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SOME FIXED POINT RESULTS FOR MULTIVALUED MAPPINGS IN $b-$MULTIPLICATIVE AND $b-$METRIC SPACE

MAZHAR MEHMOOD*, **ABDULLAH SHOAIB AND HAMZA KHALID**

Department of Mathematics and Statistics,Riphah International University, Islamabad - 44000, Pakistan

* *Corresponding author:* mazharm53@gmail.com

Abstract. The main outcome of this paper is to introduce the notion of Hausdorff b-multiplicative metric space and to present some fixed point results for multivalued mappings in this space. Moreover, we obtain some fixed point results satisfying rational type contractive condition on closed ball for multivalued mappings in b-metric space.The proven results are original in nature.

1. Introduction

Bourbaki and Bakhtin [6], were the first ones who gave the idea of b-metric. After that, Czerwik [7] gave an axiom and formally defined a b-metric space. For further results on b-metric space, see [11–13]. Ozaksar and Cevical [10] investigated multiplicative metric space and proved its topological properties. Mongkolkeha et al. [9] described the concept of multiplicative proximal contraction mapping and proved best proximity point theorems for such mappings. Recently, Abbas et al. [1] proved some common fixed points results of quasi weak commutative mappings on a closed ball in the setting of multiplicative metric spaces. They also describe the main conditions for the existence of common solution of multiplicative boundary value problem. For further results on multiplicative metric space, see [2, 3, 8]. In 2017, Ali et al. [4] introduced the notion of b-multiplicative and proved some fixed point result. As an application, they established an existence theorem for the solution of a system of Fredholm multiplicative integral equations. Shoaib et

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al. [13], discussed the result for fuzzy mappings on a closed ball in a b-metric space. For further results on closed ball, see [5, 12, 14, 15]. In this paper, we proved some fixed point results for multivalued mappings in b-multiplicative and b-metric space.

2. Preliminaries and Basic Definitions

In this section we include some basic definitions and theorems which are useful to understand the results presented in this paper. First we give definition of b-metric space and its relevant results.

Definition 2.1. *[11] Let W be a non-empty set and $s \geq 1$ be a real number. A mapping $d : W \times W \to \Re^+ \cup \{0\}$ is said to be b-metric with coefficient "s", if for all $w, y, z \in W$, the following conditions hold*

i. $d(w, y) = 0$ *if and only if* $w = y$.

ii. $d(w, y) = d(y, w)$.

iii. $d(w, z) \leq s[d(w, y) + d(y, z)]$.

The pair (W, d) is called b-metric space.

Example 2.1. *[11] Let (W, d) be a metric space. Then for a real number $k > 1$, we define a function $d_1(a, b) = (d(a, b))^k$, then d_1 is a b-metric with $b = 2^{k-1}$.*

Definition 2.2. *[11] Let (W, b) be a b-metric space.*

i. *A sequence $\{w_n\}$ in (W, b) is called convergent if and only if there exists $w \in W$ such that $b(w_n, w) \to 0$, as $n \to +\infty$.*

ii. *A sequence $\{w_n\}$ in (W, b) is a Cauchy sequence, if and only if $b(w_n, w_m) \to 0$, as $n, m \to +\infty$.*

iii. *A b-metric space (W, b) is said to be complete if every Cauchy sequence in W converge to a point of W.*

Definition 2.3. *Let (W, d) be a b-metric space. A be a non empty subset of W with some $w_0 \in W$. An element $a \in A$ is called a best approximation in A if $d(w_o, A) = d(w_o, a)$, where $d(w_0, A) = \inf_{w \in A} d(w_o, w)$, if each $w_o \in W$ has at least one best approximation in A, then A is called a proximinal set. We denote $P(W)$, the set of all closed proximinal subsets of W.*

Definition 2.4. *Let (W, d) be b-metric space. The function $H : P(W) \times P(W) \to \Re^+$,defined by*

$H(A, B) = \max\{\sup_{w \in A} d(w, B), \sup_{y \in B} d(A, y)\}$, *is called Hausdorff b-metric on $P(W)$.*

Lemma 2.1. *Let (W, d) be b-metric space. Let H be a Hausdorff b-metric on $P(W)$. Then for all $A, B \in P(W)$ and for each $w \in A$ there exist $y \in B$ satisfying $d(w, B) = d(w, y)$ then $H(A, B) \geq d(w, y)$.*

Now, we include the definition of b-multiplicative metric space and its relevant results.

