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Zeros of One and Two Variable Complex Valued Polynomial in $\mathbb{C}$

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Abstract. The Fundamental Theorem of algebra states that every polynomial $f \in \mathbb{C}[z]$ of degree n has precisely n zeros in $\mathbb{C}$, this theorem can not be directly extended to complex valued polynomial of two variable in $\mathbb{C}$. In this article, we investigate the zeros of one and two variable complex valued polynomial in $\mathbb{C}$.

1. Introduction

Let the domain $\mathbb{D} \subset \mathbb{C}$ be the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. A real valued function of the form $u(x, y)$ is said to be harmonic if it satisfies the Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A continuous complex valued function of the form $f = u(x, y) + iv(x, y)$, is said to be analytic if $u(x, y)$ and $v(x, y)$ are harmonic and necessarily satisfies the Cauchy-Riemann equation also $v(x, y)$ is harmonic conjugate of $u(x, y)$.

A harmonic function of the form $f = u(x, y) + iv(x, y)$, where u and v are real-valued harmonic functions in $\mathbb{D}$ and u and v need not satisfies Cauchy-Riemann equation can also be expressed as $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. It is known that every harmonic function $f = h + \overline{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is positive in $\mathbb{D}$, which is equivalent to the existence of the second dilatation defined by

$$|\omega(z)| = |g'(z)|/|h'(z)| < 1 \quad \text{for all } z \in \mathbb{D}.$$

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It is important to note that in complex valued function of the form $f = u(x,y) + iv(x,y)$ selection of u and v decide the behaviour of $f(z)$. As we know, a real valued polynomial $x^2+1=0$ having no solution in $\mathbb{R}$. It is interesting to note that there exist many polynomial such that $x^2+y^2+1=0$ in $\mathbb{R}^2$ having no solution. In this article we discuss the zeros of all complex valued polynomial one and two variable generated by $f = u(x,y) + iv(x,y)$ with substitution of $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$.

The complex valued polynomial associated with two variable real valued polynomial $u(x,y) = x^2 - y^2 + 1$ and $v(x,y) = 2xy$ is $f(z) = z^2+1$ and it has two zeros $z = i, -i$ in $\mathbb{C}$. If $u(x,y) = x^2 - y^2 + 1$ and $v(x,y) = -2xy$, then $f(z) = \bar{z}^2 + 1$ and it has two zeros $\bar{z} = i, -i$ in $\mathbb{C}$. It is interesting to note that $f(z) = z^2+1$ is analytic but $f(z) = \bar{z}^2+1$ is harmonic polynomial and it is not analytic. Recall that all analytic functions are harmonic but converse is not true. In these polynomials both $u(x,y)$ and $v(x,y)$ are harmonic. Analytic polynomials satisfies the Cauchy-Riemann equation but harmonic polynomials of one or two variable does not satisfies the Cauchy-Riemann equation.

2. Preliminaries

Fundamental theorem of algebra for complex valued polynomial states that all non constant polynomial of the form

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_l z^l, \quad a_k \in \mathbb{C}, k = 0, 1, \ldots, l$$

of degree l has exactly l zeros in $\mathbb{C}$, counted with its multiplicities. It is also true for the harmonic polynomials of one variables

$$p(\bar{z}) = a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \cdots + a_m \bar{z}^m, \quad a_k \in \mathbb{C}, k = 0, 1, \ldots, m.$$

of degree m has exactly m zeros in $\mathbb{C}$, counted with its multiplicities. As we know that complex valued function of the form $f = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are arbitrary harmonic that need not satisfy the Cauchy-Riemann equation can be represented as $f(z) = h(z) + \overline{g(z)}$, where h is a polynomial of degree l and g is a polynomial of degree m because the function f is in fact two variable function $f(z, \bar{z})$. Two variable harmonic polynomials having no zeros, infinitely many or finite number of zeros. It is interesting to note that only one variable non constant complex valued polynomials having equal zeros according its degree in $\mathbb{C}$, not all type of complex valued polynomials generated by $f = u(x,y) + iv(x,y)$ with selection of two variable polynomials $u(x,y)$ and $v(x,y)$.

