Int. J. Anal. Appl. (2026), 24:380

International Journal of Analysis and Applications

Uniformly Convex and Starlike Functions Associated with Generalized Mittag-Leffler Functions

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Abstract. The main object of this paper is to obtain the coefficient condition for the normalized generalized Mittag-Leffler functions of certain classes of univalent functions. We also obtain some inclusion relations. Finally, we discuss an integral operator associated with generalized Mittag-Leffler functions.

1. Introduction

Let $\mathcal{A}$ denote the collection of functions Φ defined by the expression:

$$\Phi(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \tag{1.1}$$

where these functions are analytic within the open unit disk $\mathbb{U} = \{\zeta : \zeta \in \mathbb{C} \text{ and } |\zeta| < 1\}$. These functions are normalized under the condition $\Phi(0) = \Phi'(0) - 1 = 0$. Additionally, we define $\mathcal{S}$ as a subset of $\mathcal{A}$ comprising functions of the same form as above, which are also univalent within $\mathbb{U}$.

A function $\Phi(\zeta) \in \mathcal{A}$ belongs to the class $k - UCV(\alpha)$ if it satisfies the condition:

$$\Re\left\{1 + \frac{\zeta\Phi''(\zeta)}{\Phi'(\zeta)}\right\} \geq k\left|\frac{\zeta\Phi''(\zeta)}{\Phi'(\zeta)}\right| + \alpha \tag{1.2}$$

where $0 \leq k < \infty$, $0 \leq \alpha < 1$.

Received: Feb. 13, 2025.

2020 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic function; univalent function; generalized Mittag-Leffler function.

https://doi.org/10.28924/2291-8639-24-2026-380
ISSN: 2291-8639

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This class, denoted as $k-UCV(\alpha)$, was originally introduced and examined by Bharti et al. [1]. Notably, for $\alpha = 0$, Kanas and Wisniowska [7] explored this class, and for $k = 1, \alpha = 0$, Goodman [4] conducted an investigation. Utilizing the Alexander transform, we derive the class $k - S_p(\alpha)$ as follows:

$$\Phi(\zeta) \in k - UCV(\alpha) \Leftrightarrow \zeta\Phi'(\zeta) \in k - S_p(\alpha).$$

These classes, $k - UCV(\alpha)$ and $k - S_p(\alpha)$, represent natural extensions of convex and starlike function classes, playing significant roles in geometric function theory. Goodman [5] and Kanas with Wisniowska [8] delved into the study of uniformly starlike functions, which were further generalized by various researchers, as detailed in [2, 3, 6, 9, 16]. Ponnusamy and Rønning [11] introduced the classes $\mathcal{S}^*_\lambda$ and C_λ as follows: A function $\Phi(\zeta)$ of the form described above is part of the class $\mathcal{S}^*_\lambda$ if it satisfies the analytic criteria:

$$\left|\frac{\zeta\Phi'(\zeta)}{\Phi(\zeta)} - 1\right| < \lambda \tag{1.3}$$

where $\zeta \in \mathbb{U}$ and $\lambda > 0$. Similarly, a function $\Phi(\zeta)$ of the form (1.1) is said to be in the class C_λ if it satisfy the following analytic criteria

$$\left|\frac{\zeta\Phi''(\zeta)}{\Phi'(\zeta)}\right| < \lambda \tag{1.4}$$

where $\zeta \in \mathbb{U}$ and $\lambda > 0$. It's worth noting that:

$$\Phi(\zeta) \in C_\lambda \Leftrightarrow \zeta\Phi'(\zeta) \in \mathcal{S}^*_\lambda.$$

The application of special functions such as hypergeometric functions, generalized hypergeometric functions, generalized Bessel functions, Mittag-Leffler functions, and Wright functions on univalent functions are intriguing research topics within geometric function theory. In this article, we explore a fascinating application of the generalized Mittag-Leffler function on classes of univalent functions within conic regions. First, we define the generalized Mittag-Leffler function as follows:

In 1903, Swedish Mathematician Gösta Mittag-Leffler [10] introduced the function:

$$E_\delta(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\delta n + 1)} \tag{1.5}$$

where ζ is a complex variable and $\Gamma(\delta)$ is the Gamma function, with $\delta \geqslant 0$.

