



ON THE SOLUTIONS OF FALKNER-SKAN EQUATION

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ABSTRACT. We consider the differential equation $f''' + ff'' + \beta(f'^2 - 1) = 0$, with $\beta > 0$. In order to prove the existence of solutions satisfying the boundary conditions $f(0) = a \geq 0$, $f'(0) = b \geq 0$ and $f'(+\infty) = -1$ or 1 for $0 < \beta \leq \frac{1}{2}$. We use shooting technique and consider the initial conditions $f(0) = a$, $f'(0) = b$ and $f''(0) = c$. We prove that there exists an infinitely many solutions such that $f'(+\infty) = 1$.

1. INTRODUCTION

In 1931 the Falkner-Skan equation is introduced for studying the boundary layer flow past a semi infinite wedge, it is defined by

$$f''' + ff'' + \beta(f'^2 - 1) = 0 \quad (1.1)$$

The solution of this equation have been studied by numerous authors as, for example, D. R. Hartree (1937), H. Weyl (1942), W. A. Coppel (1960), P. Hartman (1964) and G.C. Yang (2003,2004).

The more general equation of (1.1) is

$$f''' + ff'' + g(f') = 0 \quad (1.2)$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is some function. The solutions obtained are called similarity solutions.

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The most famous example is perhaps the Blasius equation (1908), which corresponds to $g(x) = 0$ and arises in the study of laminar boundary layer on a flat plate.

More recently, the equation (1.2) with $g(x) = \beta x^2$ and $g(x) = \beta x(x - 1)$ has been considered. These cases occur, for example, in the study of free convection and of mixed convection boundary layer flows over a vertical surface embedded in a porous medium.

Most of the time, associated with the equation (1.1) is the boundary value problem:

$$\begin{cases} f''' + ff'' + \beta (f'^2 - 1) = 0 \\ f(0) = a \\ f'(0) = b \\ f'(t) \rightarrow \lambda \text{ as } t \rightarrow +\infty \end{cases} \quad (\mathcal{P}_{\beta;a,b,\lambda})$$

To solve the boundary value problem $(\mathcal{P}_{\beta; a, b, \lambda})$ we will use the shooting technique. To this end, let f_c denote the solution of the initial value problem $(\mathcal{Q}_{\beta; a, b, c})$ consisting in the equation (1.1) together with the initial conditions $f_c(0) = a$, $f'_c(0) = b$ and $f''_c(0) = c$, and let $[0, T_c[$ be the right maximal interval of existence of f_c . To obtain a solution of $(\mathcal{P}_{\beta; a, b, \lambda})$ amounts to find a value of c such that $T_c = +\infty$ and $f'_c(t) \rightarrow \lambda$ as $t \rightarrow +\infty$.

We must assume that $\beta(\lambda^2 - 1) = 0$ to have solutions, in our case of Falkner-Skan equation, the only relevant conditions are $f'_c(t) \rightarrow -1$ or $f'_c(t) \rightarrow 1$ as $t \rightarrow +\infty$.

In the following, we will study the existence of concave, convex, concave-convex and convex-concave solutions to the boundary value problem $(\mathcal{P}_{\beta; a, b, -1})$ and $(\mathcal{P}_{\beta; a, b, 1})$ for $0 < \beta \leq \frac{1}{2}$, $a \geq 0$ and $b \geq 0$.

2. PRELIMINARY RESULTS

Let f be a solution of the equation (1.1) on some interval I , we consider the function $H_f : I \rightarrow \mathbb{R}$ defined by

$$H_f = f'' + f(f' - 1). \tag{2.1}$$

This function is obtained by integrating the equation (1.1), in fact, if f is a solution of (1.1) then

$$H'_f = (f' - 1)((1 - \beta)f' - \beta).$$

We give lemmas that will be useful later.

