Optimality Conditions for Set-Valued Optimization Problems

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ABSTRACT. In this paper, we first prove that the generalized subconvexlikeness introduced by Yang, Yang and Chen [1] and the presubconvexlikeness introduced by Zeng [2] are equivalent. We discuss set-valued nonconvex optimization problems and obtain some optimality conditions.

1. Introduction

Set-valued optimization is a vibrant and expanding branch of mathematics that deals with optimization problems where the objectives and/or the constraints are set-valued maps. Corley [3] pointed out that the dual problem of a multiobjective optimization involves the optimization of a set-valued map, while Klein and Thompson [4] gave some examples in Economics where it is necessary to use set-valued maps instead of single-valued maps. There are many recent developments about set-values optimization problems, e.g., [5-9].


In this paper, we first prove that the generalized subconvexlikeness introduced by Yang, Yang, and Chen [1] and the presubconvexlikeness introduced by Zeng [2] are equivalent, in locally convex...
topological spaces. And then, we deal with set-valued optimization problems and obtain some optimality conditions.

A subset $Y_+$ of a real linear topological space $Y$ is a cone if $\lambda y \in Y_+$ for all $y \in Y_+$ and $\lambda \geq 0$. We denote by $0_Y$ the zero element in the linear topological space $Y$ and simply by $0$ if there is no confusion.

A convex cone is one for which $\lambda_1 y_1 + \lambda_2 y_2 \in Y_+$ for all $y_1, y_2 \in Y_+$ and $\lambda_1, \lambda_2 \geq 0$. A pointed cone is one for which $Y_+ \cap (-Y_+) = \{0\}$. Let $Y$ be a real linear topological space with pointed convex cone $Y_+$. We denote the partial order induced by $Y_+$ as follows:

$y_1 \succ y_2$ iff $y_1 - y_2 \in Y_+$,

$y_1 \succ\succ y_2$ iff $y_1 - y_2 \in \text{int} Y_+$,

where $\text{int} Y_+$ denotes the topological interior of a set $Y_+$. Let $X, Z, W$ be real linear topological spaces and $Y$ be an ordered linear topological space with the partial order induced by a pointed convex cone $Y_+$.

We recall some notions of generalized convexity of set-valued maps. First we recall the notion of cone-convexity of a set-valued map introduced by Borwein [10].

**Definition 1.1 (Convexity)** Let $X, Y$ be real linear topological spaces, $D \subseteq X$ a nonempty convex set and $Y_+$ a convex cone in $Y$. A set-valued map $f : X \to Y$ is said to be $Y_+$-convex on $D$ if and only if $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, there holds

$$\alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(\alpha x_1 + (1-\alpha)x_2) + Y_+.$$

The following notion of generalized convexity is a set-valued map version of Ky Fan convexity [11] (Ky Fan’s definition was for vector-valued optimization problems).

**Definition 1.2 (Convexlike)** Let $X, Y$ be real linear topological spaces, $D \subseteq X$ a nonempty set and $Y_+$ be a convex cone in $Y$. A set-valued map $f : X \to Y$ is said to be $Y_+$-convexlike on $D$ if and only if $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$ such that

$$\alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(x_3) + Y_+.$$

The following concept of generalized subconvexlikeness was introduced by Yang, Yang and Chen [1] ([1] introduced subconvexlikeness for vector-valued optimization).
**Definition 1.3** (Generalized subconvexlike) Let $Y$ be a linear topological space and $D \subseteq X$ be a nonempty set and $Y_+$ be a convex cone in $Y$. A set-valued map $f: D \to Y$ is said to be generalized $Y_+$-subconvexlike on $D$ if $\exists u \in \text{int} Y_+$ such that $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, $\forall \alpha \in [0, 1]$, $\exists x_3 \in D$, $\exists \tau > 0$ there holds

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq \tau f(x_3) + Y_+.$$ 

The following Lemma 1.1 is from Chen and Rong [12, Proposition 3.1].

**Lemma 1.1** A function $f : D \to Y$ is generalized $Y_+$-subconvexlike on $D$ if $\forall u \in \text{int} Y_+$, $\forall x_1, x_2 \in D$, $\forall \alpha \in [0, 1]$, $\exists x_3 \in D$, $\exists \tau > 0$ such that

$$u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq \tau f(x_3) + Y_+.$$ 

A bounded function in a real linear topological space can be defined as following Definition 1.4 (e.g., see Yosida [13]).

