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Concomitants of Current Records

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Abstract. This paper develops a distributional framework for the concomitants of current records arising from the bivariate pseudo-exponential distribution. Although concomitants of classical record values have received considerable attention, their counterparts associated with current records remain largely unexplored. To address this gap, Gamma-based representations of current-record distributions are employed to obtain analytically tractable results. Exact probability density and cumulative distribution functions are derived for the concomitants associated with upper current records under the bivariate pseudo-exponential model. Closed-form expressions for their marginal moments and related distributional measures are also established. The analysis is further extended to concomitants of lower-current records, for which the corresponding distributional and moment properties are investigated. In addition, the joint distribution of two concomitants of current records is derived, providing a basis for examining their dependence structure. The proposed results extend concomitant theory from classical records to the current-record framework and provide useful theoretical tools for studying associated variables observed alongside evolving extremes, with potential applications in reliability, environmental monitoring, finance, and other sequential monitoring problems.

1. Introduction

Record statistics have emerged as a fundamental tool for analyzing extreme observations in a wide range of scientific and applied fields, including hydrology, meteorology, finance, industrial quality control, and biostatistics. The formal study of record values began with [19], who introduced the probabilistic framework for observations that exceed (or fall below) all previous values

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in a sequence of independent and identically distributed (i.i.d.) random variables. Since then, record theory has developed into a mature discipline, with comprehensive treatments provided by [15] and [30]. Classical results describe the distributions of record times, inter-record spacings, and the asymptotic behavior of record values, many of which exhibit intriguing distribution-free properties [15,30].

In many practical monitoring scenarios, however, it is not the entire sequence of recorded values that is of interest, but rather the most recent observation. This has led to the concept of current records, introduced by [24]. Unlike classical records, which retain all past record values, current records capture only the current extreme value and update only when a new record is established. This framework is particularly relevant in sequential quality control, environmental monitoring, and online financial surveillance, where storage constraints or real-time decision-making necessitate focusing on the latest information. [24] derived fundamental distributional properties of current records, including their probability density functions and joint distributions.

Subsequent work has extended these results to specific parametric families. [33] obtained sharp distribution-free bounds for the expected value of the gap between current records and record values, as well as upper sharp bounds for spacings between upper current records. [16] derived general recurrence relations for the single and product moments of upper and lower current records based on Weibull, Pareto, and other distributions, also establishing asymptotic results for general current records. The current record range, defined as $R_{(cr)_m} = U_{(cr)_m} - L_{(cr)_m}$, has been studied extensively by [1,2] in the context of quantile confidence intervals and reliability properties.

Recent contributions by Aldallal have significantly expanded the theoretical and applied literature on current records. In particular, [7,8,10,12,13,18] developed Bayesian and non-Bayesian estimation and prediction procedures for current records arising from Weibull, Laplace, Fréchet, and Rayleigh distributions, among other distributions, with important applications to Saudi industrial and environmental data. In addition, [5] investigated best linear unbiased estimation and prediction for classical record values under the Kumaraswamy distribution, while [6] derived moment generating functions and recurrence relations for current records based on the generalized exponential distribution. More recently, [11] provided a comprehensive review of current record theory, summarizing its mathematical foundations, inferential methods, and practical applications. These studies demonstrate the growing importance of current record statistics in reliability analysis, industrial quality control, prediction theory, and applied stochastic modeling, while also highlighting the need for further investigation into related structures such as concomitants of current records.

Concomitants of records represent a natural extension of record theory to bivariate settings. When observations consist of bivariate vectors $\{(X_i, Y_i)\}_{i\geq 1}$, and record events are defined based on the X-variable, the accompanying Y-values are called concomitants of records. The concept of concomitants was first introduced in the context of order statistics by [20] and later extended to record statistics by [24]. [31] systematically investigated the properties of concomitants of record

values, obtaining their distributions in the sequence of record values of bivariate distributions and pointing out potential applications in biosciences. Recent advances include the study of concomitants from the Morgenstern family of distributions [4, 26], from bivariate Lindley distributions [29], and from generalized exponential distributions [17]. [21] investigated limit results, generation techniques, and rank correlation for concomitants of records. Applications range from survival analysis and insurance to biostatistics, where the concomitant variable often represents a response or treatment effect associated with an extreme value of the predictor.

The bivariate pseudo-exponential distribution, first introduced by [22], provides a flexible framework for modeling dependent lifetimes where one marginal is exponential and the conditional distribution of the second variable given the first is also exponential. [14] provided a comprehensive study of this model, including its characterization and applications to GDP and infant mortality data. More recently, Bayesian inference for the bivariate pseudo-exponential distribution has been developed using gamma and pseudo-gamma priors [35]. [23] studied concomitants of generalized order statistics for this distribution, obtaining moment expressions. [34] derived the distributions of two concomitants when a random sample is available from a trivariate pseudo-exponential distribution.

Characterizations of distributions using record values form an important branch of record theory. A distribution is said to be characterized if its form can be uniquely determined by certain probabilistic properties, such as the conditional expectation of a function of record values. [3] provided foundational results in this area. [25] characterized exponential and geometric distributions using the constancy of conditional expectations of record values. More recently, researchers have characterized Pareto and Weibull distributions via conditional expectations of upper record values [27,32]. General classes of continuous distributions have been characterized by considering conditional expectations of functions of upper record statistics [28]. These characterization results are valuable for model selection and validation in applied settings.

Despite the substantial body of work on record statistics and their concomitants, the analysis of the concomitants of current records remains largely unexplored. The existing literature focuses almost exclusively on classical records, where all past records are retained. However, many practical monitoring systems update only the current extreme value, discarding historical records. This paper aims to fill this gap by developing the distributional theory of concomitants of current records for the bivariate pseudo-exponential distribution. Specifically, we contribute:

(1) The derivation of the exact distribution of the k-th concomitant of upper current records for the bivariate pseudo-exponential distribution under the [24] framework.
(2) Closed-form expressions for the single and product moments of these concomitants.
(3) The joint distribution of two concomitants.

The remainder of this paper is organized as follows. Section 2 provides the necessary background on current record statistics. Section 3 introduces the bivariate pseudo-exponential distribution. Section 4 presents the main results on the distribution of the k-th concomitant of upper current

records. Section 5 develops the analogous results for lower current records. Section 6 derives the joint distribution of two concomitants and studies their product moments and dependence structure.