Definition 2.5. *[4] Let W be a non-empty set and let $s \geq 1$ be a given real number. A mapping $m : W \times W \to [1, \infty)$ is called a b-multiplicative metric with coefficient s, if the following conditions hold:*

i. $m(w, y) > 1$ *for all* $w, y \in W$ *with* $w \neq y$ *and* $m(w, y) = 1$ *if and only if* $w = y$.

ii. $m(w, y) = m(y, w)$ *for all* $w, y \in W$.

iii. $m(w, z) \leq [m(w, y).m(y, z)]^s$ *for all* $w, y, z \in W$.

The triplet (W, m, s) is called b-multiplicative metric space.

Example 2.2. *[4] Let $W = [\,0, \infty)$. Define a mapping $m_a : W \times W \to [\,1, \infty)$, $m_a(w, y) = a^{(w-y)^2}$, where $a > 1$ is any fixed real number. Then for each a, m_a is b-multiplicative metric on W with $s = 2$. Note that m_a is not a multiplicative metric on W.*

Definition 2.6. *[4] Let (W, m) be a b-multiplicative metric space.*

i. *A sequence $\{w_n\}$ is convergent iff there exists $w \in W$, such that $m(w_n, w) \to 1$, as $n \to +\infty$.*

ii. *A sequence $\{w_n\}$ is called b-multiplicative Cauchy, iff $m(w_m, w_n) \to 1$, as $m, n \to +\infty$.*

iii. *A b-multiplicative metric space (W, m) is said to be complete if every multiplicative Cauchy sequence in W is convergent to some $y \in W$.*

Definition 2.7. *Let (W, d) be a b-multiplicative metric space. A be a non empty subset of W with some $w_o \in W$. An element $a \in A$ is called a best approximation in A if $d(w_o, A) = d(w_o, a)$, where $d(w_o, A) = \inf_{w \in A} d(w_o, w)$, if each $w_o \in W$ has at least one best approximation in A, then A is called a proximinal set. We denote $P(W)$, the set of all closed proximinal subsets of W.*

Definition 2.8. *Let (W, d) be a b-multiplicative metric space. The function $H : P(W) \times P(W) \to \Re^+$, defined by $H(A, B) = \max\{\sup_{w \in A} d(w, B), \sup_{y \in B} d(A, y)\}$, is called Hausdorff b-multiplicative metric on $P(W)$.*

Lemma 2.2. *Let (W, d) be a b-multiplicative metric. Let H be a Hausdorff b-multiplicative metric on $P(W)$. Then for all $A, B \in P(W)$ and for each $w \in A$ there exist $y \in B$ satisfying $d(w, B) = d(w, y)$ then $H(A, B) \geq d(w, y)$.*

Definition 2.9. *Let (W, d) be a complete b-multiplicative metric and $w_0 \in W$ and $S : W \to P(W)$ be the multivalued mapping on W, then there exist $w_1 \in Sw_0$ be an element such that $d(w_0, Sw_0) = d(w_0, w_1)$. Let $w_2 \in Sw_1$ be such that $d(w_1, Sw_1) = d(w_1, w_2)$. Let $w_3 \in Sw_2$ be such that $d(w_2, Sw_2) = d(w_2, w_3)$. Continuing this process, we construct a sequence $\{w_n\}$ of points in W such that $w_{n+1} \in Sw_n$, $d(w_n, Sw_n) = d(w_n, w_{n+1})$. We denote this iterative sequence by $\{WS(w_n)\}$. We say that $\{WS(w_n)\}$ is a sequence in W generated by w_0.*

