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Spectral Analysis of a 2×2 Operator Matrix with Finite-Rank Perturbation

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Abstract. The paper is devoted to the spectral investigation of a self-adjoint 2×2 operator matrix acting in the Hilbert space $\mathbb{C}\oplus L_2(\mathbb{T})$, whose off-diagonal entries generate a finite-rank perturbation. We first identify its essential spectrum and show that it is the interval $[0,2]$. The analysis of the spectrum outside this interval is reduced to a scalar equation involving an associated Fredholm determinant. We prove that the zeros of this determinant provide a complete characterization of the discrete eigenvalues. The resulting equation is then examined explicitly, yielding precisely two simple eigenvalues outside the essential spectrum: one in $(-\infty,0)$ and the other in $(2,\infty)$. Closed-form expressions for both eigenvalues are derived.

1. Introduction

Operator matrices provide a useful framework for the spectral analysis of various models arising in mathematical physics. In particular, operator-theoretic formulations of lattice quantum systems, Friedrichs-type models, and Hamiltonians with variable particle number naturally lead to block operator structures acting in direct sums of Hilbert spaces; see, for example, [1–4]. Such representations make it possible to investigate spectral properties of complicated Hamiltonians through the structure of their matrix entries.

A substantial part of the spectral theory of lattice models is devoted to understanding the structure of the essential spectrum and the behavior of isolated eigenvalues. In this direction, the essential spectrum of operator matrices has been investigated using different spectral approaches.

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In particular, the formation and location of its branches for certain classes of matrix operators were studied in [5,6], while related questions concerning families of operator matrices and their spectral components were considered in [9,10].

Another important line of research concerns the discrete spectrum and the behavior of eigenvalues near the boundaries of the essential spectrum. The existence and distribution of discrete eigenvalues for operator matrices, together with threshold spectral phenomena, were investigated in [7, 8]. These results illustrate the role played by the structure of the matrix entries and the interaction terms in determining spectral points outside or near the essential spectrum.

Spectral problems for higher-order operator matrices have also received considerable attention. Estimates for the spectral boundaries of 3×3 operator matrices and the use of Schur-complement techniques in Fock-space settings were considered in [11,12]. Further investigations of such models include the analysis of their discrete spectra and the construction of the corresponding resolvent operators; see [13, 14].

Closely related questions arise for few-particle quantum systems on lattices. Conditions guaranteeing the existence of eigenvalues for three-particle lattice Hamiltonians and for tensor sums of Friedrichs-type models with finite-rank perturbations were obtained in [15–17]. These studies provide examples of how interaction terms can generate isolated spectral points in lattice models.

The structure of the essential spectrum in the presence of nonlocal or finite-rank interactions has been investigated in several related settings. The spectrum of three-particle lattice model operators with nonlocal potentials was studied in [18], whereas the components of the essential spectrum associated with tensor sums of Friedrichs models with finite-rank nonlocal potentials were analyzed in [19].

The discrete spectrum of lattice Schrödinger operators constitutes another closely connected direction. The existence of infinitely many eigenvalues for a two-particle lattice Schrödinger operator was considered in [20], while criteria for the existence of eigenvalues of a three-particle Schrödinger operator on a lattice were established in [21].

Additional results for three-particle discrete Schrödinger operators were obtained in [22, 23], where questions related to eigenvalues and the structure of the essential spectrum were investigated for different classes of lattice interactions. Collectively, these studies show that finite-rank, nonlocal, and short-range interactions can substantially influence the spectral structure of few-particle lattice Hamiltonians.

In the present paper, we consider a self-adjoint 2×2 operator matrix acting in $\mathbb{C} \oplus L_2(\mathbb{T})$, whose off-diagonal entries define a finite-rank interaction. Unlike the more general lattice operator matrices discussed above, the particular structure of the present model allows the spectral problem outside the essential spectrum to be reduced to an explicitly solvable scalar equation. We first determine the essential spectrum by separating the diagonal part of the operator from its finite-rank interaction. We then introduce the associated Fredholm determinant and establish a correspondence between its zeros and the discrete eigenvalues of the operator matrix. This reduction

enables us to prove that the discrete spectrum consists of exactly two simple eigenvalues, situated on opposite sides of the interval $[0,2]$, and to derive closed-form expressions for both of them.

