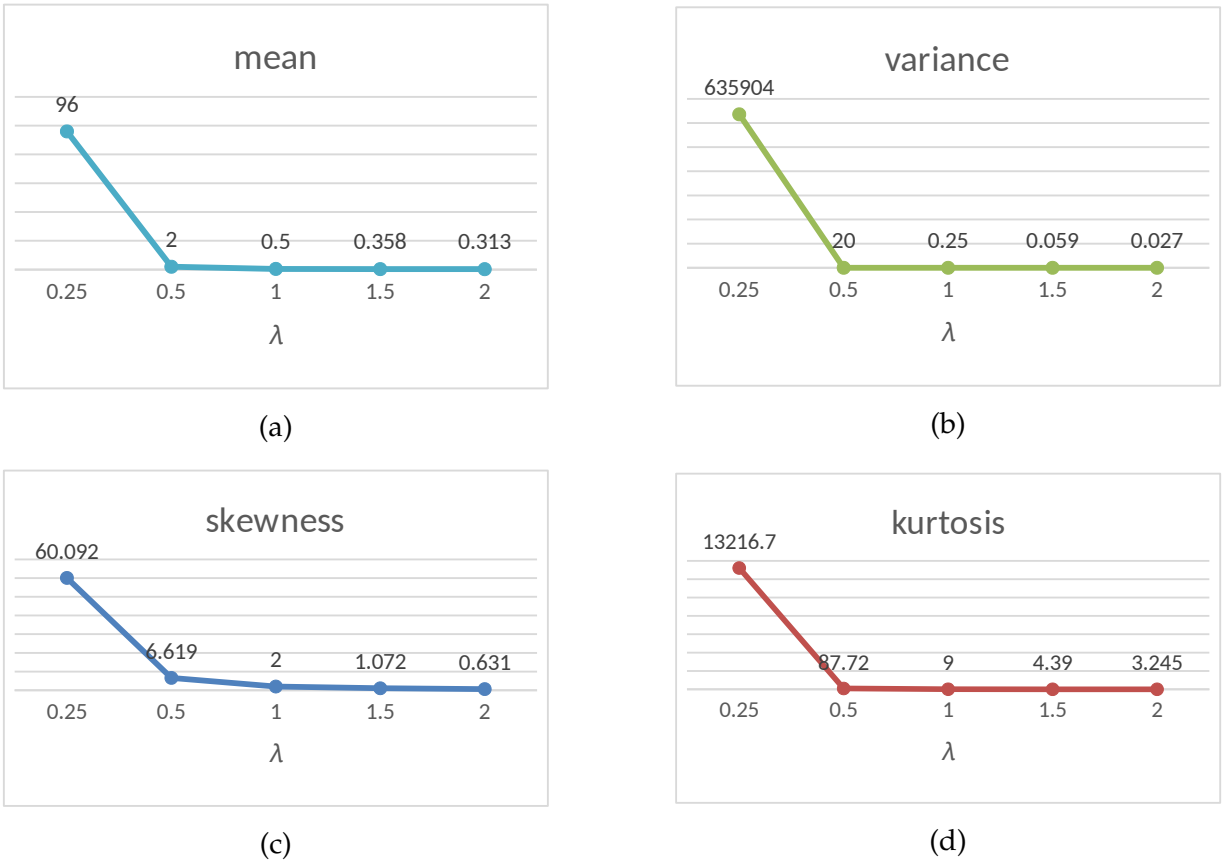


FIGURE 2. Plots of descriptive measures for different parameters values $\theta = \beta = 1$, $\alpha = 0.50$ and $\lambda = 0.25 - 2.0$.

$$h_m(u) = \begin{cases} \frac{1}{m+1}[-(1-u)^{m+1}], & m \neq -1, \\ -\ln(1-u), & m = -1 \end{cases}$$

and

$$g_m(u) = h_m(u) - h_m(0), \quad u\epsilon[0,1).$$

The model of ordered random scheme is described in the following Table 3.