**Definition 1.4** (Bounded set-valued map) A subset $M$ of a real linear topological space $Y$ is said to be a bounded subset if for any given neighbourhood $U$ of 0, $\exists$ positive scalar $\beta$ such that $\beta^{-1}M \subseteq U$, where $\beta^{-1}M = \{y \in Y; y = \beta^{-1}v; v \in M\}$. A set-valued map $f : D \to Y$ is said to be bounded map if $f(Y)$ is a bounded subset of $Y$.

The following Definition 1.5 was introduced by Zeng [2] for single-valued functions.

**Definition 1.5** (Presubconvexlike) Let $Y$ be a linear topological space and $D \subseteq X$ be a nonempty set and $Y_+$ be a convex cone in $Y$. A set-valued map $f : D \to Y$ is said to be $Y_+$-presubconvexlike on $D$ if $\forall x_1, x_2 \in D$, $\forall \alpha \in [0, 1]$, $\forall \varepsilon > 0$, $\exists x_3 \in D$, $\exists \tau > 0$, $\exists$ bounded set-valued map $u : D \to Y$ such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq \tau f(x_3) + Y_+.$$ 

It is obvious that $Y_+$-convex $\Rightarrow$ $Y_+$-convexlike $\Rightarrow$ generalized $Y_+$-subconvexlike $\Rightarrow$ $Y_+$-presubconvexlike.

It is important to note that the concept of convexlike or any weaker concepts are only nontrivial if $Y$ is not the one-dimensional Euclidean space since any real-valued function is $\mathbb{R}^+$-convexlike.
2. The Equivalence of Generalized Subconvexlikeness and Presubconvexlikeness

In this section, we are going to prove that Definition 1.4 (Generalized subconvexlikeness) and Definition 1.5 (Presubconvexlikeness) are equivalent.

Definition 2.1 (1) A subset $M$ of $Y$ is said to be convex, if $y_1, y_2 \in M$ and $0 < \alpha < 1$ implies $\alpha y_1 + (1 - \alpha) y_2 \in M$;

(2) $M$ is said to be balanced if $y \in M$ and $|\alpha| \leq 1$ implies $\alpha y \in M$;

(3) $M$ is said to be absorbing if for any given neighbourhood $U$ of 0, there exists a positive scalar $\beta$ such that $\beta^{-1} M \subseteq U$, where $\beta^{-1} M = \{ y \in Y; y = \beta^{-1} v; v \in M \}$.

Definition 2.2 A real linear topological space $Y$ is called a locally convex, linear topological space (we call it a locally convex topological space, in the sequel) if any neighborhood of 0 contains a convex, balanced, and absorbing open set.

From [13, pp.26 Theorem, pp.33 Definition 1] one has Lemma 2.1.

Lemma 2.1 Banach spaces are locally convex topological spaces, so are finite dimensional Euclidean spaces.

Proposition 2.1 Let $Y$ be a locally convex topological space and $D \subseteq X$ be a nonempty set and $Y_+$ be a convex cone in $Y$. A set-valued map $f: D \rightarrow Y$ is generalized $Y_+$-subconvexlike on $D$ if and only if $\bigcup_{t > 0} (tf(D) + \text{int} Y_+)$ is convex.

Proof. The necessity. See [1, Theorem 2.1].

The sufficiency. Assume that $\bigcup_{t > 0} (tf(D) + \text{int} Y_+)$ is convex, aim to show that $f: D \rightarrow Y$ is generalized $Y_+$-subconvexlike on $D$. From Lemma 1.1, we are going to show that, $\forall u \in \text{int} Y_+$, $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$, $\exists \tau > 0$ such that

$$u + \alpha f(x_1) + (1 - \alpha) f(x_2) \subseteq \tau f(x_3) + Y_+.$$ 

$\forall y \in \text{int} Y_+$, $\forall t > 0$, since $\text{int} Y_+$ is a cone, one has

$$ty \in \text{int} Y_+.$$ 

$\forall y_1 \in f(x_1), y_2 \in f(x_2)$, $\forall \alpha \in R$, one has

$$f(x_1) + ty, f(x_2) + ty \subseteq \bigcup_{t > 0} (tf(D) + \text{int} Y_+).$$

From the convexity of $\bigcup_{t > 0} (tf(D) + \text{int} Y_+)$, $\exists x_3 \in D, \exists y_3 \in \text{int} Y_+$, $\exists \tau > 0$ such that
\[
\alpha(f(x_1)+ty)+(1-\alpha)(f(x_2)+ty) \\
\subseteq \alpha f(x_1)+(1-\alpha)f(x_2)+ty \\
\subseteq \tau f(x_3)+y_3 \\
\subseteq \bigcup_{\tau>0} (tf(D)+\text{int}Y_+).
\]

