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SOME RESULTS OF RATIONAL CONTRACTION MAPPING VIA EXTENDED C_F-SIMULATION FUNCTION IN METRIC-LIKE SPACE WITH APPLICATION

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Abstract. In this paper, we introduce a new contraction via C_F-simulation function and prove the existence and the uniqueness of our mapping defined on a metric-like space. Our work generalizes and extends some theorems in the literature. An example and application of second type of Fredholm integral equation are given.

1. Introduction

Many problems in mathematics and other sciences such as physics, chemistry, computer science and engineering resolved by using fixed point theory. The Banach contraction mapping principle [1] is one of the essential results in fixed point theory. Thus, a huge number of mathematical researchers generalized and extended it in a lot of spaces that appeared after 1922. One of the most spaces introduced in this decade is metric-like space that was presented by Amini-Harandi [11] in 2012. After that, a lot of researchers proved (common) fixed point results by using different types of contractive conditions in the setting of metric-like spaces, for example see([2], [3], [6]- [10]).

Definition 1.1. *[11] Let χ is a nonempty set. A function $\sigma : \chi \times \chi \to [0, \infty)$ is said to be a metric like space (or dislocated metric) on χ if for any $\alpha, \nu, \xi \in \chi$, the following conditions hold:*

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(σ_1) $\sigma(\alpha, \xi) = 0 \Rightarrow \alpha = \xi$,

(σ_2) $\sigma(\alpha, \xi) = \sigma(\xi, \alpha)$,

(σ_3) $\sigma(\alpha, \xi) \leq \sigma(\alpha, w) + \sigma(w, \xi)$.

The pair (χ, σ) is called a metric-like space.

Let (χ, σ) be a metric-like space. A sequence $\{\alpha_n\}$ in χ, if and only if

$$\lim_{n\to\infty} \sigma(\alpha_n, \alpha) = \sigma(\alpha, \alpha)$$

A sequence $\{\alpha_n\}$ is called σ-Cauchy if the limit $\lim_{n,m\to\infty} \sigma(\alpha_n, \alpha_m)$ exists and is finite. The metric-like space (χ, σ) is called complete if for each σ-Cauchy sequence $\{\alpha_n\}$, there is some $\alpha \in \chi$ such that

$$\lim_{n\to\infty} \sigma(\alpha_n, \alpha) = \sigma(\alpha, \alpha) = \lim_{n,m\to\infty} \sigma(\alpha_n, \alpha_m).$$

Lemma 1.1. *[12] Let (χ, σ) be a metric-like space. Let $\{\alpha_n\}$ be a sequence in χ such that $\alpha_n \to u$ where $\alpha \in \chi$ and $\sigma(\alpha, \alpha) = 0$. Then, for all $\xi \in \chi$, we have $\lim_{n\to\infty} \sigma(\alpha_n, \xi) = \sigma(\alpha, \xi)$.*

Definition 1.2. *[24] Let χ be a nonempty set. A function $\text{ß} : \chi \times \chi \to [0, \infty)$ is a partial metric if for all $\alpha, \xi, w \in \chi$, the following conditions are satisfied:*

(1) $\alpha = \xi \Leftrightarrow \text{ß}(\alpha, \alpha) = \text{ß}(\alpha, \xi) = \text{ß}(\xi, \xi)$,

(2) $\text{ß}(\alpha, \alpha) \leq \text{ß}(\alpha, \xi)$,

(3) $\text{ß}(\alpha, \xi) = \text{ß}(\xi, \xi)$,

(4) $\text{ß}(\alpha, \xi) \leq \text{ß}(\alpha, w) + \text{ß}(w, \xi) - \text{ß}(w, w)$.

In this case, the pair $(\chi, \text{ß})$ is called a partial metric space.

It is known that each partial metric is a metric-like, but the converse is not true in general.

Example 1.1. *Let $\chi = \{0, 1\}$ and $\sigma : \chi \times \chi \to [0, \infty)$ defined by*

$$\sigma(0, 0) = 2,\ \sigma(u, v) = 1 \ \text{if } (\alpha, \xi) \neq (0, 0)$$

Then, pair (χ, σ) is a metric-like space. Note that σ is not a partial metric on χ because $\sigma(0, 0) \nleq \sigma(1, 0)$.