Several publications have examined the moments of recurrence relations of several distribution under the frame of *GOS* in the literature. The characterizations of probability distribution via different techniques have also been studied by many researchers. Detailed studies are reported in [4–15] and many more exist in literature. For other aspects of theory and applications of GOS, interested readers may also refer to [16] providing thorough insights into the wider field of *GOS* analysis. Our findings seek to enhance by introducing moments problems and characterizations formulated through recurrence relations of WRD.

The remaining part of the manuscript is as follows. Section 2 is devoted to single moments and its deductions for several models of ordered random schemes. The product moments are proved

TABLE 3. Variants of *GOS*.

S. No.	Model	k	γ_r	m_r
1	Pfeifer's record	β_n	β_r	$\beta_r - \beta_{r+1} - 1$
2	Progressively type II censored order statistics	$R_n + 1$	$(n-r+1) + \sum_{j=1}^{n} R_j$	R_γ
3	Record statistics	1	1	-1
4	Ordinary order statistics	1	$n-r+1$	0
5	Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$\gamma_r - \gamma_{r+1} - 1$

in Section 3 with its many deductions. Section 4 involves characterization results. Finally, the conclusion is presented in the last section.

2. Single Moments

We prove the exact expressions and derive the relationships between the single moments of GOS from the WRD in this section. Consider the following notation for computational convenience and clarity

$$E[U^x\,(r,\ n, \widetilde{m},\ k)\ ] = \mu^{(x)}_{r,\ n,\widetilde{m},\ k}$$

Theorem 2.1. *For WRD as given in (1.2) and let $\gamma_i \neq \gamma_j$ be satisfied for $1 \leq r \leq n,\ x = 1,\ 2, \ldots$, then*

$$\mu^{(x)}_{r,\ n,\widetilde{m},\ k} = C_{r-1}\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x}{2\lambda(1-\alpha)}}\Gamma\left(\frac{x}{2\lambda(1-\alpha)}+1\right)\frac{\sum_{i=1}^{r} a_i(r)}{\gamma_i^{\frac{x}{2\lambda(1-\alpha)}+1}} \tag{2.1}$$

Proof. Considering (1.5), we have

$$\begin{aligned}\mu^{(x)}_{r,\ n,\widetilde{m},\ k} &= C_{r-1}\int_0^\infty u^x \sum_{i=1}^{r} a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}\frac{f_U(u)}{\overline{F}_U(u)}du \\ &= C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^\infty u^x \frac{\lambda(1-\alpha)u^{2\lambda(1-\alpha)-1}}{\theta^2(\beta\alpha)^{2\lambda}} e^{-\frac{\gamma_i}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}du. \end{aligned} \tag{2.2}$$

Let $t = \frac{\gamma_i}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}$ in (2.2). Thus, we have

$$\mu^{(x)}_{r,\ n,\widetilde{m},\ k} = C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^\infty \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x}{2\lambda(1-\alpha)}}}{\gamma_i^{\frac{x}{2\lambda(1-\alpha)}}} t^{\frac{x}{2\lambda(1-\alpha)}} e^{-t}\frac{dt}{\gamma_i}.$$

Hence, relation (2.1) is proved.

Corollary 2.1. *Let $m_1 = m_2 = \ldots = m_{n-1} = m \neq 1$ is hold, then the following relationship*

$$\mu^{(x)}_{r,\ n,m,\ k} = \frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x}{2\lambda(1-\alpha)}}\Gamma\left(\frac{x}{2\lambda(1-\alpha)}+1\right)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{\gamma_i^{\frac{x}{2\lambda(1-\alpha)}+1}}$$

holds under the same conditions as described in Theorem 2.1.

Proof. From [17] for $\gamma_i \neq \gamma_j$ but $m_i = m_j = m,\ i, j = 1, 2, \ldots,\ n-1$, we have

$$a_i(r) = \frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!}$$

Hence, the corollary 2.1 can be proved easily.