Example 2.1. *If $f(z) = 2x + 4iy$, then after substitution of $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$. We get $f(z) = 3z - \bar{z}$, which is harmonic polynomial of two variable in $\mathbb{C}$.*

In this direction, it is current interest of research to find the zeros of complex valued two variable polynomials. Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic polynomial where $\deg h = l$ and $\deg g = m$ with $l \geq m \geq 1$

$$f(z) = a_l z^l + \cdots + a_0 + b_m \bar{z}^m + \cdots + b_1 \bar{z} = \sum_{j=0}^{l} a_j z^j + \sum_{j=1}^{m} \overline{b_j z^j}, \quad a_k, b_k \in \mathbb{C}. \tag{2.1}$$

Harmonic mappings techniques have been used to study and solve fluid flow problems (see [1]). T. Sheil- Small proposed the following question: What is the maximum number of zeros of the polynomial of the form (2.1)? He conjectured that this can be at most l^2. Bshouty and Hengartner [5] proved that there exist a polynomial of the form (2.1), with $l > m$, that has exactly l^2 zeros. In [7] Wilmshurst provided the examples to show that l^2 is sharp for the cases when $m = l-1$. However, for $m \leq l-2$ it was suggested in [7] that the upper bound should be much lower, in particular Wilmshurst conjectured that for $m = 1$ then number of zeros of $f(z) = h(z) + \overline{g(z)}$ does not exceed $3l-2$. Khavinson and Świątek [4] examine a subclass of harmonic polynomial $f(z) = h(z) - \overline{z}$, where $\deg h = l > 1$ and prove that it has at most $3l-2$ zeros. In [7] Wilmshurst conjectured also that the maximum number of zeros of $f(z) = h(z) + \overline{g(z)}$ with $2 \leq m \leq l-2$ is $m(m-1)+3l-2$ but in [8] it was showed that Wilmshurst's conjecture is not true in general.

To apply Bezout's theorem to the valence problem we follow the following alternative form of Bezout's theorem see [7]

Theorem 2.1. *(Alternate form of Bezout's Theorem). Let A and B be polynomials in the real variables x and y with real coefficients. If $\deg A = l$ and $\deg B = m$, then either A and B have at most lm common zeros or have infinitely many common zeros.*

Bshouty and Hengartner [5] proved the following theorem

Theorem 2.2. *There exists a harmonic polynomial $f(z) = h(z) + \overline{g(z)}$ such that $h(z)$ is analytic polynomial of degree l and $g(z)$ is polynomial of degree m with $l > m$ having exactly l^2 zeros in $\mathbb{C}$.*

In [9], Khavinson, Lee and Saez proved the following theorem

Theorem 2.3. *There exists a harmonic polynomial $f(z) = h(z) + \overline{g(z)}$ such that $h(z)$ is analytic polynomial of degree l and $g(z)$ is polynomial of degree m with $l > m \geq 1$ having exactly $3l-2$ zeros in $\mathbb{C}$.*

Theorem 2.4. *There exists a harmonic polynomial $f(z) = h(z) + \overline{g(z)}$ such that $h(z)$ is analytic polynomial of degree l and $g(z)$ is polynomial of degree m with $l > m \geq 1$ having exactly $m^2 + m + l$ zeros in $\mathbb{C}$.*

In [3] Bénéteau and Hudson proposed an open question

Problem 2.1. *If $h(z)$ is a polynomial degree $l > 3$ and $g(z)$ is a polynomial degree $1 < m < l-1$, What is the maximum roots that the equation $h(z) + \overline{g(z)} = 0$ can have?*

This article also discuss the following problems

Problem 2.2. *Is any complex valued function $f = u(x,y) + iv(x,y)$ with real valued polynomial $u(x,y) = x^2 + 1$ or $u(x,y) = x^2 + y^2 + 1$, which having no solution in $\mathbb{R}^2$, having solution in $\mathbb{C}$?.*

Problem 2.3. *How many zeros of complex valued polynomial $f = u(x,y) + iv(x,y)$ generated by real valued polynomial $u(x,y)$ and $u(x,y)$, where $u(x,y)$ and $u(x,y)$ be harmonic or not having solution in $\mathbb{C}$?.*

Since $u(x,y) = x^2 + 1$ and $x^2 + y^2 + 1$ is not harmonic and its complex valued polynomials after substituting $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ we get

$$z^2 + \bar{z}^2 + 2z\bar{z} + 4 = 0$$

$$z\bar{z} + 1 = 0$$

having no zeros in $\mathbb{C}$ for any choice of $v(x,y)$. Term $z\bar{z}$ is involving due to non harmonicity of real or imaginary part in complex valued function $f = u(x,y) + iv(x,y)$. These functions are studied by Balk [2] and his students called polyanalytic functions.