Remark 1.1. *Let $\chi = \{0, 1\}$, and $\sigma(\alpha, \xi) = 1$ for each $\alpha, \xi \in \chi$ and $\alpha_n = 1$ for each $n \in \mathbb{N}$. Then it is easy to see that $\alpha_n \to 0$ and $\alpha_n \to 1$ and so in metric-like spaces the limit of a convergent sequence is not necessarily unique.*

Definition 1.3. *[27] A function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called an extended simulation function if ζ satisfies the following conditions:*

(ζ_1) $\zeta(\alpha, \xi) < \alpha - \xi$ for all $\alpha, \xi > 0$,

(ζ_2) if $\{\alpha_n\}$ and $\{\xi_n\}$ are sequences in $(0,\infty)$ *such that* $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\xi_n = \ell \in (0,\infty) > 0$, *and* $\alpha_n > l$, $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \sup \zeta(\alpha_n, \xi_n) < 0.$$

(ζ_2)let $\{\alpha_n\}$ be a sequences in $(0,\infty)$ *such that*

$$\lim_{n\to\infty} \alpha_n = \ell \in [0,\infty) > 0, \;\; \zeta(\alpha_n, l) \geq 0, \;\; n \in \mathbb{N},$$

then $l = 0$.

Many researchers have used the above notation to prove some fixed and common fixed point results, see for example ([13], [23]).

In 2014, Ansari [26] introduced the concept of $\mathcal{C}$-class functions as follows:

Definition 1.4. *[26] A mapping* $F : [0,\infty)^2 \to \mathbb{R}$ *is called a C-class function if for any* $\alpha, \xi \in [0,\infty)$, *the following conditions hold:*

(i) $F(\alpha, \xi) \leq \alpha$,

(ii) $F(\alpha, \xi) = \alpha$ *implies that either* $\alpha = 0$ *or* $\xi = 0$.

As examples of C-*class* functions, we state:

(1) $F(\alpha, \xi) = \alpha - \xi$ for all $\alpha, \alpha \in [0,\infty)$;

(2) $F(\alpha, \xi) = l\alpha$ for all $\alpha, \xi \in [0,\infty)$ where $0<l<1$;

Definition 1.5. *[5] A mapping* $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *has the property* C_F, *if there exists* $C_F \geq 0$ *such that*

(F_i) $F(\alpha, \xi) > C_F \Rightarrow \alpha > \xi$,

(F_{ii}) $F(\xi, \xi) \leq C_F$ *for all* $\xi \in [0,\infty)$.

The following example of C-*class* functions that have property C_F

(1) $F_1(\alpha, \xi) = \frac{\alpha}{1+\xi}$, $C_F = 1, 2$.

(2) $F_2(\alpha, \xi) = \alpha - \xi$, $\;C_F = r$, $r \in [0,\infty)$.

Liu [5] linked between a C-class function and C_F-simulation function and presented it as the following:

Definition 1.6. *[5] A mapping* $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is C_F-simulation function if satisfying the following conditions:*

(ζ_i) $\zeta(0, 0) = 0$

(ζ_{ii}) $\zeta(\alpha,\xi) < F(\alpha,\xi)$, where $\alpha,\xi > 0$, with property C_F

(ζ_{iii}) if $\{\alpha_n\},\{\xi_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\xi_n > 0,$$

and $\alpha_n < \xi_n$, then

$$\limsup_{n\to\infty}\zeta(\alpha_n,\xi_n) < C_F,$$

Example 1.2. *[5] Let $\zeta : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$ be a function defined by $\zeta(\alpha,\xi) = mF(\alpha,\xi)$, where $\alpha,\xi \in [0,\infty)$ and $m \in \mathbb{R}$ be such that $m < 1$ and for each $\alpha,\xi \in [0,\infty)$. Considering $C_F = 1, \zeta$ is a C_F-simulation function.*

Choosing $F(\alpha,\xi) = \frac{\alpha}{1+\xi}$, we get $\zeta(\alpha,\xi) = \frac{m\alpha}{1+\xi}$ is also a C_F-simulation function with $C_F = 1$.