Remark 2.1. *For single moments of OS from (2.1) as follows*

$$\mu^{(x)}_{r:\,n} = C_{r:n}\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x}{2\lambda(1-\alpha)}}\Gamma\left(\frac{x}{2\lambda(1-\alpha)}+1\right)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{[n-r+i+1]^{\frac{x}{2\lambda(1-\alpha)}+1}}$$

Remark 2.2. *Single moments of k^{th} record values from (2.1) as*

$$\mu^{(x)}_{U^{(k)}_n} = \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x}{2\lambda(1-\alpha)}}\Gamma\left(\frac{x}{2\lambda(1-\alpha)}+1\right)}{\Gamma n\, k^{\frac{x}{2\lambda(1-\alpha)}}}.$$

TABLE 4. Moments of WRD based on upper record values for selected parameters

n	$\theta=\beta=1,\ \alpha=0.25$					$\theta=\beta=1,\ \alpha=0.50$				
	$\lambda=0.25$	$\lambda=0.50$	$\lambda=1.0$	$\lambda=1.5$	$\lambda=2.0$	$\lambda=0.25$	$\lambda=0.50$	$\lambda=1.0$	$\lambda=1.5$	$\lambda=2.0$
1	4.06	0.46	0.23	0.19	0.18	96	2.0	0.5	0.36	0.31
2	4.06	0.46	0.23	0.19	0.18	96	2.0	0.5	0.36	0.31
3	2.03	0.23	0.12	0.10	0.9	48	1.0	0.25	0.18	0.16
4	0.68	0.08	0.04	0.03	0.03	16	0.33	0.08	0.06	0.05
5	0.17	0.02	0.01	0.01	0.01	4	0.08	0.02	0.02	0.01

TABLE 5. Variance of WRD based on upper record values for selected parameters

n	$\theta=\beta=1,\ \alpha=0.25$					$\theta=\beta=1,\ \alpha=0.50$				
	$\lambda=0.25$	$\lambda=0.50$	$\lambda=1.0$	$\lambda=1.5$	$\lambda=2.0$	$\lambda=0.25$	$\lambda=0.50$	$\lambda=1.0$	$\lambda=1.5$	$\lambda=2.0$
1	197	0.41	0.023	0.008	0.004	635904	20	0.25	0.06	0.03
2	197	0.41	0.023	0.008	0.004	635904	20	0.25	0.06	0.03
3	98.5	0.21	0.012	0.004	0.002	317952	10	0.13	0.03	0.012
4	32.8	0.07	0.004	0.0013	0.0007	105984	3.33	0.04	0.01	0.005
5	8.21	0.02	0.0001	0.0003	0.0002	26496	0.83	0.01	0.003	0.013

Theorem 2.2. *Recurrence relation for moments of GOS for WRD is noted under the same conditions as mentioned in Theorem 2.1*

$$\mu^{(x)}_{r,\,n,\widetilde{m},\,k} - \mu^{(x)}_{r-1,\,n,\widetilde{m},\,k} = \frac{x}{\gamma_r}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x-2\lambda(1-\alpha))}_{r,\,n,\widetilde{m},\,k} \tag{2.3}$$

Proof. From [18], we have

$$E\left[\xi\left\{U\left(r,\ n,\widetilde{m},\ k\right)\right\}\right]-E\left[\xi\left\{U\left(r-1,\ n,\widetilde{m},\ k\right)\right\}\right]=C_{r-2}\int_{-\infty}^{\infty}\xi^{'}(u)\sum_{i=1}^{r}a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}du. \tag{2.4}$$

By putting $\xi(U)=u^x$ in (2.4), we get

$$\mu^{(x)}_{r,\ n,\widetilde{m},\ k}-\mu^{(x)}_{r-1,\ n,\widetilde{m},\ k}=xC_{r-2}\int_{-\infty}^{\infty}u^{x-1}\sum_{i=1}^{r}a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}dx. \tag{2.5}$$

Now, using (1.3) in (2.5), we get the required result.