For the given \(u \in \text{int}Y_+\), From Definition 2.2, \(\exists\) neighbourhood \(U\) of 0 such that \(U\) is convex, balanced, and absorbing, and \(u+U \subseteq \text{int}Y_+\), where \(u+U\) is a neighbourhood of \(u\). Therefore, we may take \(t>0\) small enough, such that \(-ty \in U\). Then,

\[-ty+u \in u+U \subseteq \text{int}Y_+.
\]

This and the convexity of \(\text{int}Y_+\) imply that

\[y_3-ty+u \in \text{int}Y_+.
\]

And so

\[u+\alpha f(x_1)+(1-\alpha)f(x_2) \subseteq \tau f(x_3)+y_3-ty \subseteq \tau f(x_3)+\text{int}Y_+ \subseteq \tau f(x_3)+Y_+.
\]

**Proposition 2.2** Let \(Y\) be a locally convex topological space and \(D \subseteq X\) be a nonempty set and \(Y_+\) be a convex cone in \(Y\). A set-valued map \(f : D \to Y\) is \(Y_+\)-presubconvexlike on \(D\) if and only if \(\bigcup_{\tau>0} (tf(D)+\text{int}Y_+)\) is convex.

**Proof:** The necessity.

Suppose that \(f\) is \(Y_+\)-presubconvexlike on \(D\), aim to show that \(\bigcup_{\tau>0} (tf(D)+\text{int}Y_+)\) is convex.

\[\forall v_1 = t_1y_1 + y_1^1, v_2 = t_2y_2 + y_2^2 \in \bigcup_{\tau>0} (tf(D)+\text{int}Y_+), \ \exists x_1, x_2 \in D\text{ such that } y_1 \in f(x_1), y_2 \in f(x_2).\] Let

\[y_+^0 = \alpha y_1 + (1-\alpha)y_2^2,
\]

then \(y_+^0 \in \text{int}Y_+\). Therefore, \(\exists\) neighbourhood \(U\) of 0 such that \(y_+^0 + U\) is a neighbourhood of \(y_+^0\) and

\[y_+^0 + U \subseteq \text{int}Y_+.
\]

By Definition 2.2, without loss of generality, we may assume that \(U\) is convex, balanced, and absorbing.

From the assumption of \(Y_+\)-presubconvexlikeness, \(\forall \epsilon>0, \exists x_3 \in D, \exists \text{ bounded function } u, \text{ and } \exists \tau>0\) such that

\[
\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} f(x_1) + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} f(x_2) \subseteq \tau f(x_3) - \epsilon u + Y_+.
\]
Therefore, \( \forall y \in f(x) \) such that

\[
\alpha v_1 + (1-\alpha)v_2 = \alpha t_1 y_1 + (1-\alpha)t_2 y_2 + \alpha y^*_1 + (1-\alpha)y^*_2
\]

\[
= (\alpha t_1 + (1-\alpha)t_2)\left[\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} y_1 + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} y_2\right] + y^*_2
\]

\[
\subseteq (\alpha t_1 + (1-\alpha)t_2)\left[\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} f(x_1) + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} f(x_2)\right] + y^*_2
\]

\[
\subseteq (\alpha t_1 + (1-\alpha)t_2)[\tau f(x_1) - \varepsilon u + Y_+] + y^*_2
\]

\[
= (\alpha t_1 + (1-\alpha)t_2)[\tau f(x_1) + (\alpha t_1 + (1-\alpha)t_2)(Y_+ - \varepsilon u) + y^*_2.
\]

Since \( U \) is convex, balanced, and absorbing, by Definition 2.2, we may take \( \varepsilon > 0 \) small enough such that

\[
-(\alpha t_1 + (1-\alpha)t_2)\varepsilon u \subseteq U.
\]

Therefore

\[
-(\alpha t_1 + (1-\alpha)t_2)\varepsilon u + y^*_2 \subseteq y^*_2 + U \subseteq \text{int} Y_+.
\]

And then

\[
(\alpha t_1 + (1-\alpha)t_2)Y_+ - (\alpha t_1 + (1-\alpha)t_2)\varepsilon u + y^*_2 \subseteq Y_+ + \text{int} Y_+ \subseteq \text{int} Y_+.
\]

Therefore

\[
\alpha v_1 + (1-\alpha)v_2 \subseteq (\alpha t_1 + (1-\alpha)t_2)\tau f(x_1) + (\alpha t_1 + (1-\alpha)t_2)(Y_+ - \varepsilon u) + y^*_2
\]

\[
\subseteq \bigcup_{\varepsilon>0}(tf(D) + \text{int} Y_+).
\]

Hence \( \bigcup_{\varepsilon>0}(tf(D) + \text{int} Y_+) \) is a convex set.

The sufficiency.

