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A Carathéodory Theorem for a Nonlocal Integral Problem for a Nonlinear Delay Differential Equation

Khloud A.M. Hamed[1,*], Ahmed M.A. El-Sayed[2], Eman M.A. Hamdallah[2], Hanaa R. Ebead[2]

[1]*Faculty of Education, Al-jufra University, Al-jufra, Libya*
[2]*Faculty of Science, Alexandria University, Alexandria, Egypt*

[*]*Corresponding author:* Khloud.Alzarog_PG@alexu.edu.eg

Abstract. In this paper, we study a nonlinear delay differential equation subject to a nonlocal integral condition. Using Carathéodory's theorem, we prove the existence of at least one solution $x \in C[0,T]$. Uniqueness is established under an additional condition. We also examine the continuous dependence of the unique solution on the problem data and establish Hyers–Ulam stability. Several examples illustrate the applicability of our results.

1. Introduction

Delay differential equations (DDEs) extend ordinary differential equations by allowing the evolution of a system to depend on both its present state and its past behavior. This makes them suitable for modeling processes in which a time delay occurs between cause and effect. For this reason, DDEs have been widely used in biology, engineering, chemistry, economics, and environmental sciences [5,6,10].

In many models, nonlocal integral conditions are used instead of classical pointwise conditions to represent information accumulated over an interval. Such conditions arise in problems involving heat conduction, diffusion, and population dynamics.

Several authors have studied differential equations with nonlocal conditions (see [1,2,4,8]). In this paper, we investigate the existence and uniqueness of solutions, their continuous dependence on the data, and the Hyers–Ulam stability of a delay differential equation with a nonlocal integral condition.

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In this work, we study the existence and uniqueness of solutions to the following delay differential equation:

$$\frac{dx(t)}{dt} = f\Big(t, \lambda x(\gamma t)\Big), \ a.e.\ t \in [0,T], \tag{1.1}$$

with the nonlocal integral condition:

$$x(0) = x_0 + \int_0^T g\Big(s, \lambda x(\gamma s)\Big)\, ds. \tag{1.2}$$

Here, $\gamma \in (0,1]$ and $\lambda > 0$. Using Carathéodory's theorem, we establish the existence of at least one solution $x \in C[0,T]$; uniqueness follows under an additional condition. We also prove that the solution depends continuously on the initial datum x_0 and the parameter λ, and establish Hyers–Ulam stability.

2. Existence of a Solution

Consider problem (1.1)–(1.2) under the following assumptions:
(i) For every $y \in R$, the functions $f(\cdot, y)$ and $g(\cdot, y)$ are measurable on I where $I = [0,T]$, and for almost every $t \in I$, the functions $f(t,\cdot)$ and $g(t,\cdot)$ are continuous on R.
(ii) There exist positive functions $m_1, m_2 \in L^1(I)$ such that

$$|f(t,y)| \le m_1(t), \qquad |g(t,y)| \le m_2(t),$$

and $\int_0^t m_1(s)\, ds \le C$ for all $t \in I$.

Let $C[0,T]$ denote the space of continuous functions on $[0,T]$. We define the Banach space $X = C[0,T] \times C[0,T]$ with norm

$$\|(x,y)\|_X = \|x\|_c + \|y\|_c \,.$$

2.1. **Formulation of the problem.** Suppose that x is a solution to problem (1.1)–(1.2). Integrating equation (1.1) gives

$$x(t) \;=\; x(0) + \int_0^t f\Big(s, \lambda x(\gamma s)\Big)\, ds.$$

Substituting the nonlocal condition for $x(0)$ gives

$$x(t) = x_0 + \int_0^T g\Big(s, \lambda x(\gamma s)\Big)\, ds + \int_0^t f\Big(s, \lambda x(\gamma s)\Big)\, ds. \tag{2.1}$$

Conversely, differentiating (2.1) yields (1.1), while setting $t = 0$ in (2.1) yields (1.2).
Thus, (2.1) is equivalent to problem (1.1)–(1.2).
Now define

$$y(t) = \lambda x(\gamma t), \tag{2.2}$$

Then (2.1) becomes

$$x(t) = x_0 + \int_0^T g(s, y(s))\, ds + \int_0^t f(s, y(s))\, ds. \tag{2.3}$$

Conversely, substituting (2.2) into (2.3) yields (2.1). Thus, the integral equation (2.1) is equivalent to the system (2.2)–(2.3).