Corollary 2.2. *For $m_1=m_2=\ldots=m_{n-1}=m\neq 1$, the expression is given as*

$$\mu^{(x)}_{r,\ n,m,\ k}-\mu^{(x)}_{r-1,\ n,m,\ k}=\frac{x}{\gamma_r}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x-2\lambda(1-\alpha))}_{r,\ n,m,\ k}$$

Proof. Corollary 2 is obtained by replacing $\widetilde{m}$ with m in (2.4).

Remark 2.3. *The following expression is deduced from (2.3) for OS*

$$\mu^{(x)}_{r:\ n}-\mu^{(x)}_{r-1:\ n}=\frac{x}{(n-r+1)}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x-2\lambda(1-\alpha))}_{r:\ n}$$

Remark 2.4. *The following expression is deduced from (2.3) for k^{th} record values*

$$\mu^{(x)}_{U^{(k)}_n}-\mu^{(x)}_{U^{(k)}_{n-1}}=\frac{x}{k}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x-2\lambda(1-\alpha))}_{U^{(k)}_n}$$

Remark 2.5. *For m ordered progressive type II censored data, the following relation obtained from (2.3) as*

$$\mu^{\widetilde{R}(x)}_{r:\ m:n}-\mu^{\widetilde{R}(x)}_{r-1:m:\ n}=\frac{x}{(n-r+1)}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{\widetilde{R}(x-2\lambda(1-\alpha))}_{r:m:\ n}$$

with relation $U^{\widetilde{R}}_{1:m:n}<\ldots<U^{\widetilde{R}}_{r:m:n}<\ldots<U^{\widetilde{R}}_{m:m:n}$ and $\gamma_r=m-r+1+\sum_{l=r}^{m}R_l$, $1\leq m\leq n$, $r=1,2,\ldots,m$.

3. Product Moments

The product moments based on *GOS* are enabled to compute the covariance of other model of ordered random variables. Throughout the derivation, we shall assume

$$E\left[U^x\left(r,n,\widetilde{m},\ k\right)U^y\left(r,n,\widetilde{m},\ k\right)\right]=\mu^{(x,y)}_{r,s,n,\widetilde{m},k}$$

Theorem 3.1. *For $1<r<s\leq n,\ \ x,\ y=1,\ 2,\ \ldots,$ and let $\gamma_i\neq\gamma_j$ be satisfied for WRD. The relation holds for product moments as follows*

$$\mu^{(x,y)}_{r,s,n,\widetilde{m},k}=\frac{C_{s-1}}{\gamma_j}\frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x+y}{2\lambda(1-\alpha)}}}{\left(\frac{x}{2\lambda(1-\alpha)}+1\right)\gamma_i^{\frac{x+y}{2\lambda(1-\alpha)}+1}}\sum_{i=1}^{r}a_i(r)\sum_{j=r+1}^{s}a^{(r)}_j(s)$$
$$\times 2F_1\left(1,\ \frac{x+y}{2\lambda(1-\alpha)}+1,\frac{x}{2\lambda(1-\alpha)}+2,\frac{\gamma_i-\gamma_j}{\gamma_i}\ \right) \tag{3.1}$$

Proof. Considering (1.6), we note that

$$\mu_{r,s,n,\widetilde{m},k}^{(x,y)} = C_{s-1}\sum_{i=1}^{r} a_i(r)\sum_{j=r+1}^{s} a_j^{(r)}(s)\int_0^\infty\int_u^\infty u^x v^y\left[\frac{\overline{F}_V(v)}{\overline{F}_U(u)}\right]^{\gamma_j}\left[\overline{F}_U(u)\right]^{\gamma_i}\frac{f_U(u)}{\overline{F}_U(u)}$$
$$\times\frac{f_V(v)}{\overline{F}_V(v)}dvdu.$$
$$= C_{s-1}\sum_{i=1}^{r} a_i(r)\sum_{j=r+1}^{s} a_j^{(r)}(s)\int_0^\infty u^x\left[\overline{F}_U(u)\right]^{\gamma_i-\gamma_i-1} f_U(u)I(u)du \tag{3.2}$$

where

$$I(u) = \int_u^\infty v^y\left[\overline{F}_V(v)\right]^{\gamma_j-1} f_V(v)dv$$
$$= \int_u^\infty v^y e^{-\frac{\gamma_i}{2\theta^2}\left(\frac{v^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}\frac{\lambda(1-\alpha)v^{2\lambda(1-\alpha)-1}}{\theta^2(\beta\alpha)^{2\lambda}}dv.$$