This function serves as a direct generalization of the exponential function, reducing to it when $\delta = 1$. Wiman [14, 15] extended $E_\delta(\zeta)$ into $E_{\delta,\beta}(\zeta)$ in 1905:

$$E_{\delta,\beta}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\delta n + \beta)} \tag{1.6}$$

where $\delta, \beta \in C; \Re(\delta) > 0, \Re(\beta) > 0$. In 1971, Prabhakar [12] generalized $E_{\delta,\beta}(\zeta)$ as follows

$$E^\gamma_{\delta,\beta}(\zeta) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\delta n + \beta)} \zeta^n, \tag{1.7}$$

where $\delta, \beta, \gamma \in C\Re(\delta) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $(\gamma)_n$ is the Pochhammer symbol.

In 2007, Shukla and Prajapati [13] generalized the function $E^{\gamma}_{\delta,\beta}(\zeta)$ as follows:

$$E^{\gamma,q}_{\delta,\beta}(\zeta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\delta n + \beta)} \frac{\zeta^n}{n!} \tag{1.8}$$

where $\delta, \beta, \gamma \in C \quad \Re(\delta) > 0, \Re(\beta) > 0 \ \Re(\gamma) > 0, q \in (0,1) \cup N = \{1,2,3,\ldots\}$ and $(\gamma)_{qn}$ is the generalized pochhammer symbol.

We normalize the function $E^{\gamma,q}_{\delta,\beta}(\zeta)$ as follows:

$$\Omega^{\gamma,q}_{\delta,\beta}(\zeta) = \Gamma(\beta)\zeta E^{\gamma,q}_{\delta,\beta}(\zeta)$$

$$\Omega^{\gamma,q}_{\delta,\beta}(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{(\gamma)_{q(n-1)}\Gamma(\beta)}{\Gamma\delta(n-1)+\beta} \frac{\zeta^n}{(n-1)!}. \tag{1.9}$$

In this paper, we establish coefficient conditions, inclusion relations, and study an integral operator for the normalized generalized Mittag-Leffler function within certain classes of univalent functions of conic.

2. Preliminary Results

To prove our main results we shall require the following lemmas.

Lemma 2.1. *([1]) A function* $\Phi(\zeta)$ *of the form* (1.1) *and satisfy the condition*

$$\sum_{n=2}^{\infty} n[n(k+1) - (k+\alpha)]|a_n| \leq 1 - \alpha \tag{2.1}$$

then $\Phi \in k - UCV(\alpha)$.

Lemma 2.2. *([1]) A function* $\Phi(\zeta)$ *of the form* (1.1) *and satisfy the condition*

$$\sum_{n=2}^{\infty} [n(k+1) - (k+\alpha)]|a_n| \leq 1 - \alpha \tag{2.2}$$

then $\Phi \in k - S_p(\alpha)$.

Lemma 2.3. *([7]) Let* $\Phi \in \mathcal{A}$ *and have the form* (1.1). *If for some* $k, 0 \leq k < \infty$, *the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2}, \tag{2.3}$$

holds, then $\Phi \in k - UCV$. *The result is sharp.*

Lemma 2.4. *([11]) Let* $\Phi \in \mathcal{A}$ *be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty} (\lambda + n - 1)|a_n| \leq \lambda, \ \lambda > 0 \tag{2.4}$$

then $\Phi \in \mathcal{S}^*_{\lambda}$.

Lemma 2.5. *([11]) Let $\Phi \in A$ be of the form* (1.1)*. If*

$$\sum_{n=2}^{\infty} n(\lambda + n - 1)|a_n| \leq \lambda, \ \lambda > 0 \tag{2.5}$$

then $\Phi \in C_\lambda$.

Remark 2.1. *The conditions* (2.1), (2.2), (2.4) *and* (2.5) *are also necessary and sufficient for belonging to the classes $k - UCV(\alpha)$, $k - S_p(\alpha)$, $\mathcal{S}_\lambda^*$ and C_λ, respectively, if $\Phi \in \mathcal{A}$ is of the form*

$$\Phi(\zeta) = \zeta - \sum_{n=2}^{\infty} |a_n|\zeta^n. \tag{2.6}$$

Definition 2.1 A function Φ of the form (1.1) is said to be in the class $\mathcal{R}^\tau(A, B)$ if it satisfies the following analytic criteria

$$\left| \frac{\Phi'(\zeta) - 1}{(A - B)\tau - B\left(\Phi'(\zeta) - 1\right)} \right| < 1 \tag{2.7}$$

where $\zeta \in \mathbb{U}, \tau \in C \setminus \{0\}, -1 \leq B < A \leq 1$ The class $\mathcal{R}^\tau(A, B)$ was extensively studied by Dixit and Pal [16].