3. Results for $b-$Multiplicative Metric Space

Definition 3.1. *Let (W, d) be a b-multiplicative metric space and $S : W \to P(W)$ be the multivalued mapping. Let $w_0 \in W$ and $\{WS(w_n)\}$ be a sequence in W generated by w_0. We define the family $M(S)$ of all functions $a : W \times W \to [0, 1)$ which satisfy the following property $a(w_n, w_{n+1}) \le a(w_0, w_1)$, for all $n \in N \cup \{0\}$. Also, if $\{WS(w_n)\} \to h$, then $a(w_n, h) \le a(w_0, h)$.*

Theorem 3.1. *Let (W, d) be a complete b-multiplicative metric space with coefficient s, $S : W \to P(W)$ be a multivalued mapping on W and $\beta, \sigma, \psi \in M(S)$. If the following relations hold:*

$$H(Sw_n, Sw_{n+1}) \le [d(w_n, w_{n+1})]^{\beta(w_n,w_{n+1})}.[d(w_n, Sw_{n+1}).d(w_{n+1}, Sw_n)]^{\sigma(w_n,w_{n+1})}$$
$$.[d(w_n, Sw_n).d(w_{n+1}, Sw_{n+1})]^{\psi(w_n,w_{n+1})}, \quad (3.1)$$

for all $w_n, w_{n+1} \in \{WS(w_n)\}$, $n \in N \cup \{0\}$, $a, b \ge 0$, and

$$s\beta(w_0, w_1) + (s^2 + s)\sigma(w_0, w_1) + (s + 1)\psi(w_0, w_1) < 1 \quad for \quad w_0, w_1 \in \{WS(w_n)\}, \quad (3.2)$$

then $\{WS(w_n)\} \to w^* \in W$. Also, if inequalities 3.1 and 3.2 hold for h, then S has a fixed point w^*.

Proof. Considering a sequence $\{WS(w_n)\}$ in W generated by w_0, then we have $w_{n+1} \in Sw_n$, where $n = 0, 1, 2, \cdots$ now by using Lemma 2.2, we can write $d(w_n, w_{n+1}) \le H(Sw_{n-1}, Sw_n)$
$\le [d(w_{n-1}, w_n)]^{\beta(w_{n-1},w_n)}.[d(w_{n-1}, Sw_n).d(w_n, Sw_{n-1})]^{\sigma(w_{n-1},w_n)}$
$.[d(w_{n-1}, Sw_{n-1}).d(w_n, Sw_n)]^{\psi(w_{n-1},w_n)}$
$\le [d(w_{n-1}, w_n)]^{\beta(w_{n-1},w_n)}.[d(w_{n-1}, w_{n+1}).d(w_n, w_n)]^{\sigma(w_{n-1},w_n)}$
$.[d(w_{n-1}, w_n).d(w_n, w_{n+1})]^{\psi(w_{n-1},w_n)}$
by using the Definition 3.1 and triangle inequality, we can write
$d(w_n, w_{n+1}) \le [d(w_{n-1}, w_n)]^{\beta(w_0,w_1)}.[d(w_{n-1}, w_n).d(w_n, w_{n+1})]^{s\sigma(w_0,w_1)}$
$.[d(w_{n-1}, w_n).d(w_n, w_{n+1})]^{\psi(w_0,w_1)}.[d(w_n, w_{n+1})]^{1-s\sigma(w_0,w_1)-\psi(w_0,w_1)}$
$\le [d(w_{n-1}, w_n)]^{\beta(w_0,w_1)+s\sigma(w_0,w_1)+\psi(w_0,w_1)}$

$$d(w_n, w_{n+1}) \le [d(w_{n-1}, w_n)]^{\dfrac{\beta(w_0, w_1) + s\sigma(w_0, w_1) + \psi(w_0, w_1)}{1 - s\sigma(w_0, w_1) - \psi(w_0, w_1)}} = [d(w_{n-1}, w_n)]^K. \quad (3.3)$$