Chanda et. al. [25] brought the concept of C_F-extended simulation function as the following:

Definition 1.7. *[25] A mapping $\zeta : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$ an extended C_F-simulation function if satisfying the following conditions:*

(ζ_1) $\zeta(\alpha,\xi) < F(\alpha,\xi)$, where $\alpha,\xi > 0$, with property C_F

(ζ_2) if $\{\alpha_n\},\{\xi_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\xi_n = l,$$

where $l \in (0,\infty)$ and $\xi_n > l$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty}\zeta(\alpha_n,\xi_n) < C_F,$$

(ζ_3) if $\{\alpha_n\}$ be a sequence $(0,\infty)$, such that

$$\lim_{n\to\infty}\alpha_n = l \in [0,\infty),\ \ \zeta(\alpha_n,l) \geq C_F \Rightarrow l = 0.$$

Example 1.3. *[25] Let $\zeta : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$ be a function defined by $\zeta(\alpha,\xi) = \frac{3}{4}\alpha - \xi$, where $\alpha,\xi \in [0,\infty)$. Considering $F(\alpha,\xi) = \alpha - \xi$ with $C_F = 1$, for all $\alpha,\xi \in [0,\infty)$, we assured that ($\zeta_1$)is proved.*

Now if $\{\alpha_n\},\{\xi_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\xi_n = l > 0$$

and $\xi_n > l$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\zeta(\alpha_n,\xi_n) &= \limsup_{n\to\infty}[\frac{3}{4}\alpha_n - \xi_n]\\ &= \frac{-l}{4}\\ &< C_F = 1.\end{aligned}$$

Thus $\zeta(\alpha,\xi)=\frac{3}{4}\alpha-\xi$ *meets* (ζ_2)*. Now, we check for* (ζ_3)*.*

We choose a sequence $\{\xi_n\}$ *in* $(0,\infty)$ *with*

$$\lim_{n\to\infty}\alpha_n=l\geq 0$$

for each $n\in\mathbb{N}$ *such that*

$$\begin{aligned}\zeta(\alpha_n,l) &\geq C_F=1\\ &= \frac{3}{4}l-\alpha_n\geq 1\\ \Rightarrow \alpha_n &\leq \frac{3}{4}l-1.\end{aligned}$$

Letting $n\to\infty$*, we have*

$$\begin{aligned} l &\leq \frac{3}{4}l-1\\ \Rightarrow \frac{1}{4}l &\leq -1\\ \Rightarrow l &= -4\end{aligned}$$

which is a contradiction to $l\geq 0$*. Hence* $\zeta(\alpha,\xi)=\frac{3}{4}\alpha-\xi$ *satisfies all conditions od Definition 1.3 and so is an extended* C_F*-simulation function.*

A functional $\varphi:[0,\infty)\to[0,\infty)$ is lower semicontinuous at a point $\alpha_0\in\chi$ if

(1) $\varphi(\alpha_0)\leq\liminf_{\alpha\to\alpha_0}\varphi(\alpha)$,

(2) $\varphi(\alpha)=0\Leftrightarrow\alpha=0$,

Lemma 1.2. *Let* (χ,σ) *be a metric like space and let* $\{w_n\}$ *be a sequence in* χ *such that* $\lim_{n\to\infty}\sigma(\alpha_n,\alpha_{n+1})=0$*. If* $\lim_{n,m\to\infty}\sigma(\alpha_n,\alpha_m)\neq 0$*, then there exist* $\epsilon>0$ *and two sequences* $\{n_l\}$ *and* $\{m_l\}$ *of positive integers with* $n_l>m_l>l$ *such that following three sequences* $\sigma(\alpha_{2n_l},\alpha_{2m_l})$*,* $\sigma(\alpha_{2n_l-1},\alpha_{2m_l})$*, and* $\sigma(\alpha_{2n_l},\alpha_{2m_l+1})$ *converge to* ϵ^+ *when* $l\to\infty$*.*

In this article, motivated by the idea of an extended C_F-simulation function due to Chanda et al 1.3, we prove the existence and the uniqueness of a common fixed point for two mappings satisfying a contraction which involve a lower semicontinuous function is established. An example and application are given to support the obtained work.

2. Main Result

Theorem 2.1. *Assume that* $p,q:\chi\to\chi$ *are two self-maps on a complete metric-like space* (χ,σ)*. Suppose that there exist an extended* C_F-*simulation function* $\zeta\in\Im^*$ *and* $\varphi\in\Delta$ *such that*

$$\zeta(\sigma(p\alpha,q\xi)+\varphi(p\alpha)+\varphi(q\xi),m(\alpha,\xi))\geq C_F \tag{2.1}$$

for all $\alpha, \xi \in \chi$, *where*

$$\begin{aligned} m(\alpha,\xi) &= \max\{\sigma(\alpha,\xi)+\varphi(\alpha)+\varphi(\xi), \sigma(\alpha,p\alpha)+\varphi(\alpha)+\varphi(p\alpha), \sigma(\xi,q\xi)+\varphi(\xi)+\varphi(q\xi), \\ &\quad \frac{\sigma(\alpha,q\xi)+\varphi(\alpha)+\varphi(q\xi)+\sigma(p\alpha,\xi)+\varphi(p\alpha)+\varphi(\xi)}{4}\}. \end{aligned} \tag{2.2}$$

Then, (p,q) *has a common fixed point* $z \in \chi$ *such that* $\sigma(z,z)=0$ *and* $\varphi(z)=0$.