The paper is organized as follows. In Section 2, the operator matrix and its diagonal counterpart are introduced, and the essential spectrum is determined. Section 3 is devoted to the discrete spectrum: the corresponding Fredholm determinant is constructed, its relation to the eigenvalue problem is established, and the eigenvalues outside $[0,2]$ are obtained explicitly. The main results are summarized in the final section.

2. The Operator Matrix and Its Essential Spectrum

In this section, we introduce the operator matrix under consideration and determine its essential spectrum. The analysis is based on separating the diagonal part of the operator from the off-diagonal interaction. Since the latter has finite rank, the essential spectrum can be identified by using the invariance of the essential spectrum under compact self-adjoint perturbations.

Let $\mathbb{T} := (-\pi, \pi]$ be the one-dimensional torus, $\mathcal{H}_0 := \mathbb{C}$ be the one-dimensional Hilbert space of complex numbers, and let $\mathcal{H}_1 := L_2(\mathbb{T})$ denote the Hilbert space of square-integrable complex-valued functions on the one-dimensional torus $\mathbb{T}$. We consider the orthogonal direct sum $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$. Thus, each element $f \in \mathcal{H}$ can be represented in the form

$$f = (f_0, f_1), \qquad f_0 \in \mathcal{H}_0, \quad f_1 \in \mathcal{H}_1.$$

On the Hilbert space $\mathcal{H}$, consider the 2×2 operator matrix

$$\mathcal{A} := \begin{pmatrix} A_{00} & A_{01} \\ A_{01}^* & A_{11} \end{pmatrix}. \tag{2.1}$$

Its diagonal entries are defined by

$$A_{00} f_0 = f_0, \qquad f_0 \in \mathcal{H}_0,$$

and

$$(A_{11} f_1)(p) = \varepsilon(p) f_1(p), \qquad f_1 \in \mathcal{H}_1,$$

where

$$\varepsilon(p) := 1 - \cos p, \qquad p \in \mathbb{T}.$$

The off-diagonal operator $A_{01} : \mathcal{H}_1 \to \mathcal{H}_0$ acts according to

$$A_{01} f_1 = \int_{\mathbb{T}} f_1(t)\, dt,$$

whereas its adjoint $A_{01}^* : \mathcal{H}_0 \to \mathcal{H}_1$ is given by

$$(A_{01}^* f_0)(p) = f_0, \qquad f_0 \in \mathcal{H}_0.$$

The operator A_{00} is bounded and self-adjoint. Since $\varepsilon(\cdot)$ is a real-valued continuous function on the compact set $\mathbb{T}$, the multiplication operator A_{11} is also bounded and self-adjoint. Moreover, by

the Cauchy–Schwarz inequality,

$$\left|\int_{\mathbb{T}} f_1(t)\,dt\right| \le \left(\int_{\mathbb{T}} 1\,dt\right)^{1/2} \|f_1\|_{L_2(\mathbb{T})},$$

which shows that A_{01} is bounded. Consequently, its adjoint A_{01}^* is bounded as well. Hence, the operator matrix $\mathcal{A}$ defines a bounded self-adjoint operator on $\mathcal{H}$.

For $f = (f_0, f_1) \in \mathcal{H}$, the action of $\mathcal{A}$ can be written explicitly as

$$\mathcal{A}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_0 + \displaystyle\int_{\mathbb{T}} f_1(t)\,dt \\ f_0 + \varepsilon(\cdot) f_1(\cdot) \end{pmatrix}.$$

This representation shows that the interaction between the two components of $\mathcal{H}$ is entirely determined by the off-diagonal entries A_{01} and A_{01}^*.