Lemma 2.1. *Let f be a solution of (1.1) on some maximal interval I . If there exists $t_0 \in I$ such that $f'(t_0) \in \{-1, 1\}$ and $f''(t_0) = 0$, then $I = \mathbb{R}$ and $f''(t) = 0$ for all $t \in \mathbb{R}$.*

Proof. See [4], proposition 3.1 item 3. □

Lemma 2.2. *Let $\beta > 0$ and f be a solution of equation (1.1) on some interval I , such that f' is not constant.*

- (1) If there exists $s < r \in I$ such that $f''(s) \leq 0$ and $(f'^2 - 1) > 0$ on $]s, r[$, then $f''(t) < 0$ for all $t \in]s, r]$.
- (2) If there exists $s < r \in I$ such that $f''(s) \geq 0$ and $(f'^2 - 1) < 0$ on $]s, r[$, then $f''(t) > 0$ for all $t \in]s, r]$.
- (3) If there exists $s < r \in I$ such that $f'' < 0$ on $]s, r[$ and $f''(r) = 0$, then $(f'^2(r) - 1) < 0$.
- (4) If there exists $s < r \in I$ such that $f'' > 0$ on $]s, r[$ and $f''(r) = 0$, then $(f'^2(r) - 1) > 0$.

Proof. Let F denote any primitive function of f . From (1.1) we deduce the relation

$$(f'' \exp F)' = -\beta (f'^2 - 1) \exp F.$$

All the assertions 1-4 follow easily from this relation and from previous lemma. Let us verify the first and the third of these assertions. For the first one, since $\psi = f'' \exp F$ is decreasing on $[s, r]$, we obtain

$$\begin{aligned} t \geq s &\implies \psi(t) \leq \psi(s) \\ &\implies f''(t) \exp F(t) \leq f''(s) \exp F(s) \\ &\implies f''(t) \leq f''(s) \exp(F(s) - F(t)) \\ &\implies f''(t) \leq 0, \forall t \in]s, r]. \end{aligned}$$

For the third one, since $\psi < 0$ on $]s, r[$ and $\psi(r) = 0$, $\psi'(t) \geq 0$ on $[s, r]$, then $\psi'(r) \geq 0$.

$$\psi'(r) = -\beta (f'^2(r) - 1) \exp F \geq 0.$$

This and lemma 2.1 imply that $(f'^2(r) - 1) < 0$. □

Lemma 2.3. Let f be a solution of (1.1) on some maximal interval $]T_-, T_+[$. If T_+ is finite, then f' and f'' are unbounded in any neighborhood of T_+ .

Proof. See [4], proposition 3.1 item 6. □

Lemma 2.4. Let $\beta \neq 0$. If f is a solution of (1.1) on some interval $]\tau, +\infty[$ such that $f'(t) \rightarrow \lambda$ as $t \rightarrow +\infty$, then $\lambda \in \{-1, 1\}$. Moreover, if f is of constant sign at infinity, then $f''(t) \rightarrow 0$ as $t \rightarrow +\infty$.

Proof. See [4], proposition 3.1 item 4 and 5. □

Lemma 2.5. Let $\beta > 0$ and f be a solution of equation (1.1) on some right maximal interval $I = [\tau, +\infty[$. If $f \geq 0$ and $f' \geq 0$ on I , then $T_+ = +\infty$ and f' is bounded on I .

Proof. Let $L = L_f$ be the function defined on I by

$$L(t) = 3f''^2(t) + \beta f'(t) (2f'^2(t) - 6). \tag{2.2}$$

Easily, using (1.1), we obtain that

$$L'(t) = -6f(t) f''^2(t) \quad \forall t \in I,$$

and, since $f \geq 0$ on I this implies that L is decreasing. Hence

$$\forall t \in I = [\tau, T_+[: t > \tau \implies L(t) \leq L(\tau)$$

$$\beta f'(t) (2f'^2(t) - 6) \leq 3f''^2(t) + \beta f'(t) (2f'^2(t) - 6) \leq L(\tau), \quad \forall t \in I.$$

It follows that f' is bounded on I and thanks to lemma (2.3) that $T_+ = +\infty$. □

Lemma 2.6. *Let $\beta > 0$ and f be a solution of equation (1.1) on some right maximal interval $I = [\tau, T_+[$. If $f(\tau) \geq 0, f'(\tau) \geq 1$ and $f''(\tau) > 0$, then there exists $t_0 \in]\tau, T_+[$ such that $f'' > 0$ on $[\tau, t_0[$ and $f''(t_0) = 0$.*

Proof. Assume for contradiction that $f'' > 0$ on I . Then $f(t) \geq 0, f'(t) \geq 1$ for all $t \in I$. Then, we have

$$f''' = -ff'' - \beta(f'^2 - 1) \leq 0. \tag{2.3}$$

It follows that $0 < f''(t) \leq c$ for all $t \in I$ and hence, by lemma (2.3), we have $T_+ = +\infty$.