Lemma 2.6. *([16]) If $\Phi \in R^\tau(A, B)$ is of the form* (1.1) *then*

$$|a_n| < \frac{(A - B)(\tau)}{n}, \quad n \geqslant 2. \tag{2.8}$$

The result is sharp.

3. Main Results

Our initial theorem provides a condition that is adequate for the normalized generalized Mittag-Leffler function, as defined in equation (1.9), to belong to the class $k - UCV(\alpha)$.

Theorem 3.1. *If $\delta, \beta, \gamma > 0, q \in (0, 1) \cup \mathbb{N}$ and the inequality*

$$\begin{aligned} &(k+1)(\gamma)_{2q}\left\{\frac{\Gamma(\beta)}{\Gamma(\beta+2\delta)} + \Omega_{\delta,\beta+2\delta}^{\gamma+2q,q}(1) - 1\right\} \\ &+(2k+3-\alpha)(\gamma)_q\left\{\frac{\Gamma(\beta)}{\Gamma(\beta+\delta)} + \Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1) - 1\right\} \\ &+(1-\alpha)\Omega_{\delta,\beta}^{\gamma,q}(1) \leq 2(1-\alpha) \end{aligned} \tag{3.1}$$

is satisfied then $\Omega_{\delta,\beta}^{\gamma,q} \in k - UCV(\alpha)$.

Proof. To prove $\Omega_{\delta,\beta}^{\gamma,q}(\zeta) \in K - UCV(\alpha)$ from Lemma 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n(k+1) - (k+\alpha)] \frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1) + \beta} \frac{\Gamma\beta}{(n-1)!} \leq 1 - \alpha.$$

Now

$$\sum_{n=2}^{\infty} n[n(k+1) - (k+\alpha)] \frac{(\gamma)_{q(n-1)}}{\Gamma(\delta(n-1) + \beta)} \frac{\Gamma\beta}{(n-1)!}$$

$$= \sum_{n=2}^{\infty}[(k+1)(n-1)(n-2)+(2k+3-\alpha)(n-1)+(1-\alpha)]\frac{(\gamma)_{q(n-1)}\Gamma\beta}{\Gamma\left(\delta(n-1)+\beta\right)(n-1)!}$$

$$= (k+1)\sum_{n=3}^{\infty}\frac{(\gamma)_{q(n-1)}\Gamma(\beta)}{\Gamma\left(\delta(n-1)+\beta\right)(n-3)!}+(2k+3-\alpha)\sum_{n=2}^{\infty}\frac{(\gamma)_{q(n-1)}}{\Gamma\left(\delta(n-1)+\beta\right)}\frac{\Gamma\beta}{(n-2)!}$$

$$+(1-\alpha)\sum_{n=2}^{\infty}\frac{(\gamma)_{q(n-1)}\Gamma\beta}{\Gamma\left(\delta(n-1)+\beta\right)(n-1)!}$$

$$= (k+1)\sum_{n=0}^{\infty}\frac{(\gamma)_{q(n+2)}\Gamma(\beta)}{\Gamma\left(\delta(n+2)+\beta\right)n!}+(2k+3-\alpha)\sum_{n=0}^{\infty}\frac{(\gamma)_{q(n+1)}}{\Gamma\left(\delta(n+1)+\beta\right)}\frac{\Gamma\beta}{n!}$$

$$+(1-\alpha)\sum_{n=1}^{\infty}\frac{(\gamma)_{qn}\Gamma\beta}{\Gamma\left(\delta n+\beta\right)n!}$$

$$= (k+1)(\gamma)_{2q}\sum_{n=0}^{\infty}\frac{(\gamma+2q)_{qn}}{\Gamma\delta n+\beta+2\delta}\frac{\Gamma\beta}{n!}+(2k+3-\alpha)(\gamma)_q\sum_{n=0}^{\infty}\frac{(\gamma+q)_{qn}}{\Gamma\delta n+\beta+\delta}\frac{\Gamma\beta}{n!}$$

$$+(1-\alpha)\left\{\Omega_{\delta,\beta}^{\gamma,q}(1)-1\right\}$$

$$= (k+1)(\gamma)_{2q}\sum_{n=0}^{\infty}\frac{(\gamma+2q)_{qn}}{\Gamma\delta n+\beta+2\delta}\frac{\Gamma\beta}{n!}+(2k+3-\alpha)(\gamma)_q\sum_{n=0}^{\infty}\frac{(\gamma+q)_{qn}}{\Gamma\delta n+\beta+\delta}\frac{\Gamma\beta}{n!}$$