To separate the interaction from the diagonal part, introduce the operator

$$\mathcal{A}_0 := \begin{pmatrix} A_{00} & 0 \\ 0 & A_{11} \end{pmatrix}.$$

Then $\mathcal{A}$ admits the decomposition

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{V},$$

where

$$\mathcal{V} := \mathcal{A} - \mathcal{A}_0 = \begin{pmatrix} 0 & A_{01} \\ A_{01}^* & 0 \end{pmatrix}.$$

Thus, $\mathcal{A}_0$ represents the diagonal part of the model, whereas $\mathcal{V}$ describes the interaction between $\mathcal{H}_0$ and $\mathcal{H}_1$.

Since $\mathcal{H}_0 = \mathbb{C}$ and the operators A_{01} and A_{01}^* have one-dimensional ranges, it follows that

$$\operatorname{rank} \mathcal{V} = \operatorname{rank}(\mathcal{A} - \mathcal{A}_0) = 2.$$

Consequently, $\mathcal{A}$ is a finite-rank perturbation of its diagonal part $\mathcal{A}_0$. In particular, $\mathcal{V}$ is compact.

We now determine the spectral properties of $\mathcal{A}_0$. Since $\mathcal{A}_0$ is diagonal with respect to the decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1,$$

we have

$$\sigma(\mathcal{A}_0) = \sigma(A_{00}) \cup \sigma(A_{11}).$$

The first diagonal entry A_{00} is the identity operator on the one-dimensional space $\mathcal{H}_0$. Hence,

$$\sigma(A_{00}) = \{1\}.$$

The spectrum of the second diagonal entry is determined by the range of the function ε. Since

$$\varepsilon(p) = 1 - \cos p, \qquad p \in \mathbb{T},$$

we have

$$0 \le \varepsilon(p) \le 2.$$

More precisely,

$$\min_{p\in\mathbb{T}} \varepsilon(p) = 0, \qquad \max_{p\in\mathbb{T}} \varepsilon(p) = 2.$$

Therefore,

$$\varepsilon(\mathbb{T}) = [0,2].$$

Since A_{11} is the multiplication operator generated by the continuous function ε, it follows that

$$\sigma(A_{11}) = \sigma_{\text{ess}}(A_{11}) = [0,2].$$

Combining the spectra of the two diagonal entries, we obtain

$$\sigma(\mathcal{A}_0) = \{1\} \cup [0,2] = [0,2],$$

and consequently

$$\sigma_{\text{ess}}(\mathcal{A}_0) = [0,2].$$

It is worth noting that the scalar spectral point 1 is an eigenvalue of $\mathcal{A}_0$. Indeed, for any $b \in \mathbb{C} \setminus \{0\}$, consider the vector

$$f = (b,0) \in \mathcal{H}_0 \oplus \mathcal{H}_1.$$

Then

$$\mathcal{A}_0 f = (A_{00}b, 0) = (b,0) = f.$$

Thus, $1 \in \sigma_{\text{p}}(\mathcal{A}_0)$. However, $1 \in (0,2) \subset \sigma_{\text{ess}}(\mathcal{A}_0)$, so this eigenvalue is embedded in the essential spectrum. In particular, it is not an isolated point of the spectrum and hence does not belong to the discrete spectrum. Therefore,

$$\sigma_{\text{disc}}(\mathcal{A}_0) = \varnothing.$$

We are now in a position to determine the essential spectrum of the full operator matrix $\mathcal{A}$. Since

$$\mathcal{A} - \mathcal{A}_0 = \mathcal{V}$$

is a finite-rank operator, it is compact. Weyl's theorem on the invariance of the essential spectrum under compact self-adjoint perturbations [4] therefore yields

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(\mathcal{A}_0).$$

Using the above description of the essential spectrum of $\mathcal{A}_0$, we conclude that

$$\sigma_{\text{ess}}(\mathcal{A}) = [0,2].$$

The preceding discussion can be summarized in the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be the operator matrix defined by* (2.1)*. Then $\mathcal{A}$ is a bounded self-adjoint operator in the Hilbert space $\mathcal{H} = \mathbb{C} \oplus L_2(\mathbb{T})$, and its essential spectrum is given by*

$$\sigma_{\text{ess}}(\mathcal{A}) = [0,2].$$

Theorem 2.1 shows that the finite-rank coupling between $\mathcal{H}_0$ and $\mathcal{H}_1$ does not alter the essential spectral interval of the diagonal operator $\mathcal{A}_0$.