The polyanalytic function is a complex valued function of the form

$$f(z,\bar{z}) = \sum_{k=0}^{m} f_k(z)\bar{z}^k \tag{2.2}$$

where $f_0, f_1, \cdots, f_m$ are analytic functions.

If $f_0, f_1, \cdots, f_m$ are analytic polynomials then $f(z,\bar{z})$ is called polyanalytic polynomials. All harmonic polynomials are polyanalytic polynomials. The purpose of this article to study the zeroes of two variable polynomials of the form (2.1),(2.2). Degree of polyanalytic polynomials with example defined by Sète and Zur in [6] as follows

Definition 2.1. *Polyanalytic polynomial of the form*

$$f(z,\bar{z}) = \sum_{k=0}^{m} f_k(z)\bar{z}^k \tag{2.3}$$

with non zero $f_k(z)$ degree of $f = max_k(deg(f_k) + k)$, the analytic degree $deg_z(f) = max_k deg(f_k)$ and the anti-analytic degree $deg_{\bar{z}}(f) = m$.

Example 2.2. *The polyanalytic polynomial of the form*

$$f(z,\bar{z}) = z^3 + z^2\bar{z}^2$$

degree $(f) = 4$, $deg_z(f) = 3$, $deg_{\bar{z}}(f) = 2$.

In [2] Balk proved the fundamental theorem of algebra for polyanalytic polynomials

Theorem 2.5. *A polyanalytic polynomial $f(z,\bar{z})$ with $deg(f) > 2deg_z f$ or $deg(f) > 2deg_{\bar{z}} f$ has at least one zero.*

3. Main Results

Theorem 3.1. *Let $f(z) = u(x,y) + iv(x,y)$, where either $u(x,y)$ or $v(x,y)$ is a non zero constant function then the generated complex valued harmonic polynomials $f(z) = h(z) + \overline{g(z)}$ has no zeros in $\mathbb{C}$.*

Proof. It is known that if $u(x,y)$ and $v(x,y)$ are arbitrary harmonic then only $f(z)$ can be represented as the form $f(z) = h(z) + \overline{g(z)}$. First if we assume $u(x,y) = c$ and $v(x,y)$ be any polynomial harmonic function then

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ &= c + iv(x,y) \end{aligned}$$

Therefore, $f(z) = 0$ if and only if $c + iv(x,y) = 0$. Hence for any pair of solution (x,y) of $v(x,y) = 0$ we can not find z for which $f(z) = 0$. Similarly we can show that for harmonic polynomial $u(x,y)$ and $v(x,y) = c$, we can not find z for which $f(z) = 0$. Hence $f(z) = u(x,y) + iv(x,y)$ having no solution in $\mathbb{C}$, if any one $u(x,y)$ or $v(x,y)$ be non zero constant. □

Theorem 3.2. *Let $f(z) = u(x,y) + iv(x,y)$ with any one $u(x,y)$ or $v(x,y)$ be zero and other is non constant harmonic polynomial, then the generated complex valued harmonic polynomials $f(z) = h(z) + \overline{g(z)}$ having infinitely many zeros in $\mathbb{C}$.*

Proof. It is known that if $u(x,y)$ and $v(x,y)$ are arbitrary harmonic polynomial then only $f(z)$ it can be represented as the form $f(z) = h(z) + \overline{g(z)}$. First if we assume $u(x,y) = 0$ and $v(x,y)$ be any polynomial harmonic function then

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ &= 0 + iv(x,y) \end{aligned}$$

Therefore, $f(z) = 0$ if and only if $0 + iv(x,y) = 0$. Hence $f(z) = u(x,y) + iv(x,y)$ having infinitely many solution in $\mathbb{C}$, if any one $u(x,y)$ or $v(x,y)$ be zero and other is non constant harmonic polynomial. □

Theorem 3.3. *Let $f(z) = u(x,y) + iv(x,y)$ with $u(x,y)$ or $v(x,y)$ be real valued harmonic polynomial of degree l then their exist complex valued harmonic polynomials of the form $f(z) = h(z) + \overline{g(z)}$ with $l \geq m \geq 1$ having all orders zero $\{0, 1, 2, 3, \cdots, l^2, ..., \infty\}$ in $\mathbb{C}$.*

Proof. We proof this theorem by induction method. Hence for $l = 1$, we consider the harmonic polynomial of the form

$$f(z) = a_1 z + b_1 \overline{z} + a_0$$

It has zeros of $\{0, 1, \infty\}$ under certain condition on parameters.