Proof. Let $\alpha_0 \in \chi$, and define a sequence $\{\alpha_n\}$ by

$$\alpha_{2n+1} = p\alpha_{2n}$$

and

$$\alpha_{2n+2} = q\alpha_{2n+1}$$

for all $n \geq 0$. If $\alpha_{2n} = \alpha_{2n+1}$ for some n, then the proof is done. Therefore, if $\alpha_{2n} \neq \alpha_{2n+1}$ and $\sigma(\alpha_{2n},\alpha_{2n+1})=0$, then by (σ_1), which is a discrepancy. Applying (2.1), we obtain

$$\begin{aligned} C_F &\leq \zeta(\sigma(p\alpha_{2n},q\alpha_{2n+1})+\varphi(p\alpha_{2n})+q\varphi(\alpha_{2n+1}), m(\alpha_{2n},\alpha_{2n+1})) \\ &= \zeta(\sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2}), m(\alpha_{2n},\alpha_{2n+1})). \end{aligned} \tag{2.3}$$

By applying (ζ_2) in (2.3), we obtain

$$C_F < F(m(\alpha_{2n},\alpha_{2n+1}), \sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2})),$$

which implies

$$\sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2}) < m(\alpha_{2n},\alpha_{2n+1}) \tag{2.4}$$

where

$$\begin{aligned} m(\alpha_{2n},\alpha_{2n+1}) &= \max\{\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1}), \sigma(\alpha_{2n},p\alpha_{2n})+\varphi(\alpha_{2n})+\varphi(p\alpha_{2n}), \sigma(\alpha_{2n+1},q\alpha_{2n+1}) \\ &\quad +\varphi(\alpha_{2n+1})+\varphi(q\alpha_{2n+1}), \frac{1}{4}(\sigma(\alpha_{2n},q\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(q\alpha_{2n+1})+\sigma(p\alpha_{2n},\alpha_{2n+1})+\varphi(p\alpha_{2n}) \\ &\quad +\varphi(\alpha_{2n+1}))\} \\ &= \max\{\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1}), \sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1}), \sigma(\alpha_{2n+1},\alpha_{2n+2}) \\ &\quad +\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2}), \sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(u\alpha_{2n+1})+\varphi(\alpha_{2n+2}), \frac{1}{4}(\sigma(\alpha_{2n},\alpha_{2n+2})+\varphi(\alpha_{2n}) \\ &\quad +\varphi(\alpha_{2n+2})+\sigma(\alpha_{2n+1},\alpha_{2n+1})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+1}))\} \\ &\leq \max\{\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1}), \sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2}), \\ &\quad \frac{1}{4}(\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1})+\sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2}))\} \\ &= \max\{\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1}), \sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2})\}. \end{aligned} \tag{2.5}$$

Thus, from (2.4), we get

$$\sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2})$$
$$< \max\{\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1}),\sigma(\alpha_{2n+1},\alpha_{2n+2})+\varphi(\alpha_{2n+1})+\varphi(\alpha_{2n+2})\}. \tag{2.6}$$

By a similar process, one can also get the following

$$\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1})$$
$$< \max\{\sigma(\alpha_{2n-1},\alpha_{2n})+\varphi(\alpha_{2n-1})+\varphi(\alpha_{2n}),\sigma(\alpha_{2n},\alpha_{2n+1})+\varphi(\alpha_{2n})+\varphi(\alpha_{2n+1})\}. \tag{2.7}$$

Therefore, from (2.6) and (2.7),

$$\sigma(\alpha_n,\alpha_{n+1})+\varphi(\alpha_n)+\varphi(\alpha_{n+1}) < \max\{\sigma(\alpha_{n-1},\alpha_n)+\varphi(\alpha_{n-1})+\varphi(\alpha_n),\sigma(\alpha_n,\alpha_{n+1})+\varphi(\alpha_n)+\varphi(\alpha_{n+1})\}, \tag{2.8}$$

for all $n\in\mathbb{N}$.