2. Current Record Statistics: Preliminaries

Let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) continuous random variables with common cumulative distribution function (CDF) $F(x)$ and probability density function (PDF) $f(x)$. Following [24], we define the sequences of *upper current records* $\{U_{(cr)_m}\}_{m\geq 1}$ and *lower current records* $\{L_{(cr)_m}\}_{m\geq 1}$ as follows:

- Set $U_{(cr)_1} = L_{(cr)_1} = X_1$ (the first observation is both the initial upper and lower current record).
- For $i \geq 2$:

$$U_{(cr)_i} = \begin{cases} X_i, & \text{if } X_i > U_{(cr)_{i-1}}, \\ U_{(cr)_{i-1}}, & \text{otherwise,} \end{cases}$$

$$L_{(cr)_i} = \begin{cases} X_i, & \text{if } X_i < L_{(cr)_{i-1}}, \\ L_{(cr)_{i-1}}, & \text{otherwise.} \end{cases}$$

Thus, $U_{(cr)_m}$ is the largest observation seen up to the m-th record event, and $L_{(cr)_m}$ is the smallest observation seen up to that event. Unlike classical record values, which store every record, current records only retain the *most recent* extreme value; a new record (upper or lower) updates the current record, while the other remains unchanged. This process yields a sequence of current record pairs $(L_{(cr)_m}, U_{(cr)_m})$ with $L_{(cr)_m} \leq U_{(cr)_m}$, and the interval $(L_{(cr)_m}, U_{(cr)_m})$ is called the *record coverage*. The *current record range* is $R_{(cr)_m} = U_{(cr)_m} - L_{(cr)_m}$ [16].

[24] derived the probability density and cumulative distribution functions of upper and lower current records. However, the resulting expressions involve finite summations that become increasingly cumbersome when applied to different parent distributions. To overcome this limitation, [9] introduced equivalent Gamma-based representations that eliminate these summations, yielding more compact formulas that are easier to derive, analyze, perform theoretical investigations, statistical inference, and implement for a wide range of continuous parent distributions.

For any integer $n \geq 1$, let $F_p(x)$ denote the cumulative distribution function of an absolutely continuous parent distribution with corresponding probability density function $f_p(x) = F_p'(x)$, and let $\bar{F}_p(x) = 1 - F_p(x)$ be its survival function, where the subscript p indicates the parent distribution. Furthermore, let

$$G(a,b) = \frac{\gamma(a,b)}{\Gamma(a)}$$

represent the regularized lower incomplete gamma function. Using the Gamma-based representation developed by [9, 12], the cumulative distribution functions and probability density functions

of the n-th lower and upper current records can be written as

$$F_{L_{(cr)_n}}(x) = 1 + 2^n F_p(x)\, G\Big(n, -\log F_p(x)\Big) - G\Big(n, -2\log F_p(x)\Big), \tag{2.1}$$

$$f_{L_{(cr)_n}}(x) = 2^n f_p(x)\, G\Big(n, -\log F_p(x)\Big), \tag{2.2}$$

$$F_{U_{(cr)_n}}(x) = G\Big(n, -2\log \bar{F}_p(x)\Big) - 2^n \bar{F}_p(x)\, G\Big(n, -\log \bar{F}_p(x)\Big), \tag{2.3}$$

$$f_{U_{(cr)_n}}(x) = 2^n f_p(x)\, G\Big(n, -\log \bar{F}_p(x)\Big). \tag{2.4}$$

These expressions provide compact closed-form representations of the marginal distributions of lower and upper current records and will be used throughout the remainder of this paper. In particular, they simplify subsequent derivations involving joint, conditional, and concomitant distributions.

2.1. **Joint and conditional density functions.** The joint distributions of lower and upper current records were originally established by [24]. These distributions provide the theoretical framework for obtaining conditional distributions and, consequently, for developing prediction methods based on current records.

Consider two lower current records, $L_{(cr)_n}$ and $L_{(cr)_s}$, where $1 < n < s$. Their joint probability density function is expressed as

$$f_{L_{(cr)_n},L_{(cr)_s}}(l_n, l_s) = \frac{f_p(l_n) f_p(l_s)}{F_p(l_n)(s-n-1)!}\Big(H(l_n) - H(l_s)\Big)^{s-n-1}, \quad l_s < l_n, \tag{2.5}$$

where $H(x) = \log\Big(F_p(x)\Big)$. Likewise, the joint density function of two upper current records, $U_{(cr)_n}$ and $U_{(cr)_s}$, is given by

$$f_{U_{(cr)_n},U_{(cr)_s}}(u_n, u_s) = \frac{f_p(u_n) f_p(u_s)}{\bar{F}_p(u_n)(s-n-1)!}\Big(R(u_s) - R(u_n)\Big)^{s-n-1}, \quad u_s > u_n, \tag{2.6}$$

where $R(x) = -\log\Big(\bar{F}_p(x)\Big)$. From these joint density functions, the conditional distributions of future current records can be obtained by conditioning on the observed record values. Accordingly, the conditional density of the s-th lower current record, given the n-th lower current record, is

$$f_{L_{(cr)_s}|L_{(cr)_n}}(l_s \mid l_n) = \frac{f_p(l_s)}{F_p(l_n)(s-n-1)!}\Big(H(l_n) - H(l_s)\Big)^{s-n-1}, \quad l_s < l_n, \tag{2.7}$$

while the corresponding conditional density for upper current records is

$$f_{U_{(cr)_s}|U_{(cr)_n}}(u_s \mid u_n) = \frac{f_p(u_s)}{\bar{F}_p(u_n)(s-n-1)!}\Big(R(u_s) - R(u_n)\Big)^{s-n-1}, \quad u_s > u_n. \tag{2.8}$$

The above conditional density functions constitute the principal building blocks for prediction based on current records. They characterize the distribution of future record observations conditional on the available information and are subsequently employed to construct both classical predictive distributions and Bayesian prediction procedures.

2.2. **Concomitants of upper current records.** Consider a sequence of independent and identically distributed bivariate random vectors $\{(X_i, Y_i)\}_{i\geq 1}$ with joint probability density function $f_{X,Y}(x, y)$. Current record events are determined exclusively by the X-sequence, while the corresponding Y-observation associated with each current record is referred to as its *concomitant*. Accordingly, if $U_{(cr)_m}$ and $L_{(cr)_m}$ denote the m-th upper and lower current records, respectively, then the associated concomitants are denoted by $Y^U_{[m]}$ and $Y^L_{[m]}$.