The coupled system (2.2)–(2.3) is also equivalent to the following integral equation:

$$\Big(x(t), y(t)\Big) = \Big(x_0 + \int_0^T g\Big(s, y(s)\Big)\, ds + \int_0^t f\Big(s, y(s)\Big)\, ds\ ,\ \lambda x(\gamma t)\Big). \tag{2.4}$$

2.2. **Existence theorem.** The following theorem establishes the existence of a solution to (2.4).

Theorem 2.1. *Let assumptions (i)–(ii) hold. Then problem (2.4) has at least one solution* $(x, y) \in C[0,T] \times C[0,T]$. *Consequently, (2.1) has at least one solution* $x \in C[0,T]$.

Proof. Consider the sequence $\{(x_n(t), y_n(t))\}$ defined by

$$(x_n, y_n) = \Big(x_0 + \int_0^T g(s, y_n(s))\, ds + \int_0^t f(s, y_n(s))\, ds\ ,\ \lambda x_n(\gamma t)\Big), n = 1, 2, ...$$

Now

$$x_n(t) = x_0 + \int_0^T g\Big(s, y_n(s)\Big)\, ds + \int_0^t f\Big(s, y_n(s)\Big)\, ds,$$

and

$$\begin{aligned}
|x_n(t)| &= |x_0 + \int_0^T g\Big(s, y_n(s)\Big)\, ds + \int_0^t f\Big(s, y_n(s)\Big)\, ds| \\
&\leq |x_0| + \int_0^T |g\Big(s, y_n(s)\Big)|\, ds + \int_0^t |f\Big(s, y_n(s)\Big)|\, ds \\
&\leq |x_0| + \int_0^T m_2(s)\, ds + \int_0^t m_1(s) ds \\
&\leq |x_0| + \int_0^T m_2(s)\, ds + C \\
&\leq |x_0| + M, \qquad M = \int_0^T m_2(s)\, ds + C.
\end{aligned}$$

This shows that the family of functions $\{x_n(t)\}$ is uniformly bounded.

Since $y_n(t) = \lambda x_n(\gamma t)$, we have

$$\begin{aligned}
|y_n(t)| &= |\lambda x_n(\gamma t)| \\
&\leq \lambda(|x_0| + M).
\end{aligned}$$

Thus, $\{y_n(t)\}$ is uniformly bounded [3], and so is the sequence $\{(x_n, y_n)\}$.

Let $t_1, t_2 \in [0,T]$ satisfy $|t_2 - t_1| < \delta$. Without loss of generality, assume that $t_1 \leq t_2$. Then

$$\begin{aligned}
|x_n(t_2) - x_n(t_1)| &= |x_0 + \int_0^T g\Big(s, y_n(s)\Big)\, ds + \int_0^{t_2} f\Big(s, y_n(s)\Big)\, ds \\
&- x_0 - \int_0^T g\Big(s, y_n(s)\Big)\, ds - \int_0^{t_1} f(s, y_n(s))\, ds| \\
&= |\int_0^{t_2} f\Big(s, y_n(s)\Big)\, ds - \int_0^{t_1} f\Big(s, y_n(s)\Big)\, ds|
\end{aligned}$$

$$\begin{aligned} &\leq \int_{t_1}^{t_2} |f\Big(s, y_n(s)\Big)|\, ds \\ &\leq \int_{t_1}^{t_2} m_1(s)\, ds \\ &\leq \varepsilon. \end{aligned}$$