$$+(1-\alpha)\left\{\Omega_{\delta,\beta}^{\gamma,q}(1)-1\right\}$$

$$= (k+1)(\gamma)_{2q}\left\{\Gamma\beta\Gamma\beta+2\delta+\Omega_{\delta,\beta+2\delta}^{\gamma+2q,q}(1)-1\right\}$$

$$+(2k+3-\alpha)(\gamma)_q\left\{\frac{\Gamma\beta}{\Gamma\beta+\delta}+\Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1)-1\right\}+(1-\alpha)\left\{\Omega_{\delta,\beta}^{\gamma,q}(1)-1\right\}$$

$$\leq 1-\alpha, \quad \text{from (3.1)}$$

Therefore, the proof of Theorem 3.1 is concluded.

□

Theorem 3.2. *If $\delta,\beta,\gamma>0,\ q\in(0,1)\cup\mathbb{N}$ and the inequality*

$$(k+1)(\gamma)_q\left\{\frac{\Gamma\beta}{\Gamma\beta+\delta}+\Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1)-1\right\}+(1-\alpha)\Omega_{\delta,\beta}^{\gamma,q}(1)\leq 2(1-\alpha) \tag{3.2}$$

is satisfied then $\Omega_{\delta,\beta}^{\gamma,q}(\zeta)\in k-S_p(\alpha)$.

Proof. The proof of the aforementioned theorem closely resembles that of Theorem 3.1, thus we opt not to delve into the intricate details. □

Theorem 3.3. *If $\delta,\beta,\gamma>0, q\in(0,1)\cup\mathbb{N}$ and the inequality*

$$(\gamma)_{2q}\left\{\frac{\Gamma(\beta)}{\Gamma(\beta+2\delta)}+\Omega_{\delta,\beta+2\delta}^{\gamma+2q,q}(1)-1\right\}+2(\gamma)_q\left\{\frac{\Gamma(\beta)}{\Gamma(\beta+\delta)}+\Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1)-1\right\}\leq\frac{1}{k+2} \tag{3.3}$$

is satisfied then $\Phi\in k-UCV$.

Proof. To prove $\Omega_{\delta,\beta}^{\gamma,q}(\zeta) \in k-UCV$ then from Lemma 2.3, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n-1)\frac{(\gamma)_{q(n-1)}}{\Gamma(\delta(n-1)+\beta)}\frac{\Gamma\beta}{(n-1)!} \leq \frac{1}{k+2}.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)\frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta}\frac{\Gamma\beta}{(n-1)!}\\
=&\sum_{n=2}^{\infty}[(n-1)(n-2)+2(n-1)]\frac{(\gamma)_{q(n-1)}}{\Gamma(\delta(n-1)+\beta)}\frac{\Gamma\beta}{(n-1)!}\\
=&\sum_{n=3}^{\infty}\frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta}\frac{\Gamma\beta}{(n-3)!}+2\sum_{n=2}^{\infty}\frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta}\frac{\Gamma\beta}{(n-2)!}\\
=&(\gamma)_{2q}\sum_{n=0}^{\infty}\frac{(\gamma+2q)_n}{\Gamma\delta n+\beta+2\delta}\frac{\Gamma\beta}{n!}+2(\gamma)_q\sum_{n=0}^{\infty}\frac{(\gamma+q)_n}{\Gamma\delta n+\beta+\delta}\frac{\Gamma\beta}{n!}\\
=&(\gamma)_{2q}\left\{\Omega_{\delta,\beta+2\delta}^{r+2q,q}(1)+\frac{\Gamma\beta}{\Gamma\beta-2\delta}\right\}\\
=&2(\gamma)_q\left\{\frac{\Gamma\beta}{\Gamma\beta+\delta}+\Omega_{\delta,\beta+2\delta}^{r+2q,q}(1)-1\right\}\\
\leq&\frac{1}{k+2},\ \text{from (3.3).}
\end{aligned}$$

Thus, the proof of Theorem 3.3 is established. □

The proof of Theorem 3.4 and 3.5 run parallel to the proof of Theorem 3.1 so we only state the results.