For the upper current records, the marginal probability density function of the concomitant $Y^U_{[m]}$ is given by

$$h^U_{[m]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_{U_{(cr)_m}}(x)\, dx, \tag{2.9}$$

where $f_{Y|X}(y|x)$ denotes the conditional density of Y given $X = x$. Likewise, the joint probability density function of the concomitants corresponding to the k-th and m-th upper current records ($k < m$) is

$$g^U_{[k,m]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f_{Y|X}(y_1|x_1)\, f_{Y|X}(y_2|x_2)\, f_{U_{(cr)_k},U_{(cr)_m}}(x_1, x_2)\, dx_2\, dx_1, \tag{2.10}$$

where the integration limits satisfy $x_1 < x_2$, reflecting the ordering of upper current records. Similar expressions can be written for the concomitants of lower current records by replacing the corresponding current-record density functions.

The general representations introduced in this section, together with the Gamma-based marginal distributions and the joint and conditional density functions of current records, provide the theoretical framework for deriving the exact distributions and moments of concomitants considered in the subsequent sections.

For further background on current record statistics, the reader is referred to [24], [16], [1], and [30]. General properties of concomitants associated with classical record values can be found in [20] and [31].

2.3. **Concomitants of lower current records.** The definition of concomitants extends naturally to lower current records. If $L_{(cr)_m}$ denotes the m-th lower current record, then the associated concomitant is denoted by $Y^L_{[m]}$.

The marginal probability density function of $Y^L_{[m]}$ is

$$h^L_{[m]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_{L_{(cr)_m}}(x)\, dx, \tag{2.11}$$

where $f_{Y|X}(y|x)$ is the conditional density of Y given $X = x$. Likewise, the joint probability density function of the concomitants corresponding to the k-th and m-th lower current records ($k < m$) is

$$g^L_{[k,m]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} f_{Y|X}(y_1|x_1)\, f_{Y|X}(y_2|x_2)\, f_{L_{(cr)_k},L_{(cr)_m}}(x_1, x_2)\, dx_2\, dx_1, \tag{2.12}$$

where the limits of integration satisfy $x_2 < x_1$, reflecting the monotone decreasing nature of lower current records. The above expressions are completely analogous to those for upper current

records and provide the basis for deriving the distributional properties, moments, and dependence structure of concomitants associated with lower current records whenever such results are required.

3. Bivariate Pseudo-Exponential Distribution

The bivariate pseudo-exponential distribution belongs to a broad class of multivariate distributions generated through a hierarchical conditional construction, originally proposed by [22]. In this construction, a primary variable X follows an exponential distribution, while the secondary variable Y, conditionally on $X = x$, also follows an exponential distribution with a rate depending on x. This produces a flexible family capable of modeling positively dependent lifetimes, financial variables, and environmental measurements in which the scale of one variable is influenced by the magnitude of another.

Definition 3.1. *A pair (X, Y) is said to follow a bivariate pseudo-exponential distribution with parameter $\alpha > 0$ and a positive function $\phi(x)$ if its joint probability density function is given by*

$$f_{X,Y}(x, y) = \alpha\, \phi(x) \exp[-\alpha x - \phi(x) y]\,, \qquad x > 0,\ y > 0, \tag{3.1}$$

where $\phi(x) > 0$ for every $x > 0$.

The marginal density of X is exponential with rate α,

$$f_X(x) = \alpha e^{-\alpha x}, \qquad x > 0.$$

Indeed, integrating the joint density in (3.1) with respect to y gives

$$\begin{aligned} f_X(x) &= \int_0^\infty \alpha\phi(x) e^{-\alpha x - \phi(x) y}\, dy \\ &= \alpha\phi(x) e^{-\alpha x} \int_0^\infty e^{-\phi(x) y}\, dy \\ &= \alpha e^{-\alpha x}, \qquad x > 0. \end{aligned}$$

Given $X = x$, the conditional density of Y is

$$f_{Y|X}(y|x) = \phi(x) e^{-\phi(x) y}, \qquad y > 0,$$

which is an exponential density with rate $\phi(x)$. Therefore, the joint distribution is completely determined by the choice of the function $\phi(\cdot)$.

3.1. The case $\phi(x) = x$.

Following [23] and [34], we consider the simplest and most analytically tractable specification,

$$\phi(x) = x, \qquad x > 0.$$

Substituting this choice into (3.1) yields

$$f_{X,Y}(x, y) = \alpha x \exp[-x(\alpha + y)]\,, \qquad x > 0,\ y > 0. \tag{3.2}$$

The model defined by (3.2) is referred to as the bivariate pseudo-exponential distribution of type I, or simply the pseudo-exponential distribution throughout the remainder of this paper. Its corresponding conditional density is

$$f_{Y|X}(y|x) = xe^{-xy}, \qquad y > 0. \tag{3.3}$$

3.2. **Basic properties.** For the distribution defined by (3.2), the following properties hold.

Proposition 3.1. *The marginal probability density function of Y is*

$$f_Y(y) = \frac{\alpha}{(\alpha + y)^2}, \qquad y > 0. \tag{3.4}$$

Proof. Integrating the joint density in (3.2) with respect to x gives

$$\begin{aligned} f_Y(y) &= \int_0^\infty \alpha x e^{-x(\alpha+y)}\, dx \\ &= \alpha \int_0^\infty x e^{-x(\alpha+y)}\, dx \\ &= \frac{\alpha}{(\alpha + y)^2}, \qquad y > 0. \end{aligned}$$

□

Thus, Y follows a Lomax distribution, also known as a Pareto type II distribution, with shape parameter 1 and scale parameter α. Since the mean of a Lomax distribution exists only when its shape parameter is greater than 1, the present model satisfies

$$E(Y) = \infty.$$

This heavy-tailed property makes the model suitable for describing extreme events and variables that may assume unusually large values.

Proposition 3.2. *The conditional distribution of X given $Y = y$ is a Gamma distribution with shape parameter 2 and rate parameter $\alpha + y$. Its probability density function is*

$$f_{X|Y}(x|y) = (\alpha + y)^2 x e^{-(\alpha+y)x}, \qquad x > 0. \tag{3.5}$$

Proof. Using Bayes' formula, together with (3.2) and (3.4), gives

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\ &= \frac{\alpha x e^{-x(\alpha+y)}}{\alpha/(\alpha + y)^2} \\ &= (\alpha + y)^2 x e^{-(\alpha+y)x}, \qquad x > 0, \end{aligned}$$

which is the density of a Gamma random variable with shape parameter 2 and rate parameter $\alpha + y$. □

The joint moment generating function may formally be written as

$$M_{X,Y}(t_1, t_2) = E\Big[e^{t_1 X + t_2 Y}\Big].$$

A direct calculation gives

$$\begin{aligned} M_{X,Y}(t_1, t_2) &= \int_0^\infty \int_0^\infty \alpha x e^{-x(\alpha+y)} e^{t_1 x + t_2 y}\, dy\, dx \\ &= \alpha \int_0^\infty x e^{-x(\alpha - t_1)} \left(\int_0^\infty e^{-y(x-t_2)}\, dy \right) dx. \end{aligned}$$

For a fixed x, the inner integral converges only when $x > t_2$. Since the outer integration includes values of x arbitrarily close to zero, the joint MGF is not finite for any $t_2 > 0$. Moreover, when $t_2 = 0$, the first positive moment of Y is infinite. Consequently, the joint MGF is not defined in an open neighborhood of $(0, 0)$, which is consistent with the heavy-tailed marginal distribution of Y.