This shows that the family $\{x_n(t)\}$ is equicontinuous.
Also,

$$|y_n(t_2) - y_n(t_1)| = |\lambda x_n\Big(\gamma t_2\Big) - \lambda x_n\Big(\gamma t_1\Big)| \leq \varepsilon.$$

This shows that $\{y_n(t)\}$ is equicontinuous [3]; hence, the sequence $\{(x_n, y_n)\}$ is equicontinuous.
By the Arzelà–Ascoli theorem [9], the family $\{(x_n(t), y_n(t))\}$ is relatively compact. Therefore, it has a subsequence $\Big\{(x_{n_k}, y_{n_k})\Big\}$ such that $(x_{n_k}(t), y_{n_k}(t)) \to (x, y)$. In particular, $x_{n_k}(t) \to x(t) \in C[0, T]$ and $y_{n_k}(t) \to y(t) \in C[0, T]$.
By the Lebesgue dominated convergence theorem [3] and the relation

$$\begin{aligned} \lim_{k\to\infty} y_{n_k}(t) &= \lim_{k\to\infty} \lambda x_{n_k}(\gamma t) \\ &= \lambda x(\gamma t) = y(t), \end{aligned}$$

we obtain:

$$\begin{aligned} x_{n_k}(t) &= x_0 + \int_0^T g\Big(s, y_{n_k}(s)\Big)\, ds + \int_0^t f\Big(s, y_{n_k}(s)\Big)\, ds \\ \lim_{k\to\infty} x_{n_k}(t) &= x_0 + \lim_{k\to\infty} \int_0^T g\Big(s, y_{n_k}(s)\Big)\, ds + \lim_{k\to\infty} \int_0^t f\Big(s, y_{n_k}(s)\Big)\, ds \\ &= x_0 + \int_0^T g\Big(s, y(s)\Big)\, ds + \int_0^t f\Big(s, y(s)\Big)\, ds \\ &= x_0 + \int_0^T g\Big(s, \lambda x(\gamma s)\Big)\, ds + \int_0^t f\Big(s, \lambda x(\gamma s)\Big)\, ds = x(t). \end{aligned} \tag{2.5}$$

Then:

$$\lim_{k\to\infty} \Big(x_{n_k}(t), y_{n_k}(t)\Big) = (x, y).$$

By Carathéodory's theorem [9], the integral equation (2.4) has at least one solution $(x, y) \in X$. Consequently, the system (2.2)–(2.3) has at least one solution $(x, y) \in X$.
By the equivalence between (2.2)–(2.3) and (2.1), equation (2.1) has at least one solution $x \in C[0, T]$. Finally, the equivalence between (2.1) and (1.1)–(1.2) implies that the nonlocal problem (1.1)–(1.2) has at least one solution.

□

3. Uniqueness of the Solution

In this section, we establish sufficient conditions for uniqueness of the solution to (1.1)–(1.2).

Assume, in addition, that $(i)^*$ the functions $f, g : I \times R \to R$ are measurable in $t \in [0, T]$ for every $x \in R$ and satisfy the Lipschitz conditions

$$|f(t,x) - f(t,\bar{x})| \leq k_1 |x - \bar{x}|,\ k_1 > 0,$$

$$|g(t,x) - g(t,\bar{x})| \leq k_2 |x - \bar{x}|,\ k_2 > 0.$$

and let $k = k_1 + k_2$.