Theorem 3.4. *If $\delta,\beta,\gamma,\lambda>0$, $q\in(0,1)\cup\mathbb{N}$ and the inequality*

$$\lambda\Omega_{\delta,\beta}^{\gamma,q}(1)+(\gamma)_q\left\{\frac{\Gamma(\beta)}{\Gamma(\beta+\delta)}+\Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1)-1\right\}\leq 2\lambda \tag{3.4}$$

is satisfied then $\Omega_{\delta,\beta}^{\gamma,q}(\zeta)\in\mathcal{S}_{\lambda}^{*}$.

Theorem 3.5. *If $\delta,\beta,\gamma,\lambda>0, q\in(0,1)\cup\mathbb{N}$ and the inequality*

$$\begin{aligned}
&(\gamma)_{2q}\left\{\frac{\Gamma\beta}{\Gamma\beta+2\delta}+\Omega_{\delta,\beta+2\delta}^{\gamma+2q,q}(1)-1\right\}\\
&+(2+\lambda)(\gamma)_q\left\{\frac{\Gamma\beta}{\Gamma\beta+\delta}+\Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1)-1\right\}\\
&+\lambda\Omega_{\delta,\beta}^{\gamma,q}(1)\leq 2\lambda
\end{aligned}\tag{3.5}$$

is satisfied then $\Omega_{\delta,\beta}^{\gamma,q}(\zeta)\in C_{\lambda}$.

In our subsequent finding, we establish the inclusion relation between the classes $R^{\tau}(A,B)$ and $k-UCV(\alpha)$. To facilitate this, we introduce the following definition of convolution:

The convolution, also known as the Hadamard product, involves two functions $\Phi(\zeta)$ of the form described in equation (1.1), and $g(\zeta)$, formulated as follows:

$$g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n \tag{3.6}$$

is given by

$$(\Phi * g)(\zeta) = \Phi(\zeta) * g(\zeta) + \sum_{n=2}^{\infty} a_n b_n \zeta^n \tag{3.7}$$

Now, we, introduce the convolution operator $I_{\delta,\beta}^{\gamma,q}\Phi(\zeta)$ as follows.

$$I_{\delta,\beta}^{\gamma,q}\Phi(\zeta) = \Omega_{\delta,\beta}^{\gamma,q}(\zeta) * \Phi(\zeta)$$

or

$$I_{\delta,\beta}^{\gamma,q}\Phi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta} \frac{\Gamma\beta}{(n-1)!} a_n \zeta^n \tag{3.8}$$

Theorem 3.6. *If $\delta, \beta, \gamma > 0, q \in (0,1) \cup \mathbb{N}$, $\Phi \in R^{\tau}(A,B), 0 \leq \alpha < 1, k \geqslant 0$ and the inequality*

$$(A-B)|\tau| \left[(k+1)(\gamma)_q \left\{ \frac{\Gamma\beta}{\Gamma\beta+\delta} + \Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1) - 1 \right\} + (1-\alpha) \left\{ \Omega_{\delta,\beta}^{\gamma,q}(1) - 1 \right\} \right] \leq 1-\alpha$$

is satisfied then $I_{\delta,\beta}^{\gamma,q}\Phi(\zeta) \in k - UCV(\alpha)$.

Proof. To prove $I_{\delta,\beta}^{\gamma,q}\Phi(\zeta) \in k - UCV(\alpha)$ from Lemma 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n(k+1) - (k+\alpha)] \frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta} \frac{\Gamma\beta}{(n-1)!} |a_n| \leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n(k+1) - (k+\alpha)] \frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1+\beta)} \frac{\Gamma\beta}{(n-1)!} |a_n| \\
&= (A-B)|\tau| \sum_{n=2}^{\infty} n[n(k+1) - (k+\alpha)] \frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1+\beta)} \frac{\Gamma\beta}{(n-1)!} \\
&= (A-B)|\tau| \left[(k+1) \sum_{n=2}^{\infty} \frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1+\beta)} \frac{\Gamma\beta}{(n-1)!} + (1-\alpha) \sum_{n=2}^{\infty} \frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta} \frac{(\beta)}{(n-1)!} \right] \\
&= (A-B)|\tau| \left[(k+1)(\gamma)_q \left\{ \frac{\beta}{(\beta+\delta)} + \Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1) - 1 \right\} + (1-\alpha) \left\{ \Omega_{\delta,\beta}^{\gamma,q}(1) - 1 \right\} \right] \\
&\leq 1-\alpha.
\end{aligned}$$

Thus, the proof of Theorem 3.6 is complete. □

Theorem 3.7. *If $\delta, \beta, \gamma > 0, q \in (0,1) \cup N$ $\Phi \in R^{\tau}(A,B), k \geq 0$ and the inequality*

$$(A-B)|\tau|(\gamma)_q \left\{ \frac{\Gamma\beta}{\Gamma\beta+\delta} + \Omega_{\delta,\beta+\delta}^{\gamma+q,q}(1) - 1 \right\} \leq \frac{1}{K+2}$$

is satisfied then $I_{\delta,\beta}^{\gamma,q}\Phi(\zeta) \in k - UCV$.