Necessarily, we obtain

$$\max\{\sigma(\alpha_{n-1},\alpha_n)+\varphi(\alpha_{n-1})+\varphi(\alpha_n),\sigma(\alpha_n,\alpha_{n+1})+\varphi(\alpha_n)+\varphi(\alpha_{n+1})\} = \sigma(\alpha_{n-1},\alpha_n)+\varphi(\alpha_{n-1})+\varphi(\alpha_n), \tag{2.9}$$

for all $n\in\mathbb{N}$.

Consequently, for all $n\in\mathbb{N}$, we have

$$\sigma(\alpha_n,\alpha_{n+1})+\varphi(\alpha_n)+\varphi(\alpha_{n+1}) < \sigma(\alpha_{n-1},\alpha_n)+\varphi(\alpha_{n-1})+\varphi(\alpha_n)$$

Therefore, we find that $\{\sigma(\alpha_n,\alpha_{n+1})++\varphi(\alpha_n)+\varphi(\alpha_{n+1})\}$ is a decreasing sequence. So, there exists $l\geq 0$ such that

$$\lim_{n\to\infty}(\sigma(\alpha_n,\alpha_{n+1})+\varphi(\alpha_n)+\varphi(\alpha_{n+1}))=l.$$

Assume that $l>0$. Then, we deal with $\{\alpha_n\}$ and $\{\xi_n\}$ with same limit where

$$\alpha_n=\sigma(p\alpha_n,p\alpha_{n+1})>0$$

and

$$\alpha_n=\sigma(q\alpha_n,q\alpha_{n+1})>0$$

for all $n\in\mathbb{N}$ and $\alpha_n>l$ for all $n\in\mathbb{N}$. Lastly we get from condition (ζ_2),

$$C_F\leq\zeta(\sigma(\alpha_n,\alpha_{n+1})+\varphi(\alpha_n)+\varphi(\alpha_{n+1}),\sigma(\alpha_{n-1},\alpha_n)+\varphi(\alpha_{n-1})+\varphi(\alpha_n)<C_F$$

which is a contradiction. Then, we conclude that $l=0$ and

$$\lim_{n\to\infty}(\sigma(\alpha_n,\alpha_{n+1})+\varphi(\alpha_n)+\varphi(\alpha_{n+1}))=0,$$

which implies

$$\lim_{n\to\infty} \sigma(\alpha_n, \alpha_{n+1}) = 0, \tag{2.10}$$

and

$$\lim_{n\to\infty} \varphi(\alpha_n) = 0. \tag{2.11}$$

Now, we will prove that $\{\alpha_n\}$ is Cauchy sequence. After that, we will prove

$$\lim_{n\to\infty} \sigma(\alpha_n, \alpha_m) = 0.$$

Assume that

$$\lim_{n\to\infty} \sigma(\alpha_n, \alpha_m) \neq 0.$$

By contradiction.Thus, that is $l = 0$. There exists $\epsilon > 0$ and two sequences $\{\alpha_{n_y}\}$ and $\{\alpha_{m_y}\}$ of $\{\alpha_n\}$ with $n_y > m_y \geq l$ such that for every y with the (smallest number satisfying the condition below)

$$\sigma(\alpha_{n_y}, \alpha_{m_y}) \geq \epsilon. \tag{2.12}$$

and

$$\sigma(\alpha_{n_y-1}, \alpha_{m_y-1}) < \epsilon. \tag{2.13}$$

By using (2.12) and (2.13) and the triangular inequality, we get

$$\epsilon \leq \sigma(\alpha_{n_y}, \alpha_{m_y}) \geq \sigma(\alpha_{n_y}, \alpha_{m_y-1}) + \sigma(\alpha_{m_y-1}, \alpha_{m_y}) < \sigma(\alpha_{m_y-1}, \alpha_{m_y}) + \epsilon.$$

By (**??**)

$$\lim_{y\to\infty} \sigma(\alpha_{n_y}, \alpha_{m_y}) = \lim_{y\to\infty} \sigma(\alpha_{n_y-1}, \alpha_{m_y-1}) = \epsilon. \tag{2.14}$$

We also have

$$\sigma(\alpha_{n_y}, \alpha_{m_y-1}) - \sigma(\alpha_{n_y}, \alpha_{n_y-1}) - \sigma(\alpha_{m_y}, \alpha_{m_y-1}) \leq \sigma(\alpha_{n_y-1}, \alpha_{m_y}), \tag{2.15}$$

and

$$\sigma(\alpha_{n_y-1}, \alpha_{m_y}) \leq \sigma(\alpha_{n_y-1}, \alpha_{n_y}) + \sigma(\alpha_{n_y}, \alpha_{m_y}). \tag{2.16}$$