Theorem 3.1. *Suppose that the assumptions of Theorem 2.1 and the additional assumption $(i)^*$ hold. If $k\lambda T < 1$, then the solution to problem (1.1)–(1.2) is unique.*

Proof. Let x and $\bar{x}$ be two solutions to (1.1)–(1.2). Then

$$\begin{aligned}
|x(t) - \bar{x}(t)| &= |x_0 + \int_0^T g\Big(s, \lambda x(\gamma s)\Big)\, ds + \int_0^t f\Big(s, \lambda x(\gamma s)\Big)\, ds \\
&- x_0 - \int_0^T g\Big(s, \lambda \bar{x}(\gamma s)\Big)\, ds - \int_0^t f\Big(s, \lambda \bar{x}(\gamma s)\Big)\, ds| \\
&\leq \int_0^T |g\Big(s, \lambda x(\gamma s)\Big) - g\Big(s, \lambda \bar{x}(\gamma s)\Big)|\, ds \\
&+ \int_0^t |f\Big(s, \lambda x(\gamma s)\Big) - f\Big(s, \lambda \bar{x}(\gamma s)\Big)|\, ds \\
&\leq k_2\lambda \int_0^T |x(\gamma s) - \bar{x}(\gamma s)|\, ds + k_1\lambda \int_0^t |x(\gamma s) - \bar{x}(\gamma s)|\, ds \\
&\leq k_2\lambda \|x - \bar{x}\| T + k_1\lambda \|x - \bar{x}\| T \\
&\leq (k_1 + k_2)\lambda \|x - \bar{x}\| T.
\end{aligned}$$

Thus,

$$\|x - \bar{x}\| \leq k\lambda T \|x - \bar{x}\|, \text{ where } k = k_1 + k_2.$$

Hence,

$$(1 - k\lambda T)\|x - \bar{x}\| \leq 0.$$

Since $k\lambda T < 1$, it follows that $x = \bar{x}$. Therefore, the solution to problem (1.1)–(1.2) is unique.

□

4. Stability Analysis

In this section, we analyze the stability of problem (1.1)–(1.2) in two ways. First, we study the continuous dependence of the solution on the initial datum and parameter, which describes how changes in the data affect the solution. We then consider Hyers–Ulam stability, a framework for assessing the behavior of approximate solutions.

4.1. **Continuous Dependence on the Initial Datum and Parameter.**

Definition 4.1. *[7] We say that the solution x of problem (1.1)–(1.2) depends continuously on the initial datum x_0 and parameter λ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\max\{|x_0 - x_0^*|, |\lambda - \lambda^*|\} < \delta \quad \textit{implies} \quad \|x - x^*\| < \epsilon,$$

where

$$\frac{dx^*(t)}{dt} = f\Big(t, \lambda^* x^*(\gamma t)\Big),\ t \in [0,T],\ \textit{a.e.}, \tag{4.1}$$

$$x^*(0) = x_0^* + \int_0^T g\Big(s, \lambda^* x^*(\gamma s)\Big)\, ds. \tag{4.2}$$

Theorem 4.1. *Suppose that the assumptions of Theorem 3.1 hold. Then the solution to problem (1.1)–(1.2) depends continuously on the initial datum x_0 and parameter λ.*

Proof. Consider the solutions x and x^* to problems (1.1)–(1.2) and (4.1)–(4.2), respectively. Then