Let

$$t=\frac{\gamma_i}{2\theta^2}\left(\frac{v^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}$$
$$= \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{y}{2\lambda(1-\alpha)}}}{\gamma_j^{\frac{y}{2\lambda(1-\alpha)}+1}}\int_{x(u)}^\infty t^{\frac{y}{2\lambda(1-\alpha)}}e^{-t}dt.$$
$$=\frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{y}{2\lambda(1-\alpha)}}}{\gamma_j^{\frac{y}{2\lambda(1-\alpha)}+1}}\Gamma\left(\frac{y}{2\lambda(1-\alpha)}+1,\ x(u)\right). \tag{3.3}$$

On using (3.3) in (3.2), we obtain

$$\mu_{r,s,n,\widetilde{m},k}^{(x,y)} = \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{y}{2\lambda(1-\alpha)}}}{\gamma_j^{\frac{y}{2\lambda(1-\alpha)}+1}}\sum_{i=1}^{r} a_i(r)\sum_{j=r+1}^{s} a_j^{(r)}(s)\int_0^\infty u^x\left[\overline{F}_U(u)\right]^{\gamma_i-\gamma_i-1} f_U(u)$$
$$\times\Gamma\left(\frac{y}{2\lambda(1-\alpha)}+1,\ \frac{\gamma_i}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}\right)du. \tag{3.4}$$

Again using (1.1) and (1.2) in (3.4), we obtain

$$= \int_0^\infty u^x\left[\overline{F}_U(u)\right]^{\gamma_i-\gamma_i}\frac{f_U(u)}{\overline{F}_U(u)}\Gamma\left(\frac{y}{2\lambda(1-\alpha)}+1,\ \frac{\gamma_i}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}\right)du.$$
$$= \int_0^\infty u^x\left(e^{-\frac{1}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}\right)^{\gamma_i-\gamma_j}\frac{\lambda(1-\alpha)u^{2\lambda(1-\alpha)-1}}{\theta^2(\beta\alpha)^{2\lambda}}\Gamma\left(\frac{y}{2\lambda(1-\alpha)}+1,\ \frac{\gamma_i}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}\right)du. \tag{3.5}$$

Let

$$z = \frac{1}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}$$

then (3.5) reduces to

$$= \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{y}{2\lambda(1-\alpha)}} C_{s-1}}{\gamma_j^{\frac{y}{2\lambda(1-\alpha)}+1}} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s)$$
$$\times \int_0^\infty z^{\frac{x}{2\lambda(1-\alpha)}} \left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x}{2\lambda(1-\alpha)}} e^{-(\gamma_i-\gamma_j)z} \Gamma\left(\frac{y}{2\lambda(1-\alpha)}+1,\ \gamma_j z\right) dz$$
$$= \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x+y}{2\lambda(1-\alpha)}} C_{s-1}}{\gamma_j^{\frac{y}{2\lambda(1-\alpha)}+1}} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s)$$
$$\times \int_0^\infty z^{\frac{x}{2\lambda(1-\alpha)}} e^{-(\gamma_i-\gamma_j)z} \Gamma\left(\frac{y}{2\lambda(1-\alpha)}+1,\ \gamma_j z\right) dz. \tag{3.6}$$

From [19] p. No. 657, we note

$$\int_0^\infty u^{\mu-1} e^{-\beta u} \Gamma(v,\ \alpha u) du = \frac{\alpha^v \Gamma\mu + v}{\mu(\alpha+\beta)^{\mu+v}}\, 2F_1\left(1, \mu+v;\ \mu+1; \frac{\beta}{\alpha+\beta}\right). \tag{3.7}$$

Now by using (3.7) in (3.6), we obtain (3.1).