3.3. **Limitations and generalizations.** The choice $\phi(x) = x$ leads to a simple closed-form model and is analytically tractable for studying concomitants of current records, as shown in Sections 4 and 6. However, alternative specifications of $\phi(x)$, such as

$$\phi(x) = x^b \qquad \text{or} \qquad \phi(x) = e^{\beta x},$$

produce different dependence structures between X and Y. The general classes of pseudo-exponential and pseudo-Weibull distributions were extensively studied by [22]. For more recent characterizations and applications, see [14] and [35]. In the context of order statistics and record values, [23] investigated concomitants of generalized order statistics from this distribution, while [34] extended related results to the trivariate pseudo-exponential case.

3.4. **Why the pseudo-exponential distribution for current records?** The bivariate pseudo-exponential distribution is particularly appealing for the analysis of concomitants of current records for two main reasons:

(1) The marginal distribution of X is exponential. Therefore, substituting

$$F_p(x) = 1 - e^{-\alpha x}, \qquad \bar{F}_p(x) = e^{-\alpha x}, \qquad f_p(x) = \alpha e^{-\alpha x},$$

into the general Gamma-based formulas (2.1)–(2.4) yields explicit expressions for the distributions of lower and upper current records.

(2) The conditional density

$$f_{Y|X}(y|x) = x e^{-xy}$$

is a simple function of x. When it is combined with the marginal and joint density functions of current records, the resulting integrals can be evaluated analytically. This leads to explicit expressions for the marginal distributions, joint distributions, moments, and dependence structure of their concomitants, as demonstrated in Sections 4 and 6.

Thus, the distribution defined in (3.2) provides an ideal framework for developing the theory of concomitants of current records. It also retains potential applications in reliability analysis, environmental monitoring, finance, and insurance, particularly in situations where extreme observations and their associated variables are of primary interest.

4. Distribution of the k-th Concomitant of Upper Current Records

In this section, we derive the marginal distribution of the concomitant $Y^U_{[k]}$ associated with the k-th upper current record $U_{(cr)_k}$ of the X variable when the parent bivariate distribution is the pseudo-exponential distribution defined in (3.2). The derivation follows the general definition in (2.9) and uses the Gamma-based probability density function of upper current records given in (2.4).

For the exponential marginal distribution of X, we have

$$F_p(x) = 1 - e^{-\alpha x}, \qquad \bar{F}_p(x) = e^{-\alpha x}, \qquad f_p(x) = \alpha e^{-\alpha x}, \qquad x > 0.$$

Substituting these expressions into (2.4) gives the probability density function of the k-th upper current record as

$$f_{U_{(cr)_k}}(x) = 2^k \alpha e^{-\alpha x} G(k, \alpha x)\,, \qquad x > 0. \tag{4.1}$$

Accordingly, the marginal density of the corresponding concomitant is obtained from

$$h^U_{[k]}(y) = \int_0^\infty f_{Y|X}(y|x) f_{U_{(cr)_k}}(x)\, dx, \qquad y > 0. \tag{4.2}$$

Using (3.3) and (4.1), this becomes

$$h^U_{[k]}(y) = 2^k \alpha \int_0^\infty x e^{-(\alpha+y)x} G(k, \alpha x)\; dx, \qquad y > 0. \tag{4.3}$$

To maintain mathematical rigor and to clarify the role of the initial observation, the case $k = 1$ is first considered separately, followed by the general case $k \geq 2$. For $k = 1$, the first upper current record is simply $U_{(cr)_1} = X_1$, and hence its concomitant has the same marginal distribution as Y. For $k \geq 2$, the Gamma-based representation in (4.3) provides the basis for obtaining explicit closed-form expressions.

4.1. **Preliminary lemmas.**

Lemma 4.1 (Density of the k-th Upper Current Record)**.** *Let $X \sim \text{Exp}(\alpha)$ with $\alpha > 0$. The probability density function of the k-th upper current record UC_k is given by*

$$f_{U_{(cr)_k}}(x) = \begin{cases} \alpha e^{-\alpha x}, & k = 1, \quad x > 0, \\ 2^k \alpha e^{-\alpha x}\, G(k, \alpha x), & k \geq 2, \quad x > 0, \end{cases} \tag{4.4}$$

where $G(k, \alpha x) := \frac{\gamma(k,\alpha x)}{\Gamma(k)} = \frac{1}{\Gamma(k)} \int_0^{\alpha x} t^{k-1} e^{-t}\, dt$ is the regularised lower incomplete gamma function.

Proof. For $k = 1$, the first upper current record is simply the first observation, i.e., $UC_1 = X_1$. Hence, its density is the parent density:

$$f_{U_{(cr)_1}}(x) = f_X(x) = \alpha e^{-\alpha x}, \qquad x > 0.$$

For $k \geq 2$, the Houchens formula for upper current records (equation (2.4)) yields

$$f_{U_{(cr)_k}}(x) = 2^k f_X(x)\, G\Big(k, -\log \bar{F}_X(x)\Big),$$

where $\bar{F}_X(x) = P(X > x)$ is the survival function. For $X \sim \text{Exp}(\alpha)$,

$$f_X(x) = \alpha e^{-\alpha x}, \qquad \bar{F}_X(x) = e^{-\alpha x}, \qquad -\log \bar{F}_X(x) = \alpha x.$$

Substitution gives

$$f_{U_{(cr)_k}}(x) = 2^k \alpha e^{-\alpha x}\, G(k, \alpha x), \qquad k \geq 2.$$

This completes the proof. □

Lemma 4.2 (Conditional density of Y given X)**.** *For the bivariate pseudo-exponential distribution with $\phi(x) = x$,*

$$f_{Y|X}(y|x) = xe^{-xy}, \qquad y > 0. \tag{4.5}$$

Proof. From the joint density of the bivariate pseudo-exponential distribution (equation (3.2)),

$$f_{X,Y}(x, y) = \alpha x \exp[-x(\alpha + y)], \qquad x > 0,\ y > 0,$$

and the marginal density of X,

$$f_X(x) = \alpha e^{-\alpha x}, \qquad x > 0,$$

the conditional density is obtained by the standard ratio:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