Next, let $s > \tau$ and $\epsilon = \beta(f'(s)^2 - 1)$. One has $\epsilon > 0$ and, coming back to (2.3), we obtain

$$f''' \leq -\epsilon \text{ on } [s, +\infty[.$$

After integrating, we get

$$\forall t \geq s, f''(t) - f''(s) \leq -\epsilon(t - s),$$

and a contradiction with the fact that $f''(t) > 0$. Consequently, there exists $t_0 \in]\tau, T_+[$ such that $f'' > 0$ on $[\tau, t_0[$ and $f''(t_0) = 0$. □

Lemma 2.7. *Let $\beta \in]0, \frac{1}{2}[$ and f be a solution of equation (1.1) on some right maximal interval $I =]T_-, T_+[$. If there exists $t_0 \in I$ such that $\frac{\beta}{1-\beta} < f'(t_0) < 1$ and $0 \leq f''(t_0) \leq f(t_0)(1 - f'(t_0))$, then $T_+ = +\infty$ and $f'(t) \rightarrow 1$ as $t \rightarrow +\infty$. Moreover $f'' > 0$ on $[t_0, +\infty[$.*

Proof. Let $\delta = \sup X(t_0)$, where

$$X(t_0) = \left\{ t \in]t_0, T_+[: f'(t_0) < f' < 1 \text{ and } f'' > 0 \text{ on }]t_0, t[\right\}.$$

The set $X(t_0)$ is not empty. This is clear if $f''(t_0) > 0$, and if $f''(t_0) = 0$ it follows from the fact that

$$f'''(t_0) = -\beta(f'^2(t_0) - 1) > 0.$$

We claim that $\delta = T_+$, assume for contradiction that $\delta < T_+$. From lemma (2.2), item 2, we get that $f''(\delta) > 0$, which implies, by definition of δ , that $f'(\delta) = 1$. Therefore, since the function H_f defined by (2.1) is nonincreasing on $[t_0, \delta]$, we obtain

$$\begin{aligned} \delta \geq t_0 &\implies H_f(\delta) \leq H_f(t_0) \\ &\implies f''(\delta) \leq f''(t_0) - f(t_0)(1 - f'(t_0)) \\ &\implies f''(\delta) < 0, \end{aligned}$$

a contradiction. Thus, we have $\delta = T_+$. From lemma 2.3, it follows that $T_+ = +\infty$. Since $f'' > 0$ on $[t_0, +\infty[$, by virtue of lemma 2.4, we get that $f'(t) \rightarrow 1$ as $t \rightarrow +\infty$. □

Lemma 2.8. *Let $\beta \in]0, \frac{1}{2}]$ and f be a solution of equation (1.1) on some right maximal interval $I =]T_-, T_+[$. If there exists $t_0 \in I$ such that $f'(t_0) > 1$ and $f(t_0)(1 - f'(t_0)) \leq f''(t_0) \leq 0$. Then $T_+ = +\infty$ and $f'(t) \rightarrow 1$ as $t \rightarrow +\infty$. Moreover $f'' < 0$ on $[t_0, +\infty[$.*

Proof. If we set $\omega = \sup Y(t_0)$, where

$$Y(t_0) = \left\{ t \in]t_0, T_+[: 1 < f' < f'(t_0) \text{ and } f'' < 0 \text{ on }]t_0, t[\right\}.$$

The conclusion will follow by proceeding in the same way as the previous proof. □

3. DESCRIPTION OF OUR APPROACH WHEN $b \geq 1$

Let $\beta > 0$, $a \geq 0$ and $b \geq 1$. As said in the introduction, the method we will use to obtain solutions of the boundary value problems $(\mathcal{P}_{\beta; a, b, -1})$ and $(\mathcal{P}_{\beta; a, b, 1})$ is the shooting technique. Specifically, for $c \in \mathbb{R}$, let us denote by f_c the solution of equation (1.1) satisfying the initial conditions

$$f_c(0) = a, \quad f'_c(0) = b \text{ and } f''_c(0) = c, \tag{3.1}$$

and let $[0, T_c[$ be the right maximal interval of existence of f_c . Hence, finding a solution of one of the problems $(\mathcal{P}_{\beta; a, b, -1})$ and $(\mathcal{P}_{\beta; a, b, 1})$ amounts to finding a value of c such that $T_+ = +\infty$ and $f'_c(t) \rightarrow -1$ or 1 as $t \rightarrow +\infty$.