Letting $y \to \infty$ in (2.15) and (2.16) and by using (2.10) and (2.14), we obtain

$$\lim_{y\to\infty} \sigma(\alpha_{n_y-1}, \alpha_{m_y}) = \epsilon. \tag{2.17}$$

Again, using the triangular inequality, we have

$$\mid \sigma(\alpha_{n_y-1}, \alpha_{m_y}) - \sigma(\alpha_{n_y-1}, \alpha_{m_y-1}) \mid \sigma(\alpha_{m_y-1}, \alpha_{m_y}). \tag{2.18}$$

Letting $y \to \infty$ in (2.18)and by using (2.17), we get

$$\lim_{y\to\infty} \sigma(\alpha_{n_y-1}, \alpha_{m_y-1}) = \epsilon. \tag{2.19}$$

From (2.34), we have

$$\begin{aligned} m(\alpha_{n_y-1}, \alpha_{m_y-1}) &= \max\{\sigma(\alpha_{n_y-1}, \alpha_{m_y-1}) + \varphi(\alpha_{n_y-1}) + \varphi(\alpha_{m_y-1}), \sigma(\alpha_{n_y-1}, p\alpha_{n_y-1}) + \varphi(\alpha_{m_y-1}) \\ &\quad +\varphi(q\alpha_{m_y-1}), \frac{1}{4}(\sigma(\alpha_{n_y-1}, q\alpha_{m_y-1}) + \varphi(\alpha_{n_y-1}) + \varphi(q\alpha_{m_y-1}) + \sigma(p\alpha_{n_y-1}, \alpha_{m_y-1}) \\ &\quad +\varphi(p\alpha_{n_y-1}) + \varphi(\alpha_{m_y-1}))\} \\ &= \max\{\sigma(\alpha_{n_y-1}, \alpha_{m_y-1}) + \varphi(\alpha_{n_y-1}) + \varphi(\alpha_{m_y-1}), \sigma(\alpha_{n_y-1}, \alpha_{n_y}) + \varphi(\alpha_{n-1}) + \varphi(\alpha_n), \\ &\quad \sigma(\alpha_{m_y-1}, \alpha_{m_y}) + \varphi(\alpha_{m_y-1}) + \varphi(\alpha_{m_y}), \frac{1}{4}(\sigma(\alpha_{n_y-1}, \alpha_{m_y}) + \varphi(\alpha_{n_y-1}) + \varphi(\alpha_{m_y}) + \\ &\quad \sigma(\alpha_{n_y}, \alpha_{m_y-1}) + \varphi(\alpha_{n_y}) + \varphi(\alpha_{m_y-1}))\}. \end{aligned} \tag{2.20}$$

Letting $y \to \infty$ in (2.20)and by (2.10),(2.11),(2.14),(2.17) and (2.19), it follows that

$$\lim_{y\to\infty} \sigma(\alpha_{n_y}, \alpha_{m_y}) = \lim_{y\to\infty} m(\alpha_{n_y-1}, \alpha_{m_y-1}) = \epsilon. \tag{2.21}$$

Applying (theta2), we get

$$C_F \leq \zeta(\sigma(\alpha_{n_y}, \alpha_{m_y}) + \varphi(\alpha_n) + \varphi(\alpha_m), m(\alpha_{n_y-1}, \alpha_{m_y-1})) < C_F$$

which is a contradiction. Hence α_n is a Cauchy sequence and hence $\lim_{n\to\infty} \alpha_n = k \in \chi$ exists because χ is complete. Since φ is lower semicontinuous,

$$\varphi(k) \leq \liminf_{n\to\infty} \varphi(\alpha_n) \leq \lim_{n\to\infty} \varphi(\alpha_n),$$

which implies

$$\varphi(k) = 0. \tag{2.22}$$

We claim that k is a common fixed point of p and q. Put $\alpha = \alpha_n$ and $\xi = k$ in (2.33) for all n, and we obtain

$$\zeta(\sigma(p\alpha_n, qk) + \varphi(p\alpha_n) + \varphi(qk), m(\alpha_n, k)) \geq C_F \tag{2.23}$$