Now,
$d(w_{n-1}, w_n) \le H(Sw_{n-2}, Sw_{n-1})$
$\le [d(w_{n-2}, w_{n-1})]^{\beta(w_{n-2},w_{n-1})}.[d(w_{n-2}, Sw_{n-1}).d(w_{n-1}, Sw_{n-2})]^{\sigma(w_{n-2},w_{n-1})}$
$.[d(w_{n-2}, Sw_{n-2}).d(w_{n-1}, Sw_{n-1})]^{\psi(w_{n-2},w_{n-1})}$
$\le [d(w_{n-2}, w_{n-1})]^{\beta(w_0,w_1)}.[d(w_{n-2}, w_{n-1}).d(w_{n-1}, w_n)]^{s\sigma(w_0,w_1)}.[d(w_{n-2}, w_{n-1}).d(w_{n-1}, w_n)]^{\psi(w_0,w_1)}$

$[d(w_{n-1},w_n)]^{1-s\sigma(w_0,w_1)-\psi(w_0,w_1)} \leq [d(w_{n-2},w_{n-1})]^{\beta(w_0,w_1)+s\sigma(w_0,w_1)+\psi(w_0,w_1)}$

$$d(w_{n-1},w_n) \leq [d(w_{n-2},w_{n-1})]^{\dfrac{\beta(w_0,w_1)+s\sigma(w_0,w_1)+\psi(w_0,w_1)}{1-s\sigma(w_0,w_1)-\psi(w_0,w_1)}} = [d(w_{n-2},w_{n-1})]^K. \tag{3.4}$$

From 3.3 and 3.4, we can write

$$d(w_n,w_{n+1}) \leq [d(w_{n-1},w_n)]^K \leq [d(w_{n-2},w_{n-1})^K]^K = [d(w_{n-2},w_{n-1})]^{K^2} \leq [d(w_{n-3},w_{n-2})]^{K^3}$$
$$\leq \cdots \leq [d(w_0,w_1)]^{K^n} \tag{3.5}$$

Now, for $m,n \in N$, with $m>n$, we have

$d(w_n,w_m) \leq d(w_n,w_{n+1})^s.d(w_{n+1},w_{n+2})^{s^2} \cdots .d(w_{m-2},w_{m-1})^{s^{m-n-1}}.d(w_{m-1},w_m)^{s^{m-n}}$

by using the inequality 3.5, we have

$d(w_n,w_m) \leq d(w_n,w_{n+1})^s.d(w_{n+1},w_{n+2})^{s^2} \cdots .d(w_{m-2},w_{m-1})^{s^{m-n-1}}.d(w_{m-1},w_m)^{s^{m-n}}$

$\leq [d(w_0,w_1)]^{sK^n}.[d(w_0,w_1)]^{s^2K^{n+1}} \cdots .[d(w_0,w_1)]^{s^{m-n-1}K^{m-2}}.[d(w_0,w_1)]^{s^{m-n}K^{m-1}}$

$\leq [d(w_0,w_1)]^{sK^n(1+sK+(sK)^2+\cdots+s^{m-n-2}K^{m-n-2}+s^{m-n-1}K^{m-n-1})}$

$\leq [d(w_0,w_1)]^{sK^n(1+sK+(sK)^2+\cdots+(sK)^{m-n-2}+(sK)^{m-n-1})}$

$< [d(w_0,w_1)]^{sK^n(1+sK+(sK)^2+\cdots)} = [d(w_0,w_1)]^{sK^n(\frac{1}{1-sK})}$.

Taking $\lim_{m,n\to\infty}$, we get $d(w_n,w_m) \to 1$.