Proof. We prove the above theorem by applying same procedure as adopted in previous theorem and using Lemma 2.3 and 2.6. □

Theorem 3.8. *If* $\delta, \beta, \gamma, \lambda > 0, q \in (0,1) \cup \mathbb{N}$ *and the inequality*

$$(A-B)|\tau|\left[(\gamma)_q\left\{\frac{\Gamma(\beta)}{\Gamma\beta+\delta}+\Omega^{\gamma+q,q}_{\delta,\beta+\delta}(1)-1\right\}+\lambda\left\{\Omega^{\gamma,q}_{\delta,\beta}(1)-1\right\}\right]\leq\lambda$$

is satisfied then $I^{\gamma,q}_{\delta,\beta}(\zeta) \in C_\lambda$

Proof. The proof of the theorem above closely resembles that of Theorem 3.6. Therefore, we choose to skip the intricate details. □

4. An Integral Operator

We now introduce the integral operator connected with the generalized Mittag-Leffler function, outlined as follows.

$$Z^{\gamma,q}_{\delta,\beta}(\zeta)=\int_0^\zeta \frac{\Omega^{\gamma,q}_{\delta,\beta}(t)}{t}dt. \tag{4.1}$$

The equation (4.1) can be expressed in the following way

$$Z^{\gamma,q}_{\delta,\beta}(\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta}\frac{\Gamma\beta}{n!}\zeta^n. \tag{4.2}$$

Theorem 4.1. *If* $\delta, \beta, \gamma > 0,\ q \in (0,1) \cup \mathbb{N}, k \geq 0,\ 0 \leq \alpha < 1$ *and the condition* (3.2) *is satisfied then* $Z^{\gamma,q}_{\delta,\beta}(\zeta) \in k - UCV(\alpha)$.

Proof. To prove $Z^{\gamma,q}_{\delta,\beta}(\zeta) \in k - UCV(\alpha)$ we have to prove that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n(k+1)-(k+\alpha)]\frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta}\frac{\Gamma\beta}{n!}\\
=&\sum_{n=2}^{\infty}[(k+1)(n-1)+(1-\alpha)]\frac{(\gamma)_{q(n-1)}}{\Gamma\delta(n-1)+\beta}\frac{\Gamma\beta}{(n-1)!}\\
=&(k+1)(\gamma)_q\left\{\frac{\Gamma\beta}{\Gamma\beta+\delta}+\Omega^{\gamma+q,q}_{\delta,\beta+\delta}(1)-1\right\}\\
&+(1-\alpha)\left(\Omega^{\gamma,q}_{\delta,\beta}(1)-1\right)\\
\leq&1-\alpha.
\end{aligned}$$

□

Theorem 4.2. *If* $\delta, \beta, \gamma > 0,\ q \in (0,1) \cup \mathbb{N}, k \geq 0,\ 0 \leq \alpha < 1$ *and the inequality*

$$(\gamma)_q\left\{\frac{\Gamma(\beta)}{\Gamma(\beta+\delta)}+\Omega^{\gamma+q,q}_{\delta,\beta+\delta}(1)-1\right\}\leq\frac{1}{k+2}$$

is satisfied then $Z^{\gamma,q}_{\delta,\beta}(\zeta) \in k - UCV$.

Proof. The proof of above theorem is much akin to the proof of Theorem 4.1. Hence, we omit the details involved. □

Theorem 4.3. *If* $\delta, \beta, \gamma > 0,\ q \in (0,1) \cup \mathbb{N}, k \geq 0,\ 0 \leq \alpha < 1$ *and the condition* (3.4) *is satisfied then* $Z_{\delta,\beta}^{\gamma,q}(\zeta) \in C_\lambda$.

Proof. The proof of the theorem above closely resembles that of Theorem 4.1. Therefore, we opt not to delve into the details. □

Funding Information: This work was supported by Directorate of Research and Innovation, Walter Sisulu University, South Africa.

Author Contributions: All authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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