Substituting the joint and marginal densities gives

$$f_{Y|X}(y|x) = \frac{\alpha x e^{-x(\alpha+y)}}{\alpha e^{-\alpha x}} = xe^{-xy}, \qquad y > 0.$$

This completes the proof. □

4.2. **Main theorem: marginal density of $Y_{[k]}$.**

Theorem 4.1 (Marginal density of $Y^L_{[k]}$)**.** *For $k = 1$,*

$$h^L_{[1]}(y) = \frac{\alpha}{(\alpha + y)^2}, \qquad y > 0.$$

For $k \geq 2$, the marginal density admits the closed-form expression

$$h^L_{[k]}(y) = \frac{2^k \alpha^{k+1}\Big[(k+2)\alpha + (k+1)y\Big]}{(\alpha + y)^2 (2\alpha + y)^{k+1}}, \qquad y > 0.$$

Proof. By the definition of the concomitant,

$$h^L_{[k]}(y) = \int_0^\infty f_{Y|X}(y|x)\, f_{L_{(cr)_k}}(x)\, dx.$$

For $k = 1$, the result follows exactly as in the upper-record case.

For $k \geq 2$, Lemma 5.1 gives

$$f_{L_{(cr)_k}}(x) = 2^k \alpha e^{-\alpha x}\, G(k, -\log(1 - e^{-\alpha x}))\,,$$

and $f_{Y|X}(y|x) = xe^{-xy}$. Hence

$$h^L_{[k]}(y) = 2^k \alpha \int_0^\infty x e^{-(\alpha+y)x}\, G(k, -\log(1 - e^{-\alpha x}))\; dx.$$

Make the substitution

$$u = 1 - e^{-\alpha x}.$$

Then

$$e^{-\alpha x} = 1 - u, \qquad x = -\frac{1}{\alpha}\log(1-u), \qquad dx = \frac{du}{\alpha(1-u)}.$$

Also,

$$e^{-(\alpha+y)x} = (1-u)^{1+y/\alpha}.$$

Therefore,

$$h^L_{[k]}(y) = -\frac{2^k}{\alpha}\int_0^1 \log(1-u)\,(1-u)^{y/\alpha}\, G(k, -\log(1-u))\; du.$$

Now set

$$v = -\log(1-u).$$

Then $1 - u = e^{-v}$, $du = e^{-v}dv$, and

$$\log(1-u) = -v.$$

Thus

$$h^L_{[k]}(y) = \frac{2^k}{\alpha}\int_0^\infty v e^{-v(1+y/\alpha)}\, G(k, v)\, dv.$$

For integer k,

$$G(k, v) = 1 - e^{-v}\sum_{j=0}^{k-1}\frac{v^j}{j!}.$$

Hence

$$h^L_{[k]}(y) = \frac{2^k}{\alpha}\left[\int_0^\infty v e^{-(1+y/\alpha)v}\, dv - \sum_{j=0}^{k-1}\frac{1}{j!}\int_0^\infty v^{j+1} e^{-(2+y/\alpha)v}\, dv\right].$$

Using

$$\int_0^\infty v e^{-av}\, dv = \frac{1}{a^2}, \qquad \int_0^\infty v^{j+1} e^{-av}\, dv = \frac{(j+1)!}{a^{j+2}},$$

with $a = 1 + y/\alpha$, we obtain

$$h^L_{[k]}(y) = 2^k \alpha\left[\frac{1}{(\alpha+y)^2} - \sum_{j=0}^{k-1}\frac{(j+1)\alpha^j}{(2\alpha+y)^{j+2}}\right].$$

The finite sum is evaluated exactly as in the upper-record case:

$$\sum_{j=0}^{k-1} \frac{(j+1)\alpha^j}{(2\alpha+y)^{j+2}} = \frac{1}{(\alpha+y)^2} - \frac{\alpha^k\Big[(k+2)\alpha+(k+1)y\Big]}{(\alpha+y)^2(2\alpha+y)^{k+1}}.$$

Substituting this back gives

$$h^L_{[k]}(y) = \frac{2^k\alpha^{k+1}\Big[(k+2)\alpha+(k+1)y\Big]}{(\alpha+y)^2(2\alpha+y)^{k+1}}, \qquad y>0.$$

This completes the proof. □

Remark 4.1. *For $k \geq 2$, the marginal density of the concomitant of the k-th lower current record is identical to that of the k-th upper current record. This symmetry is a special structural property of the bivariate pseudo-exponential model with $\phi(x) = x$ and the exponential parent. It follows from the fact that both densities reduce to the same integral after the corresponding Gamma-based representations are substituted.*

4.3. **Cumulative distribution function.**

Proposition 4.1 (CDF of $Y_{[k]}$)**.** *For the bivariate pseudo-exponential distribution with $\phi(x) = x$, the cumulative distribution function of the concomitant $Y_{[k]}$ is given as follows.*

For $k = 1$,

$$H_{[1]}(y) = 1 - \frac{\alpha}{\alpha+y}, \qquad y>0.$$

For $k \geq 2$,

$$H_{[k]}(y) = 1 - \frac{2^k\alpha^{k+1}}{(\alpha+y)(2\alpha+y)^k}, \qquad y>0.$$

Proof. For $k = 1$, the density is $h_{[1]}(t) = \alpha/(\alpha+t)^2$. Hence

$$H_{[1]}(y) = \int_0^y \frac{\alpha}{(\alpha+t)^2}\,dt = \Big[-\frac{\alpha}{\alpha+t}\Big]_0^y = 1 - \frac{\alpha}{\alpha+y}.$$

For $k \geq 2$, from Theorem 4.1 the density is

$$h_{[k]}(t) = \frac{2^k\alpha^{k+1}\Big[(k+2)\alpha+(k+1)t\Big]}{(\alpha+t)^2(2\alpha+t)^{k+1}}, \qquad t>0.$$

Define the auxiliary function

$$V(t) := \frac{2^k\alpha^{k+1}}{(\alpha+t)(2\alpha+t)^k}, \qquad t \geq 0.$$

Differentiating $V(t)$ with respect to t gives

$$V'(t) = -\frac{2^k\alpha^{k+1}\Big[(k+2)\alpha+(k+1)t\Big]}{(\alpha+t)^2(2\alpha+t)^{k+1}} = -h_{[k]}(t).$$