Hence $\{WS(w_n)\}$ is a b-multiplicative Cauchy sequence. By completeness of (W,d), we have $w_n \to w^* \in W$. Also

$$\lim_{n\to\infty} d(w_n,w^*) = 1. \tag{3.6}$$

Now,

$d(w^*,Sw^*) \leq [d(w^*,w_{n+1}).d(w_{n+1},Sw^*)]^s \leq d(w^*,w_{n+1})^s.H(Sw_n,Sw^*)^s$

$\leq d(w^*,w_{n+1})^s.[d(w_n,w^*)]^{s\beta(w_n,w^*)}.[d(w_n,Sw^*).d(w^*,Sw_n)]^{s\sigma(w_n,w^*)}.[d(w_n,Sw_n).d(w^*,Sw^*)]^{s\psi(w_n,w^*)}$

$\leq d(w^*,w_{n+1})^s.[d(w_n,w^*)]^{s\beta(w_n,w^*)}.[d(w_n,w^*).d(w^*,Sw^*)]^{s^2\sigma(w_n,w^*)}$

$.d(w^*,w_{n+1})^{s\sigma(w_n,w^*)}.[d(w_n,w_{n+1}).d(w^*,Sw^*)]^{s\psi(w_n,w^*)}$.

By using Definition 2.8, we can write

$d(w^*,Sw^*) \leq d(w^*,w_{n+1})^s.[d(w_n,w^*)]^{s\beta(w_0,w^*)}.[d(w_n,w^*).d(w^*,Sw^*)]^{s^2\sigma(w_0,w^*)}.d(w^*,w_{n+1})^{s\sigma(w_0,w^*)}$

$.[d(w_n,w_{n+1}).d(w^*,Sw^*)]^{s\psi(w_0,w^*)}$.

On taking $\lim_{n\to\infty}$ and by using inequality 3.6, we get

$d(w^*,Sw^*) \leq [d(w^*,Sw^*)]^{s^2\sigma(w_0,w^*)}.[d(w^*,Sw^*)]^{s\psi(w_0,w^*)}$

$[d(w^*,Sw^*)]^{1-s^2\sigma(w_0,w^*)-s\psi(w_0,w^*)} \leq 1$

$d(w^*,Sw^*) \leq (1)^{\dfrac{1}{1-s^2\sigma(w_0,w^*)-s\psi(w_0,w^*)}} \leq 1.$

This implies that $d(w^*,Sw^*)=1$ and hence w^* is a fixed point of mapping S. □

Theorem 3.2. *Let (W, d) be a complete b-multiplicative metric space with coefficient s, $S : W \to P(W)$ be the multivalued mapping on W and $\beta \in M(S)$. If the following relations hold:*

$$H(Sw_n, Sw_{n+1}) \leq [d(w_n, w_{n+1})]^{\beta(w_n,w_{n+1})}, \tag{3.7}$$

for all $w_n, w_{n+1} \in \{WS(w_n)\}$, $n \in N \cup \{0\}, a, b \geq 0$ and

$$s\beta(w_0, w_1) < 1 \quad for \quad w_0, w_1 \in \{WS(w_n)\}, \quad then \quad \{WS(w_n)\} \to w^* \in W. \tag{3.8}$$

Also if inequalities 3.7 and 3.8 hold for h, then S has a fixed point w^*.

Theorem 3.3. *Let (W, d) be a complete b-multiplicative metric space with coefficient s, $S : W \to P(W)$ be the multivalued mapping on W and $\sigma \in M(S)$. If the following relations hold:*

$$H(Sw_n, Sw_{n+1}) \leq [d(w_n, Sw_{n+1}).d(w_{n+1}, Sw_n)]^{\sigma(w_n,w_{n+1})}, \tag{3.9}$$

for all $w_n, w_{n+1} \in \{WS(w_n)\}$, $n \in N \cup \{0\}$, $a, b \geq 0$ and

$$(s^2 + s)\sigma(w_0, w_1) < 1 \quad for \quad w_0, w_1 \in \{WS(w_n)\}, \quad then \quad \{WS(w_n)\} \to w^* \in W. \tag{3.10}$$

Also if inequalities 3.9 and 3.10 hold for h, then S has a fixed point w^*.