$$\begin{aligned}
|x(t) - x^*(t)| =& |x_0 + \int_0^T g\Big(s, \lambda x(\gamma s)\Big)\, ds + \int_0^t f\Big(s, \lambda x(\gamma s)\Big)\, ds \\
&- x_0^* - \int_0^T g\Big(s, \lambda^* x^*(\gamma s)\Big)\, ds - \int_0^t f\Big(s, \lambda^* x^*(\gamma s)\Big)\, ds| \\
\leq& |x_0 - x_0^*| + \int_0^T |g\Big(s, \lambda x(\gamma s)\Big) - g\Big(s, \lambda^* x^*(\gamma s)\Big)| ds \\
&+ \int_0^t |f\Big(s, \lambda x(\gamma(s))\Big) - f\Big(s, \lambda^* x^*(\gamma s)\Big)| ds \\
\leq& \delta + k_2 \int_0^T |\lambda x(\gamma(s)) - \lambda^* x^*(\gamma s)| ds \\
&+ k_1 \int_0^t |\lambda x(\gamma s) - \lambda^* x^*(\gamma s)| ds \\
\leq& \delta + k_2 \int_0^T |\lambda x(\gamma(s)) - \lambda x^*(\gamma s) + \lambda x^*(\gamma s) - \lambda^* x^*(\gamma s)| ds \\
&+ k_1 \int_0^t |\lambda x(\gamma s) - \lambda x^*(\gamma s) + \lambda x^*(\gamma s) - \lambda^* x^*(\gamma s)| ds \\
\leq& \delta + k_2 \int_0^T |\lambda x(\gamma s) - \lambda x^*(\gamma s)| ds + k_2 \int_0^T |\lambda x^*(\gamma s) - \lambda^* x^*(\gamma s)| ds \\
&+ k_1 \int_0^t |\lambda x(\gamma s) - \lambda x^*(\gamma s)| ds + k_1 \int_0^t |\lambda x^*(\gamma s) - \lambda^* x^*(\gamma s)| ds \\
\leq& \delta + k_2 \lambda \int_0^T |x(\gamma s) - x^*(\gamma s)| ds + k_2 |\lambda - \lambda^*| \int_0^T |x^*(\gamma s)| ds \\
&+ k_1 \lambda \int_0^t |x(\gamma s) - x^*(\gamma s)| ds + k_1 |\lambda - \lambda^*| \int_0^t |x^*(\gamma s)| ds.
\end{aligned}$$

$$\|x - x^*\| \leq \delta + k_2 \lambda T \|x - x^*\| + k_2 \delta \|x^*\| T + k_1 \lambda T \|x - x^*\| + k_1 \delta \|x^*\| T$$

$$\leq \delta + k\lambda T\|x - x^*\| + k\delta\|x^*\|T.$$

Thus,

$$\|x - x^*\|(1 - k\lambda T) \leq \delta(1 + k\|x^*\|T).$$

This implies that:

$$\|x - x^*\| \leq \frac{\delta(1 + k\|x^*\|T)}{1 - k\lambda T} = \epsilon.$$

Therefore, the solution $x \in C[0, T]$ to the nonlocal problem (1.1)–(1.2) depends continuously on x_0 and λ.

□

4.2. **Hyers–Ulam Stability.**

Definition 4.2. *[11] Suppose that problem (1.1)–(1.2) has a unique solution. We say that it is Hyers–Ulam stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-approximate solution x_s to problem (1.1)–(1.2) satisfying*

$$| \frac{dx_s(t)}{dt} - f(t, \lambda x_s(\gamma t)) | < \delta,$$

also satisfies

$$\|x - x_s\| < \epsilon.$$

Theorem 4.2. *Suppose that the assumptions of Theorem 3.1 hold. Then problem (1.1)–(1.2) is Hyers–Ulam stable.*

Proof. Suppose that

$$\left| \frac{dx_s(t)}{dt} - f(t, \lambda x_s(\gamma t)) \right| < \delta.$$

Then

$$-\delta < \frac{dx_s(t)}{dt} - f(t, \lambda x_s(\gamma t)) < \delta,$$

Integrating with respect to t gives

$$-\delta t < x_s(t) - x_s(0) - \int_0^t f(\theta, \lambda x_s(\gamma\theta))d\theta < \delta t.$$

$$-\delta T < x_s(t) - x_0 - \int_0^T g(\theta, \lambda x_s(\gamma\theta))d\theta - \int_0^t f(\theta, \lambda x_s(\gamma\theta))d\theta \leq \delta T.$$