3. Discrete Spectrum of the Operator Matrix $\mathcal{A}$

Having determined the essential spectrum of $\mathcal{A}$, we proceed to the analysis of its spectral points outside the interval $[0,2]$. The finite-rank structure of the interaction makes it possible to reduce the corresponding eigenvalue problem to a scalar equation.

For $\lambda \in \mathbb{C} \setminus [0,2]$, define

$$\Delta(\lambda) := 1 - \lambda + I(\lambda), \tag{3.1}$$

where

$$I(\lambda) := \int_{\mathbb{T}} \frac{dt}{\lambda - 1 + \cos t}. \tag{3.2}$$

Since λ lies outside the range of the function $1 - \cos t$, the denominator in (3.2) does not vanish on $\mathbb{T}$. Hence, $I(\lambda)$ is well defined and analytic on $\mathbb{C} \setminus [0,2]$, and the same is true for $\Delta(\lambda)$. In what follows, $\Delta(\cdot)$ will be referred to as the Fredholm determinant associated with $\mathcal{A}$.

The following lemma reduces the eigenvalue problem for the operator matrix to the study of the zeros of $\Delta(\cdot)$.

Lemma 3.1. *Let $\lambda \in \mathbb{C} \setminus [0,2]$. Then λ is an eigenvalue of $\mathcal{A}$ if and only if $\Delta(\lambda) = 0$.*

Proof. Let

$$f = (f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1, \qquad f \neq 0.$$

For $\lambda \in \mathbb{C} \setminus [0,2]$, the equation

$$\mathcal{A}f = \lambda f$$

is equivalent to the system

$$\begin{cases} (1-\lambda) f_0 + \displaystyle\int_{\mathbb{T}} f_1(t)\, dt = 0, \\ f_0 + (1 - \cos p - \lambda) f_1(p) = 0. \end{cases} \tag{3.3}$$

Since $\lambda \notin [0,2]$, we have

$$1 - \cos p - \lambda \neq 0, \qquad p \in \mathbb{T}.$$

The second equation in (3.3) can therefore be solved with respect to f_1, giving

$$f_1(p) = \frac{f_0}{\lambda - 1 + \cos p}. \tag{3.4}$$

Substitution of (3.4) into the first equation of (3.3) yields

$$\left(1 - \lambda + \int_{\mathbb{T}} \frac{dt}{\lambda - 1 + \cos t}\right) f_0 = 0,$$

or equivalently,

$$\Delta(\lambda) f_0 = 0. \tag{3.5}$$

Suppose first that λ is an eigenvalue of $\mathcal{A}$. If $f_0 = 0$, then the second equation in (3.3) implies

$$(1 - \cos p - \lambda) f_1(p) = 0.$$

Since the coefficient does not vanish on $\mathbb{T}$, it follows that $f_1 = 0$. This contradicts the assumption $f \neq 0$. Consequently, $f_0 \neq 0$, and (3.5) gives $\Delta(\lambda) = 0$.

Conversely, suppose that $\Delta(\lambda) = 0$. Choose any $f_0 \in \mathbb{C} \setminus \{0\}$ and define f_1 by (3.4). Since the denominator in (3.4) is separated from zero on $\mathbb{T}$, the function f_1 belongs to $L_2(\mathbb{T})$. Hence,

$$f = (f_0, f_1) \in \mathcal{H}, \qquad f \neq 0.$$

The definition of f_1 ensures the validity of the second equation in (3.3), whereas the identity $\Delta(\lambda) = 0$ gives the first one. Thus,

$$\mathcal{A}f = \lambda f,$$

and λ is an eigenvalue of $\mathcal{A}$. □

To determine the zeros of $\Delta(\cdot)$ explicitly, we first evaluate the integral appearing in (3.2).