Theorem 3.4. *Let (W, d) be a complete b-multiplicative metric space with coefficient s, $S : W \to P(W)$ be the multivalued mapping on W and $\psi \in M(S)$. If the following relations hold:*

$$H(Sw_n, Sw_{n+1}) \leq [d(w_n, Sw_n).d(w_{n+1}, Sw_{n+1})]^{\psi(w_n,w_{n+1})}, \tag{3.11}$$

for all $w_n, w_{n+1} \in \{WS(w_n)\}$, $n \in N \cup \{0\}$, $a, b \geq 0$

$$(s + 1)\psi(w_0, w_1) < 1 \quad for \quad w_0, w_1 \in \{WS(w_n)\}, \quad then \quad \{WS(w_n)\} \to w^* \in W. \tag{3.12}$$

Also if inequalities 3.11 and 3.12 hold for h, then S has a fixed point w^*

4. Results for $b-$Metric Space

Theorem 4.1. *Let (W, d) be a complete b-metric space with coefficient s and w_0 be any point in W. Let the mapping $S : W \to P(W)$ satisfy the following relations:*

$$H(Sw_n, Sw_{n+1}) \leq a_1 d(w_n, w_{n+1}) + a_2[\frac{a + d(w_n, Sw_n)}{b + d(w_n, w_{n+1})}]d(w_n, Sw_n) + a_3[\frac{c + d(w_n, w_{n+1})}{d' + d(w_n, Sw_n)}]d(w_{n+1}, Sw_{n+1}) + a_4[d(w_n, Sw_{n+1}) + d(w_{n+1}, Sw_n)], \tag{4.1}$$

for all $w_n, w_{n+1} \in \overline{B(w_0; r)} \cap \{WS(w_0)\}$ and $a, b, c, d', a_1, a_2, a_3, a_4 > 0$ with $a \leq b$, $c \leq d'$. Also

$$d(w_0, Sw_0) \leq \beta(1 - s\beta)r, \quad where \quad \beta = \frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4}, \quad r > 0 \quad and \quad s\beta < 1. \tag{4.2}$$

Then $\{WS(w_n)\}$ is a sequence in $\overline{B(w_0;r)}$ and $\{WS(w_n)\} \to h \in \overline{B(w_0;r)}$. Also if inequality 4.1 holds for h, then S has a fixed point h in $\overline{B(w_0;r)}$.

Proof. Considering a sequence $\{WS(w_n)\}$ in W generated by w_0, then, we have $w_{n+1} \in Sw_n$, where n=0,1,2,... From 4.2, we have $d(w_0,w_1) \leq \beta(1-s\beta)r \leq r$. This implies $w_1 \in \overline{B(w_0;r)}$. Now by using Lemma 2.1 and inequality 4.1, we can write $d(w_1,w_2) = d(w_1,Sw_1) \leq H(Sw_0,Sw_1)$
$\leq a_1 d(w_0,w_1) + a_2[\frac{a+d(w_0,Sw_0)}{b+d(w_0,w_1)}]d(w_0,Sw_0) + a_3[\frac{c+d(w_0,w_1)}{d'+d(w_0,Sw_0)}]d(w_1,Sw_1)$
$+a_4[d(w_0,Sw_1)+d(w_1,Sw_0)]$
$\leq a_1 d(w_0,w_1) + a_2[\frac{a+d(w_0,w_1)}{b+d(w_0,w_1)}]d(w_0,w_1) + a_3[\frac{c+d(w_0,w_1)}{d'+d(w_0,w_1)}]d(w_1,w_2) + sa_4 d(w_0,w_1)$
$+sa_4 d(w_1,w_2)$. As $a \leq b$ and $c \leq d^{'}$, we have
$[1-a_3-sa_4]d(w_1,w_2) \leq [a_1+a_2+sa_4]d(w_0,w_1)d(w_1,w_2) \leq [\frac{a_1+a_2+sa_4}{1-a_3-sa_4}]d(w_0,w_1) \leq \beta d(w_0,w_1) \leq \beta^2(1-s\beta)r$.
Now,
$d(w_0,w_2) \leq s[d(w_0,w_1)+d(w_1,w_2)] \leq s[\beta(1-s\beta)r+\beta^2(1-s\beta)r] \leq \beta s(1-s\beta)(1+\beta)r \leq \beta s(1-s\beta)(1+s\beta)r \leq \beta s[1-(s\beta)^2]r \leq r$.
This implies $w_2 \in \overline{B(w_0;r)}$. Considering $w_3, w_4, w_5, \cdots, w_j \in \overline{B(w_0;r)}$.
Now,
$d(w_j,w_{j+1}) \leq H(Sw_{j-1},Sw_j) \leq a_1 d(w_{j-1},w_j) + a_2[\frac{a+d(w_{j-1},Sw_{j-1})}{b+d(w_{j-1},w_j)}]d(w_{j-1},Sw_{j-1}) +$
$+ a_3[\frac{c+d(w_{j-1},w_j)}{d'+d(w_{j-1},Sw_{j-1})}]d(w_j,Sw_j) + a_4[d(w_{j-1},Sw_j)+d(w_j,Sw_{j-1})]$
$\leq a_1 d(w_{j-1},w_j) + a_2[\frac{a+d(w_{j-1},w_j)}{b+d(w_{j-1},w_j)}]d(w_{j-1},w_j) + a_3[\frac{c+d(w_{j-1},w_j)}{d'+d(w_{j-1},w_j)}]d(w_j,w_{j+1})$
$+ sa_4[d(w_{j-1},w_j)+d(w_j,Sw_{j+1})]$.
As $a \leq b$ and $c \leq d^{'}$, we have
$(1-a_3-sa_4)d(w_j,w_{j+1}) \leq (a_1+a_2+sa_4)d(w_{j-1},w_j)d(w_j,w_{j+1})$
$\leq [\frac{a_1+a_2+sa_4}{1-a_3-sa_4}]d(w_{j-1},w_j) = \beta d(w_{j-1},w_j) \leq \beta[\beta d(w_{j-2},w_{j-1})] \leq \beta^2[\beta d(w_{j-3},w_{j-2})]$.