Hence for $l = 2$, Consider quadratic harmonic polynomials of the form

$$f(z) = a_2 z^2 + a_1 z + a_0 + b_2 \overline{z}^2 + b_1 \overline{z}$$

It has zeros of $\{0, 1, 2, 3, 4, \infty\}$ under certain condition on parameters.

Hence for $l = 3$, Consider cubic harmonic polynomials of the form

$$f(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 + b_3 \overline{z}^3 + b_2 \overline{z}^2 + b_1 \overline{z},$$

It has zeros of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \infty\}$ under certain condition on parameters.

Consider harmonic polynomials of order l the form

$$f(z) = a_l z^l + \cdots + a_3 z^3 + a_2 z^2 + a_1 z + a_0 + b_m \overline{z^m} + \cdots + b_3 \bar{z}^3 + b_2 \bar{z}^2 + b_1 \bar{z},$$

It has zeros of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots, l^2, \cdots, \infty\}$ under certain condition on parameters.

For sharpness we see that for one degree polynomial

Case I: For $l = 1$

(i) If $a_1 = 1, a_0 = 1, b_1 = -1,$

then $f(z) = z - \bar{z} + 1$ having no zeros in $\mathbb{C}$.

(ii) If $a_1 = 1, a_0 = 1, b_1 = 1,$

then $f(z) = z + \bar{z} + 1$ having one zeros in $\mathbb{C}$.

(iii) If $a_1 = 1, a_0 = 0, b_1 = -1,$

then $f(z) = z - \bar{z}$ having infinitely zeros in $\mathbb{C}$.

We see that for two degree polynomial

Case II: For $l = 2$

(i) If $a_2 = 1, a_1, a_0 = 0, b_1 = 1, b_2 = 0,$

then $f(z) = z^2 - \bar{z}$. By substituting $z = x + iy$ and equating to zero, we directly find that it has 4 zeros: $0, 1, -1/2 \pm i\sqrt{3}/2$.

(ii) If $a_2 = 1, b_2 = -1, a_1, a_0, b_1 = 0,$

then $f(z) = z^2 - \bar{z}^2$ which have infinitely many zeros in $\mathbb{C}$.

(iii) If $a_2 = 1, b_2 = -1, a_0 = 1, a_1, b_1 = 0,$

then $f(z) = z^2 - \bar{z}^2 + 1$ which have no zeros in $\mathbb{C}$.

(iv) If $a_2 = 1, b_2 = 0, a_0 = 1, a_1 = 0, b_1 = 1,$

then $f(z) = z^2 + \bar{z} + 1$ which have two zeros in $\mathbb{C}$.

(v) If $a_0 = 1, a_1 = 5, a_2 = 1, b_1 = -1, b_2 = 3,$

then $f(z) = z^2 + 5z + 1 - \bar{z} + 3\bar{z}^2$ which have three zeros in $\mathbb{C}$.

(vi) If $a_0 = 0, a_1 = 1, a_2 = 1, b_1 = 0, b_2 = -1,$

then $f(z) = z^2 - \bar{z}^2 + z$ which have one zeros in $\mathbb{C}$.

We see that for three degree polynomial

Case III: For $l = 3$

(i) If $a_3 = 1, a_2 = 0, b_2 = 0, a_0 = 0, a_1 = 1, b_1 = 0, b_3 = 1,$

then $f(z) = z^3 + z + \bar{z}^3$ which has one zero $z = 0$ in $\mathbb{C}$.

(ii) $a_3 = 1, a_2 = -3/2, b_2 = -1/2, a_0 = 0, a_1 = 0, b_1 = 0, b_3 = 0,$

then $f(z) = z^3 - \frac{3}{2}z^2 - \frac{1}{2}\bar{z}^2$ which have two zeros $z_1 = 0$ and $z_2 = 2$ in $\mathbb{C}$.

(iii) If $a_3 = 1, a_2 = 0, b_2 = 0, a_0 = 0, a_1 = 1, b_1 = 0, b_3 = -1,$

then $f(z) = z^3 + z - \bar{z}^3$ which has three zeros $z_1 = 0, z_2 = -i/\sqrt{2}$ and $z_3 = i/\sqrt{2}$ in $\mathbb{C}$.