Lemma 3.2. *For every $\lambda \in \mathbb{R} \setminus [0,2]$, the integral $I(\lambda)$ defined by* (3.2) *satisfies*

$$I(\lambda) = \frac{2\pi \ \mathrm{sgn}(\lambda - 1)}{\sqrt{(\lambda-1)^2 - 1}}. \tag{3.6}$$

Equivalently,

$$I(\lambda) = \begin{cases} -\dfrac{2\pi}{\sqrt{(\lambda-1)^2-1}}, & \lambda < 0, \\ \dfrac{2\pi}{\sqrt{(\lambda-1)^2-1}}, & \lambda > 2. \end{cases}$$

Proof. Set

$$a = \lambda - 1.$$

For $\lambda \in \mathbb{R} \setminus [0,2]$, one has $|a| > 1$. Hence, the classical integral identity

$$\int_{-\pi}^{\pi} \frac{dt}{a + \cos t} = \frac{2\pi \ \mathrm{sgn}(a)}{\sqrt{a^2 - 1}}, \qquad |a| > 1,$$

can be applied. Substituting $a = \lambda - 1$ gives

$$I(\lambda) = \frac{2\pi \ \mathrm{sgn}(\lambda - 1)}{\sqrt{(\lambda-1)^2 - 1}},$$

which proves (3.6). The two separate formulas follow immediately from

$$\mathrm{sgn}(\lambda - 1) = \begin{cases} -1, & \lambda < 0, \\ 1, & \lambda > 2. \end{cases}$$

□

We are now ready to give a complete description of the discrete spectrum of $\mathcal{A}$.

Theorem 3.1. *The operator matrix $\mathcal{A}$ has exactly two simple eigenvalues outside its essential spectrum $[0,2]$. More precisely, there exists a unique simple eigenvalue $\lambda_1 \in (-\infty, 0)$ and a unique simple eigenvalue $\lambda_2 \in (2, \infty)$. They are given explicitly by*

$$\lambda_1 = 1 - \sqrt{\frac{1 + \sqrt{1+16\pi^2}}{2}}, \tag{3.7}$$

and

$$\lambda_2 = 1 + \sqrt{\frac{1 + \sqrt{1+16\pi^2}}{2}}. \tag{3.8}$$

Consequently,

$$\sigma_{\text{disc}}(\mathcal{A}) = \{\lambda_1, \lambda_2\}.$$

Proof. According to Lemma 3.1, it is sufficient to determine all real zeros of $\Delta(\lambda)$ outside the essential spectral interval $[0,2]$. By Lemma 3.2, the form of the determinant depends on which connected component of $\mathbb{R} \setminus [0,2]$ contains λ. We therefore consider the two cases separately.

Case 1: $\lambda < 0$.

For $\lambda < 0$, Lemma 3.2 gives

$$\Delta(\lambda) = 1 - \lambda - \frac{2\pi}{\sqrt{(\lambda-1)^2 - 1}}.$$

Thus, the equation $\Delta(\lambda) = 0$ takes the form

$$1 - \lambda = \frac{2\pi}{\sqrt{(\lambda-1)^2 - 1}}. \tag{3.9}$$

Introduce the variable

$$x := 1 - \lambda.$$

Since $\lambda < 0$, we have $x > 1$, and

$$(\lambda - 1)^2 = x^2.$$

Therefore, (3.9) becomes

$$x = \frac{2\pi}{\sqrt{x^2 - 1}}, \qquad x > 1. \tag{3.10}$$

Both sides of (3.10) are positive. Hence, squaring the equation does not introduce a change of sign and gives