$$(1-a_3-sa_4)d(w_j,w_{j+1}) \leq \beta^j d(w_0,w_1) \quad \forall \quad j \in N \tag{4.3}$$

Now,
$d(w_0,w_{j+1}) \leq sd(w_0,w_1) + s^2 d(w_1,w_2) + \cdots + s^j d(w_j,w_{j+1}) \leq sd(w_0,w_1) + s^2\beta d(w\prime,w_1) + \cdots + s^{j+1}\beta^j d(w_0,w_1) \leq s(1+s\beta+(s\beta)^2+\cdots+(s\beta)^j d(w_0,w_1)) \leq s[\frac{1-(s\beta)^j}{1-s\beta}]\beta(1-s\beta)r \leq s\beta[1-(s\beta)^j]r \leq r$.
Thus $w_{j+1} \in \overline{B(w_0;r)}$.
Hence $w_n \in \overline{B(w_0;r)}$ for all $n \in N \cup \{0\}$, therefore $\{WS(w_n)\}$ is a sequence in $\overline{B(w_0;r)}$. Now, the inequality 4.3 can be written as

$$d(w_n,w_{n+1}) \leq \beta^n d(w_0,w_1) \quad \text{for all} \quad n \in N. \tag{4.4}$$

Hence for any $m > n$
$d(w_n,w_m) \leq sd(w_n,w_{n+1}) + s^2 d(w_{n+1},w_{n+2}) + \cdots + s^{m-1} d(w_{m-1},w_m) \leq [s\beta^n + s^2\beta^{n+1} + \cdots +$