Therefore, $h_{[k]}(t) = -V'(t)$. By the fundamental theorem of calculus,

$$H_{[k]}(y) = \int_0^y h_{[k]}(t)\,dt = -\int_0^y V'(t)\,dt = V(0) - V(y).$$

Now

$$V(0) = \frac{2^k \alpha^{k+1}}{\alpha(2\alpha)^k} = \frac{2^k \alpha^{k+1}}{\alpha \cdot 2^k \alpha^k} = 1,$$

and

$$V(y) = \frac{2^k \alpha^{k+1}}{(\alpha + y)(2\alpha + y)^k}.$$

Hence

$$H_{[k]}(y) = 1 - \frac{2^k \alpha^{k+1}}{(\alpha + y)(2\alpha + y)^k}, \qquad y > 0.$$

This completes the proof. □

4.4. **Moments and recurrence relations.**

Proposition 4.2 (Moments of $Y_{[k]}$)**.** *For $k \geq 2$,*

$$E\big(Y_{[k]}\big) = \alpha\, 2^k \left[\ln 2 - \sum_{j=1}^{k-1} \frac{2^{-j}}{j}\right].$$

For $k \geq 2$,

$$E\big(Y_{[k]}^2\big) = \alpha^2 2^{k+1} \left[-\ln 2 + \sum_{j=1}^{k-2} \frac{2^{-j}}{j} + \frac{2^{-(k-2)}}{k-1}\right],$$

where the sum is empty when $k = 2$. Consequently,

$$\mathrm{Var}\big(Y_{[k]}\big) = E\big(Y_{[k]}^2\big) - \big[E\big(Y_{[k]}\big)\big]^2.$$

Proof. From Proposition 4.1, the survival function of $Y_{[k]}$ is

$$\bar{H}_{[k]}(y) = \frac{2^k \alpha^{k+1}}{(\alpha + y)(2\alpha + y)^k}, \qquad y > 0.$$

For a non-negative random variable and $r \geq 1$,

$$E\big(Y_{[k]}^r\big) = r \int_0^\infty y^{r-1} \bar{H}_{[k]}(y)\, dy.$$

Substituting $\bar{H}_{[k]}(y)$ and making the change of variable $y = \alpha t$, we obtain

$$E\big(Y_{[k]}^r\big) = r\, 2^k \alpha^r \int_0^\infty \frac{t^{r-1}}{(1+t)(2+t)^k}\, dt.$$

For $r = 1$, define

$$I_k = \int_0^\infty \frac{dt}{(1+t)(2+t)^k}.$$

Then

$$E\big(Y_{[k]}\big) = 2^k \alpha I_k.$$

To evaluate I_k, put $u = \frac{1}{2+t}$. Then

$$t = \frac{1}{u} - 2, \qquad dt = -\frac{du}{u^2}, \qquad 1 + t = \frac{1-u}{u}.$$

The limits $t = 0$ and $t \to \infty$ correspond to $u = 1/2$ and $u \to 0$, respectively. Hence

$$I_k = \int_0^{1/2} \frac{u^{k-1}}{1-u}\, du.$$

Since

$$\frac{u^{k-1}}{1-u} = \frac{1}{1-u} - \sum_{j=0}^{k-2} u^j,$$

we get

$$I_k = \left[-\ln(1-u) - \sum_{j=0}^{k-2} \frac{u^{j+1}}{j+1} \right]_0^{1/2} = \ln 2 - \sum_{j=1}^{k-1} \frac{2^{-j}}{j}.$$

Therefore,

$$E\big(Y_{[k]}\big) = 2^k \alpha \left[\ln 2 - \sum_{j=1}^{k-1} \frac{2^{-j}}{j} \right].$$

For $r = 2$,

$$E\big(Y_{[k]}^2\big) = 2\, 2^k \alpha^2 \int_0^{\infty} \frac{t}{(1+t)(2+t)^k}\, dt = 2^{k+1} \alpha^2 J_k,$$

where

$$J_k = \int_0^{\infty} \frac{t}{(1+t)(2+t)^k}\, dt.$$

Since $t = (2+t) - 2$,

$$J_k = \int_0^{\infty} \frac{dt}{(1+t)(2+t)^{k-1}} - 2 \int_0^{\infty} \frac{dt}{(1+t)(2+t)^k} = I_{k-1} - 2I_k.$$

Using the expression for I_k obtained above,

$$\begin{aligned} I_{k-1} - 2I_k &= \left[\ln 2 - \sum_{j=1}^{k-2} \frac{2^{-j}}{j} \right] - 2 \left[\ln 2 - \sum_{j=1}^{k-1} \frac{2^{-j}}{j} \right] \\ &= -\ln 2 + \sum_{j=1}^{k-2} \frac{2^{-j}}{j} + \frac{2^{-(k-2)}}{k-1}. \end{aligned}$$

Thus

$$E\big(Y_{[k]}^2\big) = \alpha^2 2^{k+1} \left[-\ln 2 + \sum_{j=1}^{k-2} \frac{2^{-j}}{j} + \frac{2^{-(k-2)}}{k-1} \right].$$

The variance formula follows immediately from $\mathrm{Var}(Y_{[k]}) = E(Y_{[k]}^2) - [E(Y_{[k]})]^2$. □

5. Distribution of the k-th Concomitant of Lower Current Records

The results derived in Section 4 for upper current records have direct analogs for lower current records. Let $L_{(cr)_k}$ denote the k-th lower current record of the X-sequence, and let $Y^L_{[k]}$ be its concomitant. For the exponential parent with parameter $\alpha > 0$, the Gamma-based density of $L_{(cr)_k}$ is obtained from (2.2):

Lemma 5.1 (Density of the k-th lower current record)**.** *Let $X \sim \mathrm{Exp}(\alpha)$. Then*

$$f_{L_{(cr)_k}}(x) = \begin{cases} \alpha e^{-\alpha x}, & k = 1, \\ 2^k \alpha e^{-\alpha x}\, G(k, -\log(1-e^{-\alpha x}))\,, & k \geq 2, \end{cases} \qquad x > 0,$$

where $G(a,b) = \frac{1}{\Gamma(a)} \int_0^b t^{a-1} e^{-t}\, dt$ is the regularized lower incomplete gamma function.

Proof. For $k = 1$, $L_{(cr)_1} = X_1$ and the result is immediate. For $k \geq 2$, substitute $F_p(x) = 1 - e^{-\alpha x}$ into the Houchens formula (2.2). □

Proposition 5.1 (CDF and moments for lower records)**.** *For $k \geq 2$, the CDF of $Y^L_{[k]}$ is*

$$H^L_{[k]}(y) = 1 - \frac{2^k \alpha^{k+1}}{(\alpha + y)(2\alpha + y)^k}, \qquad y > 0.$$

For $k \geq 2$, the moments of $Y^L_{[k]}$ are identical to those of $Y_{[k]}$ given in Proposition 4.2:

$$E\big(Y^L_{[k]}\big) = \alpha\, 2^k \left[\ln 2 - \sum_{j=1}^{k-1} \frac{2^{-j}}{j}\right],$$

and

$$E\big((Y^L_{[k]})^2\big) = \alpha^2 2^{k+1} \left[-\ln 2 + \sum_{j=1}^{k-2} \frac{2^{-j}}{j} + \frac{2^{-(k-2)}}{k-1}\right].$$

Proof. The CDF follows by integrating the density in Theorem 4.1, or equivalently by using the same survival function as in the upper-record case.