$$x^2(x^2 - 1) = 4\pi^2.$$

Equivalently,

$$x^4 - x^2 - 4\pi^2 = 0.$$

Setting

$$y := x^2,$$

we obtain the quadratic equation

$$y^2 - y - 4\pi^2 = 0.$$

Its roots are

$$y_{\pm} = \frac{1 \pm \sqrt{1+16\pi^2}}{2}.$$

Since

$$\sqrt{1+16\pi^2} > 1,$$

the root y_- is negative and therefore cannot equal x^2. Thus, the only admissible value is

$$x^2 = \frac{1+\sqrt{1+16\pi^2}}{2}.$$

Because $x > 1$, we obtain

$$x = \sqrt{\frac{1+\sqrt{1+16\pi^2}}{2}}.$$

Recalling that $\lambda = 1 - x$, we arrive at

$$\lambda_1 = 1 - \sqrt{\frac{1+\sqrt{1+16\pi^2}}{2}}.$$

Furthermore,

$$\frac{1+\sqrt{1+16\pi^2}}{2} > 1,$$

and hence $\lambda_1 < 0$. The above reduction also shows that no other admissible solution exists in $(-\infty, 0)$. Therefore, λ_1 is the unique eigenvalue of $\mathcal{A}$ below the essential spectrum.

Case 2: $\lambda > 2$.

In this region,

$$\Delta(\lambda) = 1 - \lambda + \frac{2\pi}{\sqrt{(\lambda-1)^2-1}}.$$

Hence, $\Delta(\lambda) = 0$ is equivalent to

$$\lambda - 1 = \frac{2\pi}{\sqrt{(\lambda-1)^2-1}}. \tag{3.11}$$

Set

$$x := \lambda - 1.$$

Since $\lambda > 2$, we again have $x > 1$. Equation (3.11) is therefore reduced to

$$x = \frac{2\pi}{\sqrt{x^2-1}}.$$

This is precisely the scalar equation obtained in the first case. Consequently,

$$x = \sqrt{\frac{1+\sqrt{1+16\pi^2}}{2}}.$$

Since $\lambda = 1 + x$, we obtain

$$\lambda_2 = 1 + \sqrt{\frac{1+\sqrt{1+16\pi^2}}{2}}.$$

In particular, $\lambda_2 > 2$, and the uniqueness of the admissible positive solution of the equation for x implies that λ_2 is the only eigenvalue of $\mathcal{A}$ in $(2, \infty)$.

It remains to establish the simplicity of the two eigenvalues. For $\lambda < 0$, differentiation gives

$$\Delta'(\lambda) = -1 + \frac{2\pi(\lambda-1)}{\Big((\lambda-1)^2-1\Big)^{3/2}}.$$

Since $\lambda - 1 < 0$ in this region, it follows that

$$\Delta'(\lambda) < 0, \qquad \lambda < 0.$$

In particular,

$$\Delta'(\lambda_1) \neq 0.$$

Thus, λ_1 is a simple zero of the Fredholm determinant.

For $\lambda > 2$, we similarly obtain

$$\Delta'(\lambda) = -1 - \frac{2\pi(\lambda-1)}{\Big((\lambda-1)^2-1\Big)^{3/2}}.$$

Since $\lambda - 1 > 0$, we have

$$\Delta'(\lambda) < 0, \qquad \lambda > 2.$$

Therefore,

$$\Delta'(\lambda_2) \neq 0,$$

and λ_2 is also a simple zero of $\Delta(\cdot)$.

By Lemma 3.1, the zeros λ_1 and λ_2 correspond to the eigenvalues of $\mathcal{A}$. Hence, $\mathcal{A}$ possesses exactly two simple eigenvalues outside $[0,2]$, and

$$\sigma_{\text{disc}}(\mathcal{A}) = \{\lambda_1, \lambda_2\}.$$

□

The spectral configuration described in Theorem 3.1 is illustrated in Figure 1. The essential spectrum of $\mathcal{A}$ occupies the interval $[0,2]$, while the two simple discrete eigenvalues λ_1 and λ_2 are located below and above this interval, respectively.