$s^{m-n}\beta^{m-1}]d(w_0, w_1)$ by using 4.4
$\leq s\beta^n[1 + s\beta + (s\beta)^2 + \cdots + s^{m-n-1}\beta^{m-n-1}]d(w_0, w_1) \leq s\beta^n[1 + s\beta + (s\beta)^2 + \cdots]d(w_0, w_1) \leq [\frac{(s\beta)^n}{1-s\beta}]d(w_0, w_1) \to 0$ as m, n $\to \infty$. Thus, we prove that $\{WS(w_n)\}$ is a Cauchy sequence in $\overline{B(w_0; r)}$. As every closed ball in complete b-metric space is complete, so there exists $h \in \overline{B(w_0; r)}$ such that $\{WS(w_n)\} \to h$ or $\lim_{n\to\infty} d(w_n, h) = 0$. If $h \in Sh$, the desired result is obvious and straight forward. If $h \notin Sh$ then $d(h, Sh) = z > 0$, that is $d(h, Sh) \leq s[d(h, w_{n+1})+d(w_{n+1}, Sh)] \leq s[d(h, w_{n+1})+H(Sw_n, Sh)] \leq sd(h, w_{n+1})+sa_1d(w_n, h)+a_2[\frac{a+d(w_n,Sw_n)}{b+d(w_n,h)}]d(w_n, Sw_n)+a_3[\frac{c+d(w_n,h)}{d'+d(w_n,Sw_n)}]d(h, Sh)+a_4[d(w_n, Sh)+d(h, Sw_n)]$ $\leq sd(h, w_{n+1}) + sa_1d(w_n, h) + a_2[\frac{a+d(w_n,w_{n+1})}{b+d(w_n,h)}]d(w_n, w_{n+1}) + a_3[\frac{c+d(w_n,h)}{d'+d(w_n,w_{n+1})}]d(h, Sh) + sa_4[d(w_n, h) + d(h, Sh)]$. Taking $n \to \infty$, it follows that $d(h, Sh) \leq a_3d(h, Sh) + sa_4d(h, Sh)$
$(1 - a_3 - sa_4)d(h, Sh) \leq 0$. This implies $d(h, Sh) = z \leq 0$, a contradiction, so $h \in Sh$. Hence proved □

Corollary 4.1. *Let (W, d) be a complete metric space and w_0 be any point in W. Let $S : W \to W$ be a mapping and $w_n = Sw_{n-1}$ be a Picard sequence. If*

$$d(Sw_n, Sw_{n+1}) \leq a_1d(w_n, w_{n+1}) + a_2[\frac{a + d(w_n, Sw_n)}{b + d(w_n, w_{n+1})}]d(w_n, Sw_n) + a_3[\frac{c + d(w_n, w_{n+1})}{d' + d(w_n, Sw_n)}]d(w_{n+1}, Sw_{n+1}) + a_4[d(w_n, Sw_{n+1}) + d(w_{n+1}, Sw_n)], \quad (4.5)$$

for all $w_n, w_{n+1} \in \overline{B(w_0; r)} \cup \{w_n\}$ and $a, b, c, d', a_1, a_2, a_3, a_4 > 0$ with $a \leq b$, $c \leq d'$. Also $d(w_0, Sw_0) \leq \beta(1 - \beta)r$, where $\beta = \frac{a_1+a_2+a_4}{1-a_3-a_4}$, $\beta < 1$.
Then $\{w_n\}$ is a sequence in $\overline{B(w_0; r)}$ and $w_n \to h \in \overline{B(w_0; r)}$. Also if inequality 4.5 holds for h, then S has a fixed point h in $\overline{B(w_0; r)}$.

Corollary 4.2. *Let (W, d) be a complete b-metric space with coefficient s and w_0 be any point in W. Let the mapping $S : W \to P(W)$ satisfy the following:*

$$H(Sw_n, Sw_{n+1}) \leq a_1d(w_n, w_{n+1}) + a_2[(\frac{a + d(w_n, Sw_n)}{b + d(w_n, w_{n+1})}]d(w_n, Sw_n) + a_3[\frac{c + d(w_n, w_{n+1})}{d' + d(w_n, Sw_n)}]d(w_{n+1}, Sw_{n+1}) + a_4[d(w_n, Sw_{n+1}) + d(w_{n+1}, Sw_n)], \quad (4.6)$$

for all $w_n, w_{n+1} \in \{WS(w_n)\}$ and $a, b, c, d', a_1, a_2, a_3, a_4 > 0$ with $sa_1 + sa_2 + a_3 + (s^2 + s)a_4 < 1$, and $a \leq b$, $c \leq d'$.
Then $\{WS(w_n)\} \to h \in W$. Also if inequality 4.6 holds for h, then S has a fixed point h in W.

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