Corollary 3.1. *Suppose $m_i = m_j = m,\ i, j = 1, 2, \ldots,\ n-1$ is satisfied. Then the following relation noted as*

$$\mu_{r,s,n,\widetilde{m},k}^{(x,y)} = \frac{C_{s-1}}{\gamma_{s-j}(r-1)!(s-r-1)!} \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x+y}{2\lambda(1-\alpha)}}}{\left(\frac{x}{2\lambda(1-\alpha)}+1\right)\gamma_{r-i}^{\frac{x+y}{2\lambda(1-\alpha)}+1}} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times 2F_1\left(1,\ \frac{x+y}{2\lambda(1-\alpha)}+1, \frac{x}{2\lambda(1-\alpha)}+2, \frac{\gamma_{r-i}-\gamma_{s-j}}{\gamma_{r-i}}\right).$$

under the same conditions noted in Theorem 3.1.

Proof. The corollary 3.1 can be proved in the view of [17].

Remark 3.1. *The expression for product moment of OS from (3.1) as*

$$\mu_{r,s:n}^{(x,y)} = \frac{C_{r,\,s:n}}{\gamma_{s-j}} \frac{\left[2\theta^2(\beta\alpha)^{2\lambda}\right]^{\frac{x+y}{2\lambda(1-\alpha)}}}{\left(\frac{x}{2\lambda(1-\alpha)}+1\right)(n-r+i+1)^{\frac{x+y}{2\lambda(1-\alpha)}+1}} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times 2F_1\left(1,\ \frac{x+y}{2\lambda(1-\alpha)}+1, \frac{x}{2\lambda(1-\alpha)}+2, \frac{s-r+i-j}{n-r+i+1}\right)$$

Where $\mu_{r,s:n}^{(x,y)} = E\left(U_{r:n}^x,\ U_{s:n}^y\right)$ and $C_{r,\,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Theorem 3.2. *The recurrence relations of product moments of GOS for (1.2) is given as*

$$\mu_{r,s,n,\widetilde{m},k}^{(x,y)} - \mu_{r-1,s:n,\,\widetilde{m},k}^{(x,y)} = \frac{y}{\gamma_s} \frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)} \mu_{r,s,n,\widetilde{m},\,k}^{(x,y-2\lambda(1-\alpha))} \tag{3.8}$$

under the same conditions reported in Theorem 3.1.

Proof. We have by [18]

$$E\left[\xi\left\{U\left(r,\ n,\widetilde{m},\ k\right),U\left(s,\ n,\widetilde{m},\ k\right)\right\}\right]-E\left[\xi\left\{U\left(r,\ n,\widetilde{m},\ k\right),\ U\left(s-1,\ n,\widetilde{m},\ k\right)\right\}\right]$$
$$=C_{s-2}\int_0^\infty\int_u^\infty\frac{\partial}{\partial v}\xi(u,v)\left[\sum_{j=r+1}^{s}a_i(r)\left[\frac{\overline{F}_V(v)}{\overline{F}_U(u)}\right]^{\gamma_j}\right]\left[\sum_{i=1}^{r}a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}\right]\frac{f_U(u)}{\overline{F}_U(u)}dvdu. \tag{3.9}$$

Let $\xi(u,v)=\xi_1(u)\xi_2(v)=u^xv^y$ in (3.9). Thus, we have

$$=yC_{s-2}\int_0^\infty\int_u^\infty u^xv^{y-1}\left[\sum_{j=r+1}^{s}a_i(r)\left[\frac{\overline{F}_V(v)}{\overline{F}_U(u)}\right]^{\gamma_j}\right]\left[\sum_{i=1}^{r}a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}\right]\frac{f_U(u)}{\overline{F}_U(u)}dvdu. \tag{3.10}$$

Now, the relations (3.8) can be proved by using (1.3) in (3.10).