(*iv*) If $a_3 = 1/2, a_2 = -3/2, b_2 = 1, a_0 = 0, a_1 = 0, b_1 = 0, b_3 = 0,$

then $f(z) = \frac{1}{2}z^3 - \frac{3}{2}z^2 + \bar{z}^2$ which have four zeros in $\mathbb{C}$.

(*v*) If $a_3 = 1, a_2 = 0, b_2 = 0, a_0 = 0, a_1 = 0, b_1 = -1, b_3 = 0,$

then $f(z) = z^3 - \bar{z}$ which has five zeros in $\mathbb{C}$.

(*vi*) If $a_3 = 1, a_2 = 0, b_2 = -1, a_0 = 0, a_1 = 0, b_1 = 0, b_3 = 0,$

then $z^3 - \bar{z}^2$ which has six zeros in $\mathbb{C}$.

(*vii*) If $a_3 = 1/2, a_2 = 0, b_2 = 0, a_0 = 0, a_1 = -3/2, b_1 = 1, b_3 = 0,$

then $f(z) = \frac{1}{2}z^3 - \frac{3}{2}z + \bar{z}$ having seven zeros in $\mathbb{C}$.

(*viii*) If $a_3 = 1, a_2 = 0, b_2 = -1, a_0 = 0, a_1 = -1/9, b_1 = 0, b_3 = 0,$

then $f(z) = z^3 + \frac{1}{9}z - \overline{z^2}$ having nine zeros in $\mathbb{C}$.

(*ix*) If $a_3 = 2, a_2 = -3, a_1 = 3, a_0 = 0, b_2 = 4, b_1 = -3, b_3 = -1,$

then $f(z) = 2z^3 + 3z - 3z^2 - \overline{z^3} - 3\bar{z} + 4\overline{z^2}$ having eight zeros in $\mathbb{C}$.

there exist a polynomial of degree $k+1$, having no or infinitely many and finitely many $(k+1)^2$zeros. To prove this we have the set of polynomial $l = k+1$

$$f(z) = \begin{cases} z^{k+1} - \overline{z^{-(k+1)}} + 1 = 0, \\ z^{k+1} - \overline{z^{-(k+1)}} = 0. \end{cases}$$

having no or infinitely zeros in $\mathbb{C}$, respectively.

Hence by induction, we see that there exist harmonic polynomial for degree l, such that $f(z) = (z-1)^l + z^l + \overline{i(z-1)^l} - \overline{iz}^l$ having l^2 zeros in $\mathbb{C}$.

□

Theorem 3.4. *Let $f(z) = u(x,y) + iv(x,y)$ with any one $u(x,y)$ or $v(x,y)$ be zero and other be non harmonic polynomial with no solution in $\mathbb{R}^2$ then the generated complex valued polyanalytic polynomials of the form* (2.2) *$f(z,\bar{z})$ involving the term $z\bar{z}$, having no zeros in $\mathbb{C}$.*

Proof. It is known that the polyanalytic polynomials of the form $f(z,\bar{z})$ can be two type as we know all harmonic polynomials are polyanalytic polynomial, here we consider the polyanalytic polynomial involving the term $z\bar{z}$, which are not harmonic. First if we assume $u(x,y) = 0$ and $v(x,y)$ be any non harmonic polynomial with no solution in $\mathbb{R}^2$ then

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ &= 0 + iv(x,y) \end{aligned}$$

Therefore, $f(z) = 0$ if and only if $0 + iv(x,y) = 0$. After substituting $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ we get polyanalytic polynomials of the form (2.2) $f(z,\bar{z})$. Hence $f(z) = u(x,y) + iv(x,y)$ having no solution in $\mathbb{C}$ for any value of z. Similarly, for $v(x,y) = 0$ and $u(x,y)$ be any non harmonic polynomial with no solution in $\mathbb{R}^2$. Polyanalytic polynomials $f(z) = u(x,y) + iv(x,y)$ having no solution in $\mathbb{C}$. □

Following theorem gives new upper bound for the zeros of complex valued polynomials

Theorem 3.5. *There exist a complex valued harmonic and polyanalytic polynomials with finitely many zeros has at most* $\frac{l^2+m^2+l+m}{2}$ *zeros in* $\mathbb{C}$.