Assume that \( \bigcup_{\varepsilon>0}(tf(D) + \text{int} Y_+) \) is convex. From Lemma 1.1 and Proposition 2.1, \( \exists u \in \text{int} Y_+ \) such that for all \( \forall x_1, x_2 \in D, \ \forall \alpha \in [0,1], \ \forall \varepsilon > 0, \ \exists \tau > 0 \) there holds

\[
\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq \tau f(x_1) + Y_+.
\]

The given \( u \in \text{int} Y_+ \) can be consider as a bounded function.

By Propositions 1 and 2 one has Theorem 2.1.
Theorem 2.1 Let $Y$ be a locally convex topological space and $D \subseteq X$ be a nonempty set, and $Y_+$ a convex cone in $Y$. A set-valued map $f : D \to Y$ is generalized $Y_+$-subconvexlike on $D$ if and only if $f$ is $Y_+$-presubconvexlike on $D$.

3. Optimal Conditions

We consider the following optimization problem with set-valued maps:

$$(VP) \quad Y_+ \min f(x)$$

s.t. $g_i(x) \cap (-Z_{i+}) \neq \emptyset, i = 1, 2, \cdots, m$

$0 \in h_j(x), j = 1, 2, \cdots, n$

$x \in D$

where $f : X \to Y$, $g_i : X \to Z_i$, $h_j : X \to W_j$ are set-valued maps, $Z_{i+}$ is a closed convex cone in $Z_i$, and $D$ is a nonempty subset of $X$.

For a set-valued map $f : X \to Y$, we denote by $f(D) = \bigcup_{x \in D} f(x)$.

We now explain the kind of optimality we consider here. Let $F$ be the feasible set of $(VP)$, i.e.

$F := \{ x \in D : g_i(x) \cap (-Z_{i+}) \neq \emptyset, i = 1, 2, \cdots, m; 0 \in h_j(x), j = 1, 2, \cdots, n \}$.

We are looking for a weakly efficient solution of $(VP)$ defined as follows.

**Definition 3.1 (Weakly Efficient Solution)** A point $\bar{x} \in F$ is said to be a weakly efficient solution of $(VP)$ with a weakly efficient value $\bar{y} \in f(\bar{x})$ if for every $x \in F$, there exists no $y \in f(x)$ satisfying $\bar{y} \gg y$.

Consider the set-valued optimization problem $(VP)$. From now on we assume that $Y_+, Z_{i+}$ are pointed convex cones with nonempty interior of $\text{int} Y_+$, $\text{int} Z_{i+}$, respectively. The following three assumptions will be used in this paper.

(A1) Generalized Convexity Assumption. There exist $u_0 \in \text{int} Y_+$, $u_i \in \text{int} Z_{i+}$ such that for all $x_i, x_2 \in D$, $\varepsilon > 0$, $\alpha \in [0, 1]$, there exist $x_3 \in D$, $\tau_i > 0 (i = 1, 2, \cdots, m), t_j > 0 (j = 1, 2, \cdots, n)$ such that

$$\varepsilon u_0 + \alpha f(x_1) + (1 - \alpha) f(x_2) \subseteq \tau_0 f(x_3) + Y_+$$

$$\varepsilon u_i + \alpha g_i(x_1) + (1 - \alpha) g_i(x_2) \subseteq \tau_i g_i(x_3) + Z_{i+}$$

$$\alpha h_j(x_1) + (1 - \alpha) h_j(x_2) \subseteq t_j h_j(x_3)$$

(A2) Interior Point Assumption.
\[
\text{int } h_j(D) \neq \emptyset, \quad (j = 1, 2, \ldots, n).
\]

(A3) Finite Dimension Assumption. \( W_j (j = 1, 2, \ldots, n) \) are finite dimensional spaces.

Similar to the proof of Propositions 2.1 or 2.2, one has Proposition 3.1.

**Proposition 3.1** Assumption (A1) is satisfied if and only if the following set is convex:

\[
B := \{(y, z, w) \in Y \times \prod_{i=1}^{m} Z_i \times \prod_{j=1}^{n} W_j : \exists x \in D, \tau_i, t_j > 0, s.t., y \in \tau_i f(x) + \text{int } Y_+, z_i \in \tau_i g_i(x) + \text{int } Z_{\tau_i}, w_j \in t_j h_j(x) \}.
\]

**Proposition 3.2** (Alternative Theorem) Assume that the assumption (A1) and either (A2) or (A3) are satisfied. Consider the following generalized inequality-equality systems:

[**System 1**]

\[
\exists x \in D, \text{s.t. } f(x) \cap (-\text{int } Y_+) \neq \emptyset, g_i(x) \cap (-Z_{\tau_i}) \neq \emptyset, 0 \in h_j(x).
\]