Corollary 3.2. *Let $m_1=m_2=\ldots=m_{n-1}=m\neq 1$ is satisfied and similar conditions considered in Theorem 3.1, then*

$$\mu^{(x,y)}_{r,s,n,m,k}-\mu^{(x,y)}_{r-1,s:n,\ m,k}=\frac{y}{\gamma_s}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x,\ y-2\lambda(1-\alpha))}_{r,s,n,m,\ k}$$

Remark 3.2. *Relation for product moments of OS from (3.8) as follows*

$$\mu^{(x,y)}_{r,\ s:\ n}-\mu^{(x,y)}_{r-1,\ s:\ n}=\frac{y}{(n-r+1)}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x,y-2\lambda(1-\alpha))}_{r,s,n,m,\ k}$$

Remark 3.3. *Relation for product moments of k^{th} upper record values from (3.8) as below*

$$\mu^{(x,y)}_{U^{(k)}_{n,m}}-\mu^{(x,y)}_{U^{(k)}_{n-1,m}}=\frac{y}{k}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x,yq-2\lambda(1-\alpha))}_{U^{(k)}_{n,m}}$$

Remark 3.4. *For $1\leq m\leq n$ and $\gamma_i=\sum_{j=i}^{m}(R_i+1)$, $\widetilde{R}=(R_1,\ R_2,\ldots,R_m)$ in (3.8), we note the relation of progressive type II right censored OS for product moment*

$$\mu^{\widetilde{R}(x)}_{r,\ s:\ m:n}-\mu^{\widetilde{R}(x)}_{r,s-1:m:\ n}=\frac{x}{\gamma_r}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{\widetilde{R}(x-2\alpha(1-\lambda))}_{r,s:m:\ n}$$

4. Characterization

When constructing a probabilistic model for a specific problem, a researcher must determine whether the proposed model satisfies the criteria of an appropriate underlying probability distribution. In statistical theory, many functions are inherently linked to probability distributions and serve to uniquely define them—these are known as characterizations. Researchers rely on such characterizations when selecting or validating a distribution. Before applying a model to empirical data, it is essential to ensure that the probability model fulfills the fundamental properties implied by its characterization. Studying the characterization features of probability distributions is vital in various areas of applied probability and statistics. For example, in lifetime analysis, these properties are used to assess the suitability of particular lifetime distributions. Different features of a lifetime distribution may be revealed through distinct characterizations, each highlighting a unique feature of the distribution.

In light of the significance of characterizing probability distributions, we develop characterizations between moments of *GOS* using recurrence relations.

Theorem 4.1. *Let x be a non-negative integer. A necessary and sufficient condition for a RV U to be distributed in (1.2) that*

$$\mu^{(x)}_{r,\, n,\widetilde{m},\, k} = \mu^{(x)}_{r-1,\, n,\widetilde{m},\, k} + \frac{x}{\gamma_r}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x-2\lambda(1-\alpha))}_{r,\, n,\widetilde{m},\, k} \tag{4.1}$$

Proof. From (2.3), the necessary condition satisfied.

For sufficiency.

If the relation in (4.1) is satisfied, then using [18] for $\xi(u) = u^x$, we have

$$\begin{aligned} &xC_{r-2}\int_0^\infty u^{x-1}\sum_{i=1}^{r} a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i} du \\ &= \frac{xC_{r-1}}{\gamma_r}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\int_0^\infty u^{x-2\lambda(1-\alpha)}\sum_{i=1}^{r} a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_{i-1}} f_U(u)du. \end{aligned}$$

This implies that

$$\frac{xC_{r-1}}{\gamma_r}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\int_0^\infty u^{x-1}\sum_{i=1}^{r} a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_{i-1}}\left\{\frac{\lambda(1-\alpha)}{\theta^2(\beta\alpha)^{2\lambda}}\overline{F}_U(u) - x^{1-2\lambda(1-\alpha)}f_U(u)\right\}du. \tag{4.2}$$