To this end, let us partition \mathbb{R} into the four sets C_0, C_1, C_2 and C_3 defined as follows.

Let $C_0 =]0, +\infty[$ and, according to the notations used in [4], let us set

$$C_1 = \left\{ c \leq 0; 1 \leq f'_c \leq b \text{ and } f''_c \leq 0 \text{ on } [0, T_c[\right\},$$

$$C_2 = \left\{ \begin{array}{l} c \leq 0; \exists t_c \in [0, T_c[, \exists \epsilon_c > 0 \text{ such that } f'_c > 1 \text{ on }]0, t_c[, \\ f'_c < 1 \text{ on }]t_c, t_c + \epsilon_c[\text{ and } f''_c \leq 0 \text{ on } [0, t_c + \epsilon_c[\end{array} \right\},$$

$$C_3 = \left\{ \begin{array}{l} c \leq 0; \exists r_c \in [0, T_c[, \exists \eta_c > 0 \text{ such that } f''_c < 0 \text{ on }]0, r_c[\\ f''_c > 0 \text{ on }]r_c, r_c + \eta_c[\text{ and } f'_c > 1 \text{ on }]0, r_c + \eta_c[\end{array} \right\}.$$

This is obvious that C_0, C_1, C_2 and C_3 are disjoint sets and that their union is the whole line of real numbers. Thanks to lemma 2.3 and 2.4 if $c \in C_1$ then $T_+ = +\infty$ and $f'_c(t) \rightarrow 1$ as $t \rightarrow +\infty$. In fact, C_1 is the set of values of c for which f_c is a concave solution of $(\mathcal{P}_{\beta; a, b, 1})$. Since $\beta > 0$, the study done in [4] (specially in section 5.2) says, on the one hand, that $C_3 = \emptyset$ (which can easily be deduced from lemma 2, item 1) and, on the other hand, that either $C_1 = \emptyset$ and $C_2 =]-\infty, 0]$, or there exist $c^* \leq 0$ such that $C_1 = [c^*, 0]$ and $C_2 =]-\infty, c^*[$.

In addition, from the lemma 5.16 in [4], if $\beta \in]0, \frac{1}{2}]$ then we are in the second case and $c^* \leq -a(b - 1)$.

In order to complete the study, let us divide the set C_2 into the following two subsets

$$C_{2,1} = \left\{ c \in C_2 : f'_c > -1 \text{ on } [0, T_c[\right\},$$

$$C_{2,2} = \left\{ c \in C_2 : \exists s_c \in]0, T_c[\text{ such that } f'_c > -1 \text{ on } [0, s_c[\text{ and } f'_c(s_c) = -1 \right\}.$$

And let us give properties of each of them that hold for all $\beta \in]0, \frac{1}{2}]$.

Lemma 3.1. *If $c \in \mathbb{R}$ such that $f'_c > 0$ on $[0, T_c[$, then $T_c = +\infty$ and f'_c is bounded. Moreover, if $c \leq 0$, then $f'_c \leq \max\{b, \sqrt{3}\}$ on $[0, +\infty[$.*

Proof. Let $c \in \mathbb{R}$ is such that $f'_c > 0$ on $[0, T_c[$, then $f_c \geq a \geq 0$ on $[0, T_c[$ and thanks to lemma 2.5, it follows that $T_c = +\infty$ and f'_c is bounded.

It remains to show that $f'_c \leq \max\{b, \sqrt{3}\}$ in the case where $c \leq 0$. As in [4], let us define the function L_c on $[0, +\infty[$ by

$$L_c(t) = 3f''_c{}^2(t) + \beta f'_c(t) (2f''_c{}^2(t) - 6),$$

and since $f_c \geq 0$, it implies that L_c is decreasing.

If $f_c'' \leq 0$ on $]0, +\infty[$, then $f_c' \leq b$. Otherwise, there exists t_0 such that $f_c'' < 0$ on $]0, t_0[$ and $f_c''(t_0) = 0$. By lemma 2.2 item 3, it follows that $f_c' < 1$, and thus $L_c(t_0) < 0$. Then, $L_c < 0$ on $]t_0, +\infty[$ which implies that $f_c' \leq \sqrt{3}$ on $]t_0, +\infty[$. Since $f_c' \leq b$ on $]0, t_0[$, the proof is complete. \square