[**System 2**]

\[
\exists (\xi, \eta, \zeta) \in (Y^+ \times \prod_{i=1}^{m} Z_i^+ \times \prod_{j=1}^{n} W_j^+) \setminus \{0\}, \text{s.t. } \forall x \in D \\
\xi(f(x)) + \sum_{i=1}^{m} \eta_i(g_i(x)) + \sum_{j=1}^{n} \zeta_j(h_j(x)) \geq 0.
\]

Then if System 1 has no solution \( x \), then System 2 has a solution \( (\xi, \eta, \zeta) \). If System 2 has a solution \( (\xi, \eta, \zeta) \) with \( \xi \neq 0 \), then System 1 has no solution.

**Proof.** Suppose that System 1 has no solution, then \( 0 \notin B \). Since (A1) holds, the set \( B \) is convex. By assumption, \( \prod_{i=1}^{m} Z_i \) is infinite dimensional and (A2) holds (which is equivalent to saying that \( \text{int } B \neq \emptyset \)) or \( \prod_{j=1}^{n} W_j \) is finite dimensional. Therefore by the separation theorem, \( \exists \) nonzero vector \( (\xi, \eta, \zeta) \in Y^+ \times \prod_{i=1}^{m} Z_i^+ \times \prod_{j=1}^{n} W_j^+ \) such that

\[
\xi(t_0 y + y_0) + \sum_{i=1}^{m} \eta_i(t_i z_i + z_i^0) + \sum_{j=1}^{n} \zeta_j(t_j w_j) \geq 0
\]

for all

\[
x \in D, y \in f(x), z_i \in g_i(x), w_j \in h_j(x), y_0 \in \text{int } Y_+, z_i^0 \in \text{int } Z_{\tau_i}, t_i > 0, t_j > 0.
\]

Since \( \text{int } Y_+, \text{int } Z_{\tau_i} \) are convex cones, we have

\[
\xi(t_0 y + s_i y_0) + \sum_{i=1}^{m} \eta_i(t_i z_i + s_i z_i^0) + \sum_{j=1}^{n} \zeta_j(t_j w_j) \geq 0
\]

For all

\[
x \in D, y \in f(x), z_i \in g_i(x), w_j \in h_j(x), y_0 \in \text{int } Y_+, z_i^0 \in \text{int } Z_{\tau_i}, t_i > 0, t_j > 0, s_i > 0 (i = 0, 1, 2, \ldots, m).
\]
Taking $\tau_i \to 0, t_j \to 0, s_i \to 0 (i = 0, 1, 2, \cdots, m)$, we obtain
\[ \xi (y_0) > 0, \forall y_0 \in \text{int } Y_+, \]
and consequently
\[ \xi (y_0) \geq 0, \forall y_0 \in Y_+ \subseteq cY_+ = c\text{ int } Y_+, \]
where $cY_+$ is the topological closure of the set $Y_+$. Similarly, we have
\[ \eta_i (z_i) \geq 0, \forall z_i \in Z_{i+}, \]
and hence $\xi \in Y^*, \eta_i \in Z_{i+}$.

Let $\tau_i = 1 (i = 1, 2, \cdots, m), t_j = 1 (j = 1, 2, \cdots, n)$ and take $s_i \to 0 (i = 0, 1, 2, \cdots, m)$, we have
\[ \xi (y) + \sum_{i=1}^{m} \eta_i (z_i) + \sum_{j=1}^{n} \zeta_j (w_j) \geq 0 \]
For $x \in D, y \in f(x), z_i \in g_i (x), w_j \in h_j (x)$. Hence, System 2 has a solution $(\xi, \eta, \zeta)$.

Conversely, suppose that System 2 has a solution $(\xi, \eta, \zeta)$ with $\xi \neq 0$. If System 1 has a solution
$\forall x \in D$, there would exist $y \in f(x), z_i \in g_i (x), w_j \in h_j (x)$ such that
\[ y \in -\text{int } Y_+, z_i \in -Z_{i+}, w_j = 0. \]
Thus,
\[ \xi (y) < 0, \eta_i (z_i) \leq 0, \zeta_j (w_j) = 0, \]
i.e.,
\[ \xi (y) + \sum_{i=1}^{m} \eta_i (z_i) + \sum_{j=1}^{n} \zeta_j (w_j) < 0, \]
which is a contradiction and hence System 1 does not have a solution.