Proof. It is know that any complex valued polynomial can be written the form $f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ be finite degree harmonic and non harmonic polynomials. In [5], it is proved that there exist a harmonic polynomial of degree l having l^2 zeros. As we know that the maximum valence of any harmonic polynomial of degree $l > m \geq 1$, having l^2 zeros. It is proved in [9], that there exist a harmonic polynomial of degree l having $m^2 + l + m$ zeros. It is proved in Theorem 3.3 that there exist a harmonic polynomials for every $k \in \{0, 1, 2, 3, \cdots, l^2, \infty\}$. Hence it also have the zeros of $\frac{l^2+m^2+l+m}{2} \leq l^2$ zeros in $\mathbb{C}$.

For $l \geq 1$ polyanalytic polynomials can be written the form $f(z) = u(x,y) + iv(x,y)$ and by Bezout's theorem 2.1 it has at most l^2 zeros, hence by using above method we can prove that polyanalytic polynomials with finitely many zeros has at most $\frac{l^2+m^2+l+m}{2}$ zeros in $\mathbb{C}$. □

Following example discuss the sharp bound of Theorem 3.5

Example 3.1. *Two degree harmonic polynomial of the form* $f(z) = z^2/2 + \bar{z}$ *having* 4 *zeros in* $\mathbb{C}$.

In the following corollary, we give the answer of Problem 2.1

Corollary 3.1. *If* $h(z)$ *is a polynomial degree* $l > 3$ *and* $g(z)$ *is a polynomial degree* $1 < m < l-1$, *the maximum roots that the equation* $h(z) + \overline{g(z)} = 0$ *can have at most* $\frac{l^2+m^2+l+m}{2}$ *zeros in* $\mathbb{C}$.

Proof. If $l = 4$ and $m = 2$, Wilmshurst conjectured also that the maximum number of zeros of $f(z) = h(z) + \overline{g(z)}$ with $2 \leq m \leq l-2$ is 12 zeros in $\mathbb{C}$. In Theorem 2.4 Khavinson proved that if $h(z)$ is analytic polynomial of degree l and $g(z)$ is polynomial of degree m with $l > m \geq 1$ having exactly $m^2 + m + l$ zeros, for $l = 4$ and $m = 2$ it has 10 zeros in $\mathbb{C}$. If $l = 4$ and $m = 2$ by Theorem 3.5 it has 13 zeros in $\mathbb{C}$. □

4. Examples

Following examples established the results of Theorems 3.1, 3.2, 3.3, 3.4.

Example 4.1. $f(z) = u(x,y) + iv(x,y)$ *with* $u(x,y) = x$ *and* $v(x,y) = 1$, *then the generated complex valued harmonic polynomials* $f(z) = z + \bar{z} + 2i$ *having no zeros in* $\mathbb{C}$.

Example 4.2. $f(z) = u(x,y) + iv(x,y)$ *with* $u(x,y) = 0$ *and* $v(x,y) = y$, *then the generated complex valued harmonic polynomials* $f(z) = z - \bar{z}$ *having infinitely many zeros in* $\mathbb{C}$.

Example 4.3. $f(z) = u(x,y) + iv(x,y)$ *with* $v(x,y) = 0$ *and* $u(x,y) = x^2 + 1$. *We get the polyanalytic polynomial of the form*

$$f(z,\bar{z}) = z^2 + \bar{z}^2 + 2z\bar{z} + 4$$

Here $f(z,\bar{z}) = 0$ *having no solution in* $\mathbb{C}$.

Example 4.4. *$f(z) = u(x,y) + iv(x,y)$ with $v(x,y) = 0$ and $u(x,y) = x^2 + y^2 + 1$. We get the polyanalytic polynomial of the form*

$$f(z,\bar{z}) = z\bar{z} + 1$$

Here $f(z,\bar{z}) = 0$ having no solution in $\mathbb{C}$.

Example 4.5. *$f(z) = u(x,y) + iv(x,y)$ with $v(x,y) = xy$ and $u(x,y) = x^2$. We get the polyanalytic polynomial of the form*

$$f(z,\bar{z}) = z^2 + z\bar{z}$$

Here $f(z,\bar{z}) = 0$ having infinitely solution in $\mathbb{C}$.

Example 4.6. *$f(z) = u(x,y) + iv(x,y)$ with $v(x,y) = y(1-x)$ and $u(x,y) = x^2 + x$. We get the polyanalytic polynomial of the form*

$$f(z,\bar{z}) = z\bar{z} + z$$

Here $f(z,\bar{z}) = 0$ having two solution in $\mathbb{C}$.

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