$$\begin{aligned} m(\alpha_n, k) &= \max\{\sigma(\alpha_n, k) + \varphi(\alpha_n) + \varphi(k), \sigma(\alpha_n, pu_n) + \varphi(\alpha_n) + \varphi(p\alpha_n), \sigma(k, qk) + \varphi(k) + \varphi(qk), \\ &\quad \frac{1}{4}(\sigma(\alpha_n, qk) + \varphi(\alpha_n) + \varphi(qk) + \sigma(p\alpha_n, k) + \varphi(p\alpha_n) + \varphi(k))\} \\ &= \max\{\sigma(\alpha_n, k) + \varphi(\alpha_n) + \varphi(k), \sigma(\alpha_n, u_{n+1}) + \varphi(\alpha_n) + \varphi(\alpha_{n+1}), \sigma(k, qk) + \varphi(k) + \varphi(qk), \\ &\quad \frac{1}{4}(\sigma(\alpha_n, qk) + \varphi(\alpha_n) + \varphi(qk) + \sigma(\alpha_{n+1}, k) + \varphi(\alpha_{n+1}) + \varphi(k))\}. \end{aligned}$$

Let $n \to \infty$ in (2.23) and using (2.22), we have

$$\begin{aligned} C_F &\leq \zeta(\sigma(k, qk) + \varphi(qk), \sigma(k, qk) + \varphi(qk)) \\ &< F(\sigma(k, qk) + \varphi(qk), \sigma(k, qk) + \varphi(qk)) \end{aligned} \tag{2.24}$$

$\Rightarrow$

$$\sigma(k, qk) + \varphi(qk) < \sigma(k, qk) + \varphi(qk) \tag{2.25}$$

which is absurd. Hence $\sigma(k, qk) + \varphi(qk) = 0$, and hence

$$k = qk \text{ and } \varphi(qk) = 0. \tag{2.26}$$

Similarly, when we take $\alpha = \alpha_n$ and $\xi = k$ in (2.33) for all n we get

$$k = pk \text{ and } \varphi(pk) = 0. \tag{2.27}$$

Equations (2.26) and (2.27) show that k is a common fixed point of p and q. To prove the uniqueness of the common fixed point, we suppose that h is another fixed point of p and q. We argue by contradiction. Assume that there exists $h \neq k$(so $\sigma(h, k) > 0$.) such that

$$\zeta(\sigma(ph, qk) + \varphi(ph) + \varphi(qk), m(h, k)) \geq C_F, \tag{2.28}$$

where

$$\begin{aligned} m(h, k) &= \max\{\sigma(h, k) + \varphi(h) + \varphi(k), \sigma(h, ph) + \varphi(h) + \varphi(ph), \sigma(k, qk) + \varphi(k) + \varphi(qk), \\ &\quad \frac{\sigma(h, qk) + \varphi(h) + \varphi(qk) + \sigma(ph, k) + \varphi(ph) + \varphi(k)}{4}\} \\ &= \sigma(h, qk). \end{aligned} \tag{2.29}$$

Hence from (2.30), we obtain

$$\begin{aligned} C_F &\leq \zeta(\sigma(h, k), \sigma(h, k)) \\ &< F(\sigma(h, k), \sigma(h, k)) \\ &< C_F, \end{aligned} \tag{2.30}$$

which is absurd and hence $h = k$. □

We will use the same manner in 2.1 to obtain the following result.

Theorem 2.2. *Assume that $p, q : \chi \to \chi$ are two self-maps on a complete partial metric space (χ, σ). Suppose that there exists a extended C_F-simulation function $\zeta \in \Im^*$ and $\varphi \in \Delta$ such that*

$$\zeta(\sigma(p\alpha, q\xi) + \varphi(p\alpha) + \varphi(q\xi), m(\alpha, \xi)) \geq C_F \tag{2.31}$$

for all $u, v \in \chi$, *where*

$$m(\alpha,\xi) = \max\{d_{par}(\alpha,\xi)+\varphi(\alpha)+\varphi(v), d_{par}(\alpha,p\alpha)+\varphi(\alpha)+\varphi(p\alpha), d_{par}(\xi,q\xi)+\varphi(\xi)+\varphi(q\xi),$$
$$\frac{d_{par}(\alpha,q\xi)+\varphi(\alpha)+\varphi(q\xi)+d_{par}(p\alpha,\xi)+\varphi(p\alpha)+\varphi(\xi)}{2}\}. \tag{2.32}$$

Then, (p,q) *has a common fixed point* $z \in \chi$ *such that* $\sigma(z,z)=0$ *and* $\varphi(z)=0$.