**Theorem 3.1** [Fritz John Type Necessary Optimality Condition] Assume that the generalized convexity assumption (A1) is satisfied and either (A2) or (A3) holds. If $\overline{x} \in F$ is a weakly efficient solution of (VP) with $\overline{y} \in f(\overline{x})$, $\exists$ nonzero vector $(\xi, \eta, \zeta) \in Y^* \times \prod_{i=1}^{m} Z_{i+}^* \times \prod_{j=1}^{n} W_{j+}^*$ such that
\[ \xi (\overline{y}) = \min_{x \in D} [\xi (f(x)) + \sum_{i=1}^{m} \eta_i (g_i (x)) + \sum_{j=1}^{n} \zeta_j (h_j (x))] \]
\[ \min \sum_{i=1}^{m} \eta_i (g_i (\overline{x})) = 0, \]
where $\min \sum_{i=1}^{m} \eta_i (g_i (\overline{x})) \leq \min_{z : g_i (\overline{x})} \sum_{i=1}^{m} \eta_i (z_i)$.

**Proof.** Since $\overline{x} \in F$ is a weakly efficient solution of (VP) with $\overline{y} \in f(\overline{x})$, by definition the following system
\[ x \in D, (f(x) - y) \cap (-\text{int} Y_+) \neq \emptyset, g_i(x) \cap (-Z_i^+) \neq \emptyset, 0 \in h_j(x) \]

has no solution. By Proposition 2.2, there exists a nonzero vector \((\xi, \eta, \zeta) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*\) such that \(\forall x \in D\) there holds

\[
\xi(f(x) - y) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \zeta_j(h_j(x)) \geq 0.
\]

Since \(\bar{x} \in F\), there exists \(z_i \in g_i(\bar{x})\) such that \(\bar{z}_i \in -Z_i^+\). For such \(\bar{z}_i\), it follows \(\eta_i \in Z_i^+\) that \(\eta_i(\bar{z}_i) \leq 0\). On the other hand, taking \(x = \bar{x}\) we get

\[
\xi(f(x) - y) + \sum_{i=1}^m \eta_i(\bar{z}_i) + \sum_{j=1}^n \zeta_j(h_j(\bar{x})) \geq 0,
\]

and noticing that \(\bar{y} \in f(\bar{x})\) and \(0 \in h_j(\bar{x})\) we obtain

\[
\sum_{i=1}^m \eta_i(\bar{z}_i) \geq 0,
\]

and hence \(\eta_i(\bar{z}_i) = 0\).

Since

\[
\xi(\bar{y}) + \sum_{i=1}^m \eta_i(\bar{z}_i) + \sum_{j=1}^n \zeta_j(0) = \xi(y),
\]

taking \(x = \bar{x}\) again we get

\[
\xi(f(x) - y) + \sum_{i=1}^m \eta_i(g_i(\bar{x})) + \sum_{j=1}^n \zeta_j(h_j(\bar{x})) \geq 0.
\]

Noticing that \(\bar{y} \in f(\bar{x})\) and \(0 \in h_j(\bar{x})\), we obtain

\[
\sum_{i=1}^m \eta_i(g_i(\bar{x})) \geq 0.
\]

We have shown previously that there exists \(z_i \in g_i(\bar{x})\) such that \(\eta_i(\bar{z}_i) = 0\). Therefore

\[
\min \sum_{i=1}^m \eta_i(g_i(\bar{x})) = 0.
\]

**Theorem 3.2** (Sufficient Optimality Condition) Let \(\bar{x} \in F\) and \(\bar{y} \in f(\bar{x})\). If there exists a \((\xi, \eta, \zeta) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*\) with \(\xi \neq 0\) such that

\[
\xi(\bar{y}) \leq \min_{x \in D} [\xi(f(x)) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \zeta_j(h_j(x))],
\]

then \(\bar{x}\) is a weakly efficient solution of \((VP)\) with \(\bar{y} \in f(\bar{x})\).

**Proof.** By contradiction, we assume that \(\bar{x} \in F\) is not a weakly efficient solution of \((VP)\) with \(\bar{y} \in f(\bar{x})\). Then by definition, \(\exists x^0 \in F\) and \(\exists y^0 \in f(x^0)\) such that \(y - y^0 \in \text{int} Y_+\), which implies
that $\xi(y - y^0) < 0$. Since $x^0 \in F$, $0 \in h_j(x^0)$ and $\exists z_i^0 \in g_i(x^0)$ such that $z_i^0 \in -Z_+$, and hence $\eta_i(z_i^0) \leq 0$. Consequently,

$$\xi(y^0 - \bar{y}) + \sum_{i=1}^{m} \eta_i(z_i^0) + \sum_{j=1}^{n} \zeta_j(0) < 0.$$ 

Hence $\bar{x}$ is a weakly efficient solution of $(\text{VP})$ with $\bar{y} \in f(\bar{x})$.