Now consider

$$\begin{aligned} | x(t) - x_s(t) | &= | x_0 + \int_0^T g(\theta, \lambda x(\gamma\theta))\, d\theta + \int_0^t f(\theta, \lambda x(\gamma\theta))\, d\theta - x_s(t)| \\ &= | x_0 + \int_0^T g(\theta, \lambda x(\gamma\theta))\, d\theta + \int_0^t f(\theta, \lambda x(\gamma\theta))\, d\theta \\ &\quad - x_0 - \int_0^T g(\theta, \lambda x_s(\gamma\theta))\, d\theta - \int_0^t f(\theta, \lambda x_s(\gamma\theta))\, d\theta \end{aligned}$$

$$
\begin{aligned}
&+ x_0 + \int_0^T g(\theta, \lambda x_s(\gamma\theta))\, d\theta + \int_0^t f(\theta, \lambda x_s(\gamma\theta))\, d\theta - x_s(t)| \\
=&| \int_0^T g(\theta, \lambda x(\gamma\theta)) - g(\theta, \lambda x_s(\gamma\theta))\, d\theta + \int_0^t f(\theta, \lambda x(\gamma\theta)) - f(\theta, \lambda x_s(\gamma\theta))\, d\theta \\
&- \Big(x_s(t) - x_0 - \int_0^T g(\theta, \lambda x_s(\gamma\theta))\, d\theta - \int_0^t f(\theta, \lambda x_s(\gamma\theta))\, d\theta\Big)| \\
&\leq k_1 T\lambda \|x - x_s\| + k_2 T\lambda \|x - x_s\| + \delta T \\
&\leq kT\lambda \|x - x_s\| + \delta T,
\end{aligned}
$$

Therefore,

$$
\|x - x_s\| \leq \frac{\delta T}{1 - k\lambda T} = \epsilon.
$$

$$
\|x - x_s\| \leq \epsilon.
$$

Problem (1.1)–(1.2) is Hyers–Ulam stable. □

5. Examples

Example 1. Consider the problem:

$$
\frac{dx}{dt} = m(t).\Big(e^{-|\lambda x(\gamma t)|}\Big), \ a.e.\ t \in [0, T], \tag{5.1}
$$

with the nonlocal condition:

$$
x(0) = x_0 + \int_0^T (D(s) + e^{-|\lambda x(\gamma s)|})\, ds, \tag{5.2}
$$

where $0 < \gamma \leq 1$, $0 < \lambda T < \frac{1}{3}$, and $m(t) = D(t) + 1$, with D defined by

$$
D(t) = \begin{cases} 1, & t \in Q \\ 0, & t \notin Q \end{cases},
$$

Here, D is the Dirichlet function on $[0, T]$, and Q denotes the set of rational numbers.
Set $y = \lambda x(\gamma t)$. Then

$$
\begin{aligned}
f(t, y) &= m(t).\Big(e^{-|y|}\Big), \\
g(t, y) &= D(t) + e^{-|y|}.
\end{aligned}
$$

The following bounds hold:

$$
\begin{aligned}
|f(t, y)| &\leq |m(t)|, \\
|g(t, y)| &\leq 2.
\end{aligned}
$$

Since $0 \leq D(t) \leq 1$, we have $|m(t)| = |D(t) + 1| \leq 2$. For all $y, \bar{y} \in \mathbb{R}$,

$$
\Big|e^{-|y|} - e^{-|\bar{y}|}\Big| \leq \Big||y| - |\bar{y}|\Big| \leq |y - \bar{y}|.
$$

Consequently,

$$
|f(t, y) - f(t, \bar{y})| \leq |m(t)|\, |y - \bar{y}| \leq 2|y - \bar{y}|,
$$

$$|g(t,y)-g(t,\bar{y})| \leq |y-\bar{y}|.$$

Thus, we can take $k_1 = 2$, $k_2 = 1$, and $k = 3$. The hypotheses of Theorems 2.1 and 3.1 are satisfied. Hence, problem (5.1)–(5.2) has a unique solution.