Applying the extension Theorem of [20] to (4.2), we obtain

$$\frac{\overline{F}_U(u)}{f_U(u)} = \frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}u^{1-2\lambda(1-\alpha)}$$

which implies

$$f_U(u) = \frac{\lambda(1-\alpha)u^{2\lambda(1-\alpha)-1}}{\theta^2(\beta\alpha)^{2\lambda}}e^{-\frac{1}{2\theta^2}\left(\frac{u^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}, u > 0.$$

Theorem 4.2. *Let x, y are non-negative integers. A necessary and sufficient condition for a RV U to be distributed in (1.2) that*

$$\mu^{(x,y)}_{r,s,n,\widetilde{m},k} = \mu^{(x,y)}_{r-1,s:n,\,\widetilde{m},k} + \frac{y}{\gamma_s}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\mu^{(x,y-2\lambda(1-\alpha))}_{r,s,n,\widetilde{m},\,k} \tag{4.3}$$

Proof. From (3.8), the necessary condition hold.

For sufficient part.

If the relation in (4.3) is satisfied, then using [18] for $\xi(u,v) = u^x v^y$, we note

$$\begin{aligned} &yC_{s-2}\int_0^\infty\int_u^\infty u^x v^{y-1}\left[\sum_{j=r+1}^{s} a_i(r)\left[\frac{\overline{F}_V(v)}{\overline{F}_U(u)}\right]^{\gamma_j}\right] \\ &\times\left[\sum_{i=1}^{r} a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}\right]\frac{f_U(u)}{\overline{F}_U(u)}\frac{f_V(v)}{\overline{F}_V(v)}dvdu \end{aligned}$$

$$= \frac{yC_{s-1}}{\gamma_s}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\int_0^\infty\int_u^\infty u^x v^{y-2\lambda(1-\alpha)}\left[\sum_{j=r+1}^{s} a_i(r)\left[\frac{\overline{F}_V(v)}{\overline{F}_U(u)}\right]^{\gamma_j}\right]$$
$$\times\left[\sum_{i=1}^{r} a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}\right]\frac{f_U(u)}{\overline{F}_U(u)}\frac{f_V(v)}{\overline{F}_V(v)}dvdu.$$

This implies that

$$= \frac{yC_{s-1}}{\gamma_s}\frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}\int_0^\infty\int_u^\infty u^x v^{y-1}\left[\sum_{j=r+1}^{s} a_i(r)\left[\frac{\overline{F}_V(v)}{\overline{F}_U(u)}\right]^{\gamma_j}\right]\left[\sum_{i=1}^{r} a_i(r)\left[\overline{F}_U(u)\right]^{\gamma_i}\right]\frac{f_U(u)}{\overline{F}_U(u)}$$
$$\times\left\{\frac{\lambda(1-\alpha)}{\theta^2(\beta\alpha)^{2\lambda}} - v^{1-2\lambda(1-\alpha)}\right\}dvdu. \tag{4.4}$$

Applying the extension Theorem of [20] to (4.4), we obtain

$$\frac{\overline{F}_V(v)}{f_V(v)} = \frac{\theta^2(\beta\alpha)^{2\lambda}}{\lambda(1-\alpha)}v^{1-2\lambda(1-\alpha)},$$

which implies

$$f_V(v) = \frac{\lambda(1-\alpha)y^{2\lambda(1-\alpha)-1}}{\theta^2(\beta\alpha)^{2\lambda}}e^{-\frac{1}{2\theta^2}\left(\frac{v^{1-\alpha}}{\beta\alpha}\right)^{2\lambda}}, v > 0.$$

Hence, Theorem 4.2 is proved.

5. Conclusion

This paper explores the generalized order statistics of the Weibull–Rayleigh distribution and its special cases. The study broadens the scope of statistical theory by deepening the understanding and applicability of the Weibull–Rayleigh distribution in both theoretical and practical contexts.

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