From Theorem 3.2 and 3.3 one has Theorem 3.3.

**Theorem 3.3 (Strong Duality)** Suppose all assumptions in Theorem 3.1 hold and there is no nonzero vector $(\eta, \zeta) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfying the system:

$$\min_{x \in D} \left[ \sum_{i=1}^{m} \eta_i g_i(x) + \sum_{j=1}^{n} \zeta_j h_j(x) \right] = 0$$

$$\eta_i g_i(\bar{x}) = 0.$$ 

Let $\bar{x}$ be a solution of problem $(\bar{P})$. Then the strong duality holds. That is,

$$f(\bar{x}) = \min_{g(x) \leq 0, h(x) = 0, x \in D} f(x) = \max_{\eta \geq 0} \min_{x \in D} \left[ f(x) + \sum_{i=1}^{m} \eta_i g_i(x) + \sum_{j=1}^{n} \zeta_j h_j(x) \right].$$

4. Applications to Single-Valued Optimization Problems

Consider the optimization problem:

$$(\bar{P}) \quad \min f(x)$$

s.t. $g_i(x) \leq 0$ ($i = 1, 2, \ldots, m$)

$h_j(x) = 0, (j = 1, 2, \ldots, n)$

$x \in D$

where $f$, $g_i$, $h_j$: $X \to \mathbb{R}$ are functions and $D$ is a nonempty subset of $X$.

Applying Theorem 3.1 to the above single-valued optimization problem we have the following Fritz John type necessary optimality condition.

**Theorem 4.1** Let $\bar{x}$ be an optimal solution of $(\bar{P})$. Suppose the following generalized convexity assumption holds: $\exists u_i > 0, (i = 0, 1, 2, \ldots, m)$ such that $\forall x_1, x_2 \in D$, $\forall \epsilon > 0$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$, $\exists \tau_i > 0, (i = 1, 2, \ldots, m)$, $\exists t_j > 0, (j = 1, 2, \ldots, n)$ there holds

$$\epsilon u_0 + \alpha f(x_1) + (1-\alpha) f(x_2) \leq \tau_0 f(x_3) + R,$$

$$\epsilon u_i + \alpha g_i(x_1) + (1-\alpha) g_i(x_2) \leq \tau_i g_i(x_3) + R,$$

$$\alpha h_j(x_1) + (1-\alpha) h_j(x_2) = t_j h_j(x_3)$$

$(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n).$
Then, \( \exists \) nonzero vector \((\lambda, \eta, \zeta)\in \mathbb{R}_+ \times \mathbb{R}_n^m \times \mathbb{R}^n\) such that
\[
\lambda f(\bar{x}) = \min_{x \in D}[\lambda f(x) + \sum_{i=1}^{m} \eta_i g_i(x) + \sum_{j=1}^{n} \zeta_j h_j(x)]
\]
\[
\min \sum_{i=1}^{m} \eta_i g_i(x) = 0.
\]

We now study some cases where the generalized convexity holds and consequently the Fritz John condition in the above theorem holds.

**Theorem 4.2** Let \( \bar{x} \) be an optimal solution of \((P)\). Suppose one of the following set of assumptions hold.

(I) All functions \( g_i \) are nonnegative on the set \( D \) and \( n = 0 \) (i.e. there is no equality constraints).

(II) All functions \( f, g_i \) are nonnegative on the set \( D \) and \( n = 1 \). Then, \( \exists \) non-zero vector

\[
(\lambda, \eta, \zeta)\in \mathbb{R}_+ \times \mathbb{R}_n^m \times \mathbb{R}^n
\]
such that
\[
\lambda f(\bar{x}) = \min_{x \in D}[\lambda f(x) + \sum_{i=1}^{m} \eta_i g_i(x) + \sum_{j=1}^{n} \zeta_j h_j(x)]
\]
\[
\min \sum_{i=1}^{m} \eta_i g_i(x) = 0.
\]

**Proof.** From Theorem 4.1, it suffices to prove that the generalized convexity assumption holds.

First assume that assumption (I) holds. Let \( x_1, x_2 \in D, \alpha \in [0,1] \).