4. Conclusion

In this paper, we have obtained a complete description of the essential and discrete spectra of the self-adjoint 2×2 operator matrix $\mathcal{A}$ acting in $\mathbb{C}\oplus L_2(\mathbb{T})$. The decomposition of $\mathcal{A}$ into its diagonal part and a finite-rank off-diagonal perturbation made it possible to identify the essential spectrum as

$$\sigma_{\text{ess}}(\mathcal{A}) = [0,2].$$

The spectral analysis outside this interval was reduced to the study of the zeros of the Fredholm determinant associated with $\mathcal{A}$. An explicit evaluation of the integral entering this determinant transformed the eigenvalue problem into a scalar algebraic equation. As a result, the discrete

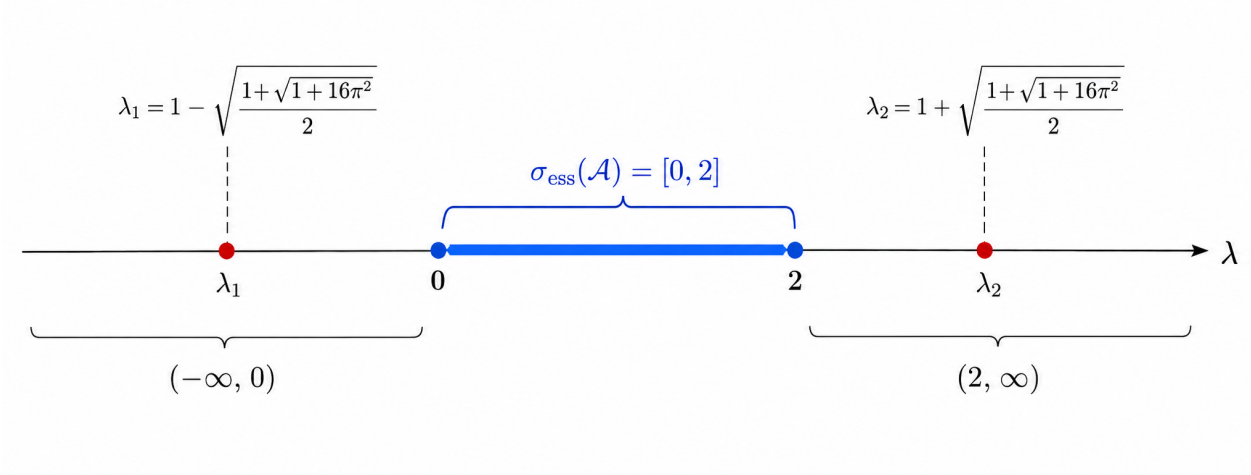


FIGURE 1. Geometric representation of the spectrum of the operator matrix $\mathcal{A}$.

spectrum was determined completely and shown to consist of exactly two simple eigenvalues, namely,

$$\sigma_{\text{disc}}(\mathcal{A}) = \{\lambda_1, \lambda_2\},$$

where

$$\lambda_1 < 0, \qquad \lambda_2 > 2.$$

Thus, the two discrete eigenvalues are located on opposite sides of the essential spectral interval, and both admit explicit closed-form representations.

The obtained results demonstrate that, although the finite-rank off-diagonal interaction preserves the essential spectrum, it changes the spectral structure of the diagonal operator by producing isolated eigenvalues outside the essential spectrum. For the particular model considered here, the combination of finite-rank perturbation methods and the Fredholm determinant provides not only a qualitative description of the spectrum but also an explicit determination of its discrete component.

Consequently, the spectrum of $\mathcal{A}$ is completely described by

$$\sigma(\mathcal{A}) = \{\lambda_1\} \cup [0, 2] \cup \{\lambda_2\}.$$

This representation gives a concise spectral picture of the model and emphasizes the effect of the off-diagonal coupling on the formation of discrete eigenvalues.

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