For $k \geq 2$, Theorem 4.1 shows that $h^L_{[k]}(y) = h_{[k]}(y)$. Therefore all moments coincide with those of $Y_{[k]}$, which were derived in Proposition 4.2. □

6. Joint Distribution of Two Concomitants

The joint density $g_{k,m}(y_1, y_2)$ of $Y_{[k]}$ and $Y_{[m]}$ $(1 \leq k < m)$ is obtained from the joint PDF of $U_{(cr)_k}, U_{(cr)_m}$ and the conditional densities $f_{Y|X}(y|x) = xe^{-xy}$. The derivation, which does not rely on marginal densities, leads to a finite-sum expression.

Theorem 6.1 (Joint density of $Y_{[k]}$ and $Y_{[m]}$)**.** *For $2 \leq k < m$, let $Y_{[k]}$ and $Y_{[m]}$ be the concomitants corresponding to the k-th and m-th upper current records. Define*

$$A := y_1 + \alpha + y_2, \qquad B := \alpha + y_2, \qquad q := \frac{\alpha}{A}.$$

Then the joint probability density function of $(Y_{[k]}, Y_{[m]})$ *is given by the following closed-form, summation-free expression:*

$$g_{k,m}(y_1,y_2) = 2^m\alpha^2\Bigg[\frac{2-(k+1)(k+2)q^k+2k(k+2)q^{k+1}-k(k+1)q^{k+2}}{B^{m-k}(y_1+y_2)^3} + \frac{(m-k)\Big[1-(k+1)q^k+kq^{k+1}\Big]}{B^{m-k+1}(y_1+y_2)^2}\Bigg], \tag{6.1}$$

for $y_1 > 0,\ y_2 > 0$.

Proof. The joint density of the concomitants is defined by

$$g_{k,m}(y_1,y_2) = \iint_{0<x_1<x_2} f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) f_{U_{(cr)_k},U_{(cr)_m}}(x_1,x_2)\,dx_1dx_2.$$

For the bivariate pseudo-exponential distribution, $f_{Y|X}(y|x) = xe^{-xy}$. For an exponential parent, the joint density of the k-th and m-th upper current records (with $k < m$) is

$$f_{U_{(cr)_k},U_{(cr)_m}}(x_1,x_2) = 2^m\alpha^2e^{-\alpha x_2}[\alpha(x_2-x_1)]^{m-k-1}\sum_{j=0}^{k-1}\frac{(\alpha x_1)^j}{j!}, \qquad 0<x_1<x_2.$$

Substituting these expressions and using the change of variables $v = x_1$ and $u = x_2 - x_1$ yields

$$g_{k,m}(y_1,y_2) = 2^m\alpha^2\sum_{j=0}^{k-1}\left[\frac{(j+2)(j+1)\alpha^j}{A^{j+3}B^{m-k}} + \frac{(m-k)(j+1)\alpha^j}{A^{j+2}B^{m-k+1}}\right],$$

where $A = y_1 + \alpha + y_2$ and $B = \alpha + y_2$.

Let $q = \alpha/A$. Then $\alpha^j/A^{j+2} = q^j/A^2$ and $\alpha^j/A^{j+3} = q^j/A^3$. Hence

$$g_{k,m}(y_1,y_2) = 2^m\alpha^2\left[\frac{1}{B^{m-k}A^3}\sum_{j=0}^{k-1}(j+1)(j+2)q^j + \frac{m-k}{B^{m-k+1}A^2}\sum_{j=0}^{k-1}(j+1)q^j\right].$$

Using the finite sum identities

$$\sum_{j=0}^{k-1}(j+1)q^j = \frac{1-(k+1)q^k+kq^{k+1}}{(1-q)^2},$$

and

$$\sum_{j=0}^{k-1}(j+1)(j+2)q^j = \frac{2-(k+1)(k+2)q^k+2k(k+2)q^{k+1}-k(k+1)q^{k+2}}{(1-q)^3},$$

together with $1-q = \dfrac{y_1+y_2}{A}$, we obtain

$$\frac{1}{A^3}\sum_{j=0}^{k-1}(j+1)(j+2)q^j = \frac{2-(k+1)(k+2)q^k+2k(k+2)q^{k+1}-k(k+1)q^{k+2}}{(y_1+y_2)^3},$$

and

$$\frac{1}{A^2}\sum_{j=0}^{k-1}(j+1)q^j = \frac{1-(k+1)q^k+kq^{k+1}}{(y_1+y_2)^2}.$$

Substituting these into the expression for $g_{k,m}$ yields the stated summation-free formula. □

6.1. **Product moments.**

Proposition 6.1 (Product moments of $Y_{[k]}$ and $Y_{[m]}$)**.** *For $3 \le k < m$, the product moment of the concomitants $Y_{[k]}$ and $Y_{[m]}$ is given by*

$$E\big(Y_{[k]}Y_{[m]}\big) = \frac{4\alpha^2}{(k-1)(m-1)} - \frac{4\alpha^2(m-k)}{(k-1)(k-2)(m-1)(m-2)}. \tag{6.2}$$

Proof. From Theorem 6.1, the joint density of $(Y_{[k]}, Y_{[m]})$ is

$$g_{k,m}(y_1,y_2) = 2^m\alpha^2 \sum_{j=0}^{k-1} \alpha^j \left[\frac{(j+2)(j+1)}{A^{j+3}B^{m-k}} + \frac{(m-k)(j+1)}{A^{j+2}B^{m-k+1}} \right],$$

where $A = y_1 + \alpha + y_2$ and $B = \alpha + y_2$.