**Case 1:** \( f(x_1) > f(x_2) \). Then
\[
\alpha f(x_1) + (1-\alpha)f(x_2) - f(x_1) = (1-\alpha)(f(x_2) - f(x_1)) \geq 0.
\]
Let \( x_3 = x_1 \). Then
\[
\alpha f(x_3) + (1-\alpha)f(x_2) \subseteq f(x_3) + R_v.
\]
Since \( g \) is nonnegative on set \( D \), for small enough \( \tau \in (0,\alpha] \) one has
\[
\alpha g(x_1) + (1-\alpha)g(x_2) - \tau g(x_1) = (\alpha - \tau)g(x_1) + (1-\alpha)g(x_2) \geq 0.
\]
That is,
\[
\alpha g(x_1) + (1-\alpha)g(x_2) \in \tau g(x_1) + R_v^m.
\]

**Case 2.** \( f(x_1) \leq f(x_2) \). In this case by choosing \( x_3 = x_2 \) similarly as in case 1 we can prove (1) and (2). Hence the generalized convexity assumption holds.

Now assume that assumption (II) holds. Let \( x_1, x_2 \in D \) and \( \alpha \in [0,1] \). If \( h(x_2) = 0 \) then
\[
\alpha h(x_1) + (1-\alpha)h(x_2) = \alpha h(x_1).
\]
Let $x_3 = x_1$. Then since $f, g_i$ are nonnegative, similarly as in (1) one can find a small enough $\tau_0 > 0$ and $\tau_1 > 0$ such that

$$\alpha f(x_i) + (1 - \alpha)f(x_3) \in \tau_0 f(x_3) + R, \quad \alpha g(x_i) + (1 - \alpha)g(x_3) \in \tau_1 g(x_3) + R^m. \quad (3)$$

Otherwise if $h(x_2) \neq 0$, then one can find $\tau_2 > 0$ such that

$$\alpha h(x_i) + (1 - \alpha)h(x_2) = \tau_2 h(x_2).$$

Let $x_3 = x_2$. Then since $f, g_i$ are nonnegative, similarly as in (1) one can find a small enough $\tau_0 > 0$ and $\tau_1 > 0$ such that (3) and (4) hold. Hence the generalized convexity assumption holds.

**Theorem 4.3 (Kuhn-Tucker Type Necessary Optimality Condition)** Let $\bar{x}$ be an optimal solution of $(P)$. Suppose all assumptions in Theorem 4.2 hold and there is no nonzero vector $(\eta, \zeta) \in R^m \times R^n$ satisfying the system:

$$\min_{x \in B(U(\bar{x}}) [\sum_{i=1}^{m} \eta_i g_i(x) + \sum_{j=1}^{n} \zeta_j h_j(x)] = 0$$

$$\eta_i g_i(\bar{x}) = 0.$$ 

where $U(\bar{x})$ is a neighbourhood of $\bar{x}$, then $\exists (\eta, \zeta) \in R^m \times R^n$ such that

$$f(\bar{x}) = \min_{x \in B(U(\bar{x}}) [f(x) + \sum_{i=1}^{m} \eta_i g_i(x) + \sum_{j=1}^{n} \zeta_j h_j(x)]$$

$$\eta_i g_i(\bar{x}) = 0.$$ 

5. Conclusion Remark

Yang, Yang and Chen [1] defined the following generalized subconvexlike functions. ([1] introduced subconvexlikeness for vector-valued optimization).

Let $Y$ be a topological vector space and $D \subseteq X$ be a nonempty set and $Y_+$ be a convex cone in $Y$. A set-valued map $f: D \rightarrow Y$ is said to be generalized $Y_+$-subconvexlike on $D$ if $\exists u \in \text{int} Y_+$, such that $\forall x_1, x_2 \in D$, $\forall \epsilon > 0$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$, $\exists \tau > 0$ there holds

$$\epsilon u + \alpha f(x_1) + (1 - \alpha)f(x_2) \subseteq \tau f(x_3) + Y_+. \quad (5)$$

And Zeng [2] introduced the presubconvexlikeness as follows.
Let $Y$ be a topological vector space and $D \subseteq X$ be a nonempty set and $Y_+$ be a convex cone in $Y$. A set-valued map $f: D \rightarrow Y$ is said to be $Y_+$-presubconvexlike on $D$ if there exists a bounded set-valued map $u: D \rightarrow Y$ such that

$$
\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+.
$$

The inclusions (5) and (6) may be written as

$$
\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \prec \tau f(x_3),
$$

by the partial order induced by the convex cone $Y_+$.

In this paper, we proved that the above two generalized convexities are equivalent.

And then, we worked with nonconvex set-valued optimization problems and attained some optimality conditions. Our Fritz John Type Necessary Optimality Condition (Theorem 3.1) and Kuhn-Tucker Type Necessary Optimality Condition (Theorem 4.3) extend the classic results in Clarke [14]. Our Proposition 3.1 are modifications of the alternative theorems in [15, 16]. Our Theorem 3.2 (sufficient optimality condition) extends Theorem 23 in [9]. Our Strong Duality Theorem (Theorem 3.3) extends Theorem 7 in Li and Chen [8].

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