Proposition 3.1. *Let $c_* = \inf (C_1 \cup C_{2,1})$. Then c_* is finite.*

Proof. let $c \in C_1 \cup C_{2,1}$. By definition of C_1 and $C_{2,1}$, and thanks to lemma 2.3, we have $T_c = +\infty$ and $0 < f_c' < d$ on $[0, +\infty[$ where $d = \max \{b, \sqrt{3}\}$. Since

$$\begin{aligned} (f_c'' + f_c f_c')' &= f_c''' + f_c f_c'' + f_c'^2 \\ &= -\beta (f_c'^2 - 1) + f_c'^2 \\ &= -\beta f_c'^2 + \beta + f_c'^2 \leq \beta + d^2. \end{aligned}$$

By integrating, we then have

$$\forall t \geq 0, f_c''(t) + f_c(t) f_c'(t) \leq c + ab + (\beta + d^2)t.$$

Integrating once again, for all $t \geq 0$, we get

$$0 < f_c'(t) \leq f_c'(t) + \frac{1}{2} f_c^2(t) \leq b + \frac{1}{2} a^2 + (c + ab) + \frac{1}{2} (\beta + d^2) t^2.$$

Which implies that

$$c \geq -ab - \sqrt{(2b + a^2)(\beta + d^2)}.$$

\square

Remark 3.1. *As we have seen above, if $C_1 \neq \emptyset$, then $C_1 = [c^*, 0]$ and thus $C_{2,1} \subset [c_*, c^*]$.*

4. THE CASE $\beta \in]0, \frac{1}{2}]$ AND $b \geq 1$

In this section we assume that $\beta \in]0, \frac{1}{2}]$, $a \geq 0$ and $b \geq 1$.

Proposition 4.1. *If $c > 0$, then $T_c = +\infty$ and $f_c'(t) \rightarrow 1$ as $t \rightarrow +\infty$.*

Proof. From lemma 2.6, there exists $t_0 \in]0, T_c[$ such that $f_c'' > 0$ on $[0, t_0[$ and $f_c''(t_0) = 0$. Since $f_c(t_0) > 0$ and $f_c'(t_0) > b > 1$. Thus $f_c(t_0) (1 - f_c'(t_0)) \leq f_c''(t_0) = 0$, the conclusion follows from lemma 2.8. \square

Remark 4.1. *Thanks to the previous proposition, we see that f_c is a convex-concave solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c > 0$.*

Proposition 4.2. *There exists $c^* \leq -a(b - 1)$ such that $C_1 = [c^*, 0]$.*

Proof. If $b = 1$ then $C_1 = \{0\}$.

If $b > 1$, as we already said in the previous section, this result is proven in [4] (see corollary 5.13 and lemma 5.16), let us recall briefly the main arguments which were used to get it. On the one hand, from lemma 2.8 with $t_0 = 0$ (or lemma 5.16 of [4]), it follows that $[-a(b - 1), 0] \subset C_1$. On the other hand, lemma 5.12 of [4] implies that C_2 is an interval of the type $]-\infty, c^*[$. This completes the proof since $C_1 =]-\infty, 0] \setminus C_2$. \square

Remark 4.2. From the previous proposition, we have that $0 \notin C_{2.2}$.

Proposition 4.3. If $c \in C_{2.1}$, then $T_c = +\infty$ and f'_c has a finite limit at infinity, equal either to -1 or to 1 .

Proof. Let $c \in C_{2.1}$. By proposition 4.2, we have $c < 0$.

Assume first that $f''_c < 0$ on $]0, +\infty[$. Then f'_c is decreasing, and thus f'_c has a finite limit λ at infinity. Moreover, by definition of the set $C_{2.1}$ we get

$$\exists t_c \in [0, +\infty[\text{ such that } f'_c(t_c) = 1$$

and by lemma 2.4, we finally get that $\lambda = -1$.

Assume now that f''_c vanishes on $]0, +\infty[$, let t_0 be the first point where f''_c vanishes. Thanks to lemma 2.2 item 3, we have $0 < f'_c(t_0) < 1$, and the conclusion follows from lemma 2.7 ($\lambda = 1$). \square

Remark 4.3. If $c \in C_{2.1}$ then either f_c is a concave solution of $(\mathcal{P}_\beta; a, b, -1)$ or f_c is a concave-convex solution of $(\mathcal{P}_\beta; a, b, 1)$.

Proposition 4.4. Let c be a point of the boundary of $C_{2.2}$. Then $c \in C_{2.1}$ and $f'_c(t) \rightarrow -1$ as $t \rightarrow +\infty$.