Example 2. Consider the problem:

$$\frac{dx}{dt} = m(t)\,\frac{\lambda x(\gamma t)}{1+|\lambda x(\gamma t)|}\,, a.e.\ t \in [0,T], \tag{5.3}$$

with the nonlocal condition:

$$x(0) = x_0 + \int_0^T (D(s) + \frac{\lambda x(\gamma s)}{1+|\lambda x(\gamma s)|})ds, \tag{5.4}$$

where $0 < \gamma \leq 1$ and $0 < \lambda T < \frac{1}{3}$.
Set $y = \lambda x(\gamma t)$. Then

$$\begin{aligned} f(t,y) &= m(t).\Big(\frac{y}{1+|y|}\Big), \\ g(t,y) &= D(t) + \Big(\frac{y}{1+|y|}\Big). \end{aligned}$$

The following bounds hold:

$$\begin{aligned} |f(t,y)| &\leq |m(t)|, \\ |g(t,y)| &\leq 2 \end{aligned}$$

For the derivative estimates,

$$\begin{aligned} \frac{\partial f(t,y)}{\partial y} &= m(t)(\frac{1}{(1+|y|)^2}) \\ |\frac{\partial f(t,y)}{\partial y}| &\leq |m(t)| \leq 2. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial g(t,y)}{\partial y} &= \frac{1}{(1+|y|)^2} \\ |\frac{\partial g(t,y)}{\partial y}| &\leq 1 \end{aligned}$$

Therefore,

$$|f(t,y)-f(t,\bar{y})| \leq 2|y-\bar{y}|$$

$$|g(t,y)-g(t,\bar{y})| \leq |y-\bar{y}|.$$

Thus, $k_1 = 2$, $k_2 = 1$, and $k = 3$. The hypotheses of Theorems 2.1 and 3.1 are satisfied. Hence, problem (5.3)–(5.4) has a unique solution.

Example 3. Consider the following problem:

$$\frac{dx}{dt} = m(t).\frac{e^{-\lambda x(\gamma t)}}{1+e^{-\lambda x(\gamma t)}},\ a.e.\ t \in [0,T], \tag{5.5}$$

with the nonlocal condition

$$x(0) = x_0 + \int_0^T (D(s) + \frac{e^{-\lambda x(\gamma s)}}{1+e^{-\lambda x(\gamma s)}})ds. \tag{5.6}$$

where $0 < \gamma \leq 1$ and $0 < \lambda T < \frac{1}{2}$.

Set $y = \lambda x(\gamma t)$. Then

$$\begin{aligned} f(t,y) &= m(t).\Big(\frac{e^{-y}}{1+e^{-y}}\Big), \\ g(t,y) &= D(t) + \frac{e^{-y}}{1+e^{-y}}. \end{aligned}$$

The following bounds hold:

$$\begin{aligned} |f(t,y)| &\leq |m(t)| \\ |g(t,y)| &\leq 2. \end{aligned}$$

For the derivative estimates,

$$\begin{aligned} \frac{\partial f(t,y)}{\partial y} &= m(t)(\frac{-e^{-y}}{(1+e^{-y})^2}) \\ |\frac{\partial f(t,y)}{\partial y}| &\leq \frac{|m(t)|}{4} \leq \frac{1}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial g(t,y)}{\partial y} &= (\frac{-e^{-y}}{(1+e^{-y})^2}) \\ |\frac{\partial g(t,y)}{\partial y}| &\leq |\frac{e^{-y}}{(1+e^{-y})^2}| \leq \frac{1}{4}. \end{aligned}$$

Therefore,

$$|f(t,y) - f(t,\bar{y})| \leq \frac{1}{2}|y-\bar{y}|$$

$$|g(t,y) - g(t,\bar{y})| \leq \frac{1}{4}|y-\bar{y}|.$$

Thus, $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{4}$, and $k = \frac{3}{4}$. The hypotheses of Theorems 2.1 and 3.1 are satisfied. Hence, problem (5.5)–(5.6) has a unique solution.

6. Conclusion

In this paper, we studied a delay differential equation subject to a nonlocal integral condition under Carathéodory-type assumptions. We established existence and, under the stated additional condition, uniqueness of solutions. We also proved the equivalence between the original problem and its integral formulation, and established continuous dependence on the initial data x_0 and parameter λ, as well as Hyers–Ulam stability. The examples illustrate the applicability of our results.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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