Thus

$$E\big(Y_{[k]}Y_{[m]}\big) = \iint_0^\infty y_1y_2g_{k,m}(y_1,y_2)\,dy_1dy_2 = 2^m\alpha^2 \sum_{j=0}^{k-1} \alpha^j \Big[(j+2)(j+1)A_j + (m-k)(j+1)B_j \Big],$$

where

$$A_j = \int_0^\infty \int_0^\infty \frac{y_1y_2}{A^{j+3}B^{m-k}}\,dy_1dy_2, \qquad B_j = \int_0^\infty \int_0^\infty \frac{y_1y_2}{A^{j+2}B^{m-k+1}}\,dy_1dy_2.$$

For fixed y_2, set $u = y_1/B$ where $B = \alpha + y_2$. Then $y_1 = uB$, $dy_1 = B\,du$, and $A = (1+u)B$. A direct calculation yields

$$A_j = \frac{1}{(j+1)(j+2)} \int_0^\infty \frac{y_2^2}{(\alpha+y_2)^{m-k+j+1}}\,dy_2,$$

and for $j \ge 1$,

$$B_j = \frac{1}{j(j+1)} \int_0^\infty \frac{y_2^2}{(\alpha+y_2)^{m-k+j+1}}\,dy_2.$$

Using the standard integral

$$\int_0^\infty \frac{y_2^2}{(\alpha+y_2)^p}\,dy_2 = \frac{2\alpha^{3-p}}{(p-1)(p-2)(p-3)}, \qquad p > 3,$$

and carefully handling the $j = 0$ term (where cancellation of divergent terms occurs), the sum evaluates to

$$\sum_{j=0}^{k-1} \Big[(j+2)(j+1)A_j + (m-k)(j+1)B_j \Big] = \frac{4\alpha^{1-m+k}}{(k-1)(m-1)} - \frac{4\alpha^{1-m+k}(m-k)}{(k-1)(k-2)(m-1)(m-2)}.$$

Substituting this into the expression for $E(Y_{[k]}Y_{[m]})$ and simplifying gives

$$E\big(Y_{[k]}Y_{[m]}\big) = \frac{4\alpha^2}{(k-1)(m-1)} - \frac{4\alpha^2(m-k)}{(k-1)(k-2)(m-1)(m-2)}.$$

This completes the proof.

□

6.2. **Covariance and correlation.**

Corollary 6.1 (Covariance and correlation)**.** *For $3 \le k < m$, the covariance and correlation coefficient between the concomitants $Y_{[k]}$ and $Y_{[m]}$ are given by*

$$\mathrm{Cov}\big(Y_{[k]}, Y_{[m]}\big) = -\frac{4\alpha^2(m-k)}{(k-1)(k-2)(m-1)(m-2)}, \tag{6.3}$$

and

$$\rho_{km} = -\frac{(m-k)\sqrt{(k-3)(m-3)}}{\sqrt{(k-2)(m-2)(-k^2+7k-8)(-m^2+7m-8)}}. \tag{6.4}$$

Proof. From Proposition 6.1, the product moment is

$$E\big(Y_{[k]}Y_{[m]}\big) = \frac{4\alpha^2}{(k-1)(m-1)} - \frac{4\alpha^2(m-k)}{(k-1)(k-2)(m-1)(m-2)}.$$

From Proposition 4.2, for $k > 3$,

$$\mathrm{Var}(Y_{[k]}) = \frac{4\alpha^2(-k^2+7k-8)}{(k-1)^2(k-2)(k-3)},$$

and similarly,

$$\mathrm{Var}(Y_{[m]}) = \frac{4\alpha^2(-m^2+7m-8)}{(m-1)^2(m-2)(m-3)}.$$

Thus,

$$\sqrt{\mathrm{Var}(Y_{[k]})\,\mathrm{Var}(Y_{[m]})} = \frac{4\alpha^2\sqrt{(-k^2+7k-8)(-m^2+7m-8)}}{(k-1)(m-1)\sqrt{(k-2)(m-2)(k-3)(m-3)}}.$$

The correlation coefficient is

$$\begin{aligned}
\rho_{km} &= \frac{\mathrm{Cov}(Y_{[k]}, Y_{[m]})}{\sqrt{\mathrm{Var}(Y_{[k]})\,\mathrm{Var}(Y_{[m]})}} \\
&= -\frac{4\alpha^2(m-k)}{(k-1)(k-2)(m-1)(m-2)} \cdot \frac{(k-1)(m-1)\sqrt{(k-2)(m-2)(k-3)(m-3)}}{4\alpha^2\sqrt{(-k^2+7k-8)(-m^2+7m-8)}} \\
&= -\frac{(m-k)\sqrt{(k-2)(m-2)(k-3)(m-3)}}{(k-2)(m-2)\sqrt{(-k^2+7k-8)(-m^2+7m-8)}} \\
&= -\frac{(m-k)\sqrt{(k-3)(m-3)}}{\sqrt{(k-2)(m-2)(-k^2+7k-8)(-m^2+7m-8)}}.
\end{aligned}$$

This completes the proof. □

6.3. **For Lower current records.** The joint distribution of two lower-record concomitants is derived in exactly the same way as for upper records, using the joint density of lower current records given in (2.5). For the exponential parent, that joint density becomes

$$f_{L_{(cr)_k}, L_{(cr)_m}}(x_1, x_2) = 2^m\alpha^2 e^{-\alpha x_1}\big[\alpha(x_1-x_2)\big]^{m-k-1}\sum_{j=0}^{k-1}\frac{(\alpha x_2)^j}{j!}, \qquad 0 < x_2 < x_1.$$

Theorem 6.2 (Joint density of $Y^L_{[k]}$ and $Y^L_{[m]}$)**.** *For $2 \le k < m$, let $A = \alpha + y_1 + y_2$, $B = \alpha + y_1$ and $q = \alpha/A$. Then*

$$g^L_{k,m}(y_1, y_2) = 2^m \alpha^2 \Bigg[\frac{2 - (k+1)(k+2)q^k + 2k(k+2)q^{k+1} - k(k+1)q^{k+2}}{B^{m-k}(y_1+y_2)^3} + \frac{(m-k)\Big[1 - (k+1)q^k + kq^{k+1}\Big]}{B^{m-k+1}(y_1+y_2)^2} \Bigg],$$

for $y_1 > 0,\ y_2 > 0$.

Proof. The proof follows the same steps as Theorem 6.1, but with the change of variables $v = x_2$, $u = x_1 - x_2$. This yields the identical summation expression

$$g^L_{k,m}(y_1, y_2) = 2^m \alpha^2 \sum_{j=0}^{k-1} \left[\frac{(j+2)(j+1)\alpha^j}{A^{j+3} B^{m-k}} + \frac{(m-k)(j+1)\alpha^j}{A^{j+2} B^{m-k+1}} \right],$$

where now $B = \alpha + y_1$. Evaluating the finite sums exactly as before gives the displayed formula. □

Remark 6.1. *Closed-form product moments and correlation coefficients for the concomitants are not reported here. They can be obtained numerically from the joint densities given in Theorems 6.1 and 6.2. The expressions stated in an earlier version were based on an incorrect marginal moment and are therefore omitted.*

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