Proof. See [3] in the case of the mixed convection equation. \square

Proposition 4.5. There exists at most one c such that $f'_c(t) \rightarrow -1$ as $t \rightarrow +\infty$.

Proof. From proposition 4.2 and 4.3, we see that if c is such that $f'_c(t) \rightarrow -1$ as $t \rightarrow +\infty$, then $c < 0$ and $f''_c < 0$. By the change of variable, as done in [4], section 4, we can define a function $v :]0, b^2] \rightarrow \mathbb{R}$ such that

$$\forall t \geq 0, \quad v(f_c'^2(t)) = f_c(t). \tag{4.1}$$

By setting $y = f_c'^2(t)$, we get

$$f_c(t) = v(y), \quad f'_c(t) = \sqrt{y}, \quad f''_c(t) = \frac{1}{2v'(y)} \quad \text{and} \quad f'''_c(t) = -\frac{v''(y)\sqrt{y}}{2v'^3(y)},$$

and using (1.1) we obtain

$$\forall y \in]0, b^2], \quad v''(y) = \frac{v(y)v'(y)}{\sqrt{y}} + \frac{2\beta(y-1)}{\sqrt{y}}v'(y)^3. \tag{4.2}$$

From (3.1), we deduce that $v(b^2) = a$ and $v'(b^2) = \frac{1}{2c}$. Moreover, since f_c is bounded, it is so for v .

Assume that there exists $c_1 > c_2$ such that $f'_{c_1}(t) \rightarrow 0$ and $f'_{c_2}(t) \rightarrow 0$ as $t \rightarrow +\infty$, and denote by v_1 and v_2 the functions associated to f_{c_1} and f_{c_2} by (4.1). If we set $w = v_1 - v_2$ then $w(b^2) = 0$ and $w'(b^2) < 0$. We claim that $w' < 0$ on $]0, b^2]$. For contradiction, assume there exists $x \in]0, b^2[$ such that $w' < 0$ on $]0, x[$ and $w'(x) = 0$. Hence we have $w''(x) \leq 0$ and $w(x) > 0$. But thanks to (4.1), we have

$$w''(x) = \frac{w(x)}{\sqrt{x}}v_1^2(x),$$

and a contradiction.

Now, let us set $V_i = \frac{1}{v'_i}$ for $i = 1, 2$ and $W = V_1 - V_2$. Then $W(b^2) = 2(c_1 - c_2) > 0$ and $W(y) \rightarrow 0$ as $y \rightarrow 0$. In the other hand, thanks to (4.2); we have

$$\forall y \in]0, b^2], \quad W'(y) = -\frac{w(y)}{\sqrt{y}} - 2\beta \frac{y-1}{\sqrt{y}}w'(y).$$

Therefore, we have

$$\begin{aligned} W(b^2) &= \int_0^{b^2} W'(y) dy \\ &= -\int_0^{b^2} \left(\frac{w(y)}{\sqrt{y}} + 2\beta \frac{y-1}{\sqrt{y}}w'(y) \right) dy \\ &= -2[\sqrt{y}w(y)]_0^{b^2} + 2 \int_0^{b^2} \left((1-\beta)\sqrt{y} + \frac{2\beta}{\sqrt{y}} \right) w'(y) dy \\ &= 2 \int_0^{b^2} \left((1-\beta)\sqrt{y} + \frac{2\beta}{\sqrt{y}} \right) w'(y) dy, \end{aligned} \tag{4.3}$$

the last equality following from the fact that $w(y)$ tends to finite limit as $y \rightarrow 0$. Since $w' < 0$, we finally obtain $W(b^2) < 0$ and a contradiction. □

Remark 4.4. *The change of variable (4.1) is particularly efficient to obtain some uniqueness results. In [4], it is used for the general equation (1.2) (cf. Section 4, lemma 5.4 and lemma 5.17). The case we examined in the previous proposition is part of lemma 5.17 of [4] with $\lambda = -1$.*

Corollary 4.1. *One has $C_{2,2} =]-\infty, c_*[$ and $C_{2,1} = [c_*, c^*[$.*

Proof. From remark 4.2, proposition 4.4 and 4.5, we see that $C_{2,2}$ is open, contains $]-\infty, c_*[$. Therefore, since $c_* = \inf(C_1 \cup C_{2,1})$, we necessarily have $C_{2,2} =]-\infty, c_*[$ and $C_{2,1} = [c_*, c^*[$. □

To finish this section, let us express the results of proposition 4.1, proposition 4.2 and corollary 4.1 in terms of the boundary problems $(\mathcal{P}_{\beta; a, b, -1})$ and $(\mathcal{P}_{\beta; a, b, 1})$.

Theorem 4.1. *Let $\beta \in]0, \frac{1}{2}]$, $a \geq 0$ and $b \geq 1$. There exists $c_* < 0$ such that:*

- (1) f_c is not defined on the whole interval $[0, +\infty[$ if $c < c_*$;
- (2) f_{c_*} is a concave solution of $(\mathcal{P}_{\beta; a, b, -1})$;
- (3) f_c is a solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c \in]c_*, +\infty[$

Moreover, there exists $c^* \in]c_*, -a(b-1)]$;

- (1) f_c is a convex-concave solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c \in]0, +\infty[$;
- (2) f_c is a concave solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c \in [c^*, 0]$;
- (3) f_c is a concave-convex solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c \in]c_*, c^*[$;

5. THE CASE $\beta \in]0, \frac{1}{2}]$ AND $-1 < b < 1$

Let $\beta \in]0, \frac{1}{2}]$, $a \geq 0$ and $-1 < b < 1$. In this situation, it is easy to see that \mathbb{R} can be partitioned into the four sets $C'_{0,1}$, $C'_{0,2}$, C'_1 and C'_2 where

$$C'_{0,1} = \left\{ c < 0 : f'_c > -1 \text{ on } [0, T_c[\right\},$$

$$C'_{0,2} = \left\{ c < 0 : \exists s_c \in]0, T_c[\text{ such that } f'_c > -1 \text{ on } [0, s_c[\text{ and } f'_c(s_c) = -1 \right\},$$

$$C'_1 = \left\{ c \geq 0; b \leq f'_c \leq 1 \text{ and } f''_c \geq 0 \text{ on } [0, T_c[\right\},$$

$$C'_2 = \left\{ \begin{array}{l} c \geq 0; \exists t_c \in [0, T_c[, \exists \epsilon_c > 0 \text{ such that } f'_c < 1 \text{ on }]0, t_c[, \\ f'_c > 1 \text{ on }]t_c, t_c + \epsilon_c[\text{ and } f''_c > 0 \text{ on }]0, t_c + \epsilon_c[\end{array} \right\}.$$

The arguments used in the previous section, can be applied here.

First, since $g(x) = \beta(x^2 - 1) < 0$ for $x \in]-1, b]$ where $b \in]-1, 0[$, the function g is nonincreasing on $]-1, b]$, it follows from theorem 5.5 of [4] that there exists a unique c_* such that f_{c_*} is a concave solution of $(\mathcal{P}_{\beta; a, b, -1})$. Moreover, we have $c_* < 0$. As in the previous section, this implies that $C'_{0,2} =]-\infty, c_*[$. Hence $C'_{0,1} = [c_*, 0[$, and if $c \in]c_*, 0[$, then f''_c vanishes at a first point where $f'_c < 1$.

Next, in the same way as in the proof of proposition 3.1, we can proof that $c^* = \sup C'_1$ is finite, and hence that $C'_1 = [0, c^*]$ and $C'_2 =]c^*, +\infty[$. Moreover, from lemma 2.7, we have $c^* \geq a(1-b)$. On the other hand, it follows from lemma 2.6 that, if $c \in C'_2$, then f''_c vanishes at a first point where $f'_c > 1$.

All this, combined with an appropriate use of lemmas 2.7 and 2.8 allows to state the following theorem.

Theorem 5.1. Let $\beta \in]0, \frac{1}{2}]$ and $-1 < b < 1$. There exist $c_* < 0$ and $c^* \geq a(1 - b)$ such that:

- (1) f_c is not defined on the whole interval $[0, +\infty[$ if $c < c_*$;
- (2) f_{c_*} is a concave solution of $(\mathcal{P}_{\beta; a, b, -1})$ if $b \in]-1, 0[$;
- (3) f_c is a concave-convex solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c \in]c_*, 0[$;
- (4) f_c is a convex solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c \in [0, c^*]$;
- (5) f_c is a convex-concave solution of $(\mathcal{P}_{\beta; a, b, 1})$ for all $c \in]c^*, +\infty[$.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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