If we put $q = p$ in 2.1, we have the following Corollary

Corollary 2.1. *Assume that* $p : \chi \to \chi$ *be self-map on a complete metric-like space* (χ,σ)*. Suppose that there exists a extended* C_F-*simulation function* $\zeta \in \Im^*$ *and* $\varphi \in \Delta$ *such that*

$$\zeta(\sigma(p\alpha,p\xi)+\varphi(p\alpha)+\varphi(p\xi), m(\alpha,\xi)) \geq C_F \tag{2.33}$$

for all $\alpha,\xi \in \chi$, *where*

$$m(\alpha,\xi) = \max\{\sigma(\alpha,\xi)+\varphi(\alpha)+\varphi(\xi), \sigma(\alpha,p\alpha)+\varphi(\alpha)+\varphi(p\alpha), \sigma(\xi,p\xi)+\varphi(\xi)+\varphi(p\xi),$$
$$\frac{\sigma(\alpha,p\xi)+\varphi(\alpha)+\varphi(p\xi)+\sigma(p\alpha,\xi)+\varphi(p\alpha)+\varphi(\xi)}{4}\}. \tag{2.34}$$

Then, p *has a unique fixed point* $z \in \chi$ *such that* $\sigma(z,z)=0$ *and* $\varphi(z)=0$.

Corollary 2.2. *Assume that* $p,q : \chi \to \chi$ *are two self-maps on a complete metric-like space* (χ,σ)*. Suppose that there exists a extended* C_F-*simulation function* $\zeta \in \Im^*$ *and* $\varphi \in \Delta$ *such that*

$$\zeta(\sigma(p\alpha,q\xi)+\varphi(p\alpha)+\varphi(q\xi), \sigma(\alpha,\xi)+\varphi(\alpha)+\varphi(\xi)) \geq C_F \text{ for all } \alpha,\xi \in \chi. \tag{2.35}$$

Then, (p,q) *has a unique common fixed point* $z \in \chi$ *such that* $\sigma(z,z)=0$ *and* $\varphi(z)=0$.

Corollary 2.3. *Assume that* $p : \chi \to \chi$ *be self-map on a complete metric-like space* (χ,σ)*. Suppose that there exists a extended* C_F-*simulation function* $\zeta \in \Im^*$ *and* $\varphi \in \Delta$ *such that*

$$\zeta(\sigma(p\alpha,p\xi)+\varphi(p\alpha)+\varphi(p\xi), \sigma(\alpha,\xi)+\varphi(\alpha)+\varphi(\xi)) \geq C_F \text{ for all } \alpha,\xi \in \chi. \tag{2.36}$$

Then, p *has a unique fixed point* $z \in \chi$ *such that* $\sigma(z,z)=0$ *and* $\varphi(z)=0$.

If we take $\varphi(t) = 0$ in 2.1 and 2.2, we obtain the following two corollaries.

Corollary 2.4. *Assume that* $p,q : \chi \to \chi$ *are two self-maps on a complete metric-like space* (χ,σ)*. Suppose that there exists a extended* C_F-*simulation function* $\zeta \in \Im^*$ *and* $\varphi \in \Delta$ *such that*

$$\zeta(\sigma(p\alpha,q\xi), m(\alpha,\xi)) \geq C_F \tag{2.37}$$

for all $\alpha,\xi \in \chi$, *where*

$$m(\alpha,\xi) = \max\{\sigma(\alpha,\xi), \sigma(\alpha,p\alpha), \sigma(\xi,q\xi), \frac{\sigma(\alpha,q\xi)+\sigma(p\alpha,\xi)}{4}\}. \tag{2.38}$$

Then, (p,q) has a unique common fixed point $z \in \chi$ such that $\sigma(z,z)=0$.

Corollary 2.5. *Assume that $p:\chi\to\chi$ be self-map on a complete metric-like space (χ,σ). Suppose that there exists a extended C_F-simulation function $\zeta\in\Im^*$ and $\varphi\in\Delta$ such that*

$$\zeta(\sigma(p\alpha,q\xi),m(\alpha,\xi))\geq C_F \tag{2.39}$$

for all $\alpha,\xi\in\chi$, where

$$m(\alpha,\xi) \quad = \quad \max\{\sigma(\alpha,\xi),\sigma(\alpha,p\alpha),\sigma(\xi,q\xi),\frac{\sigma(\alpha,q\xi)+\sigma(p\alpha,\xi)}{4}\}. \tag{2.40}$$

Then, p has a unique fixed point $z\in\chi$ such that $\sigma(z,z)=0$.

Example 2.1. *Let $\chi=[0,1]$ be equipped with the metric-like mapping $\sigma(\alpha,\xi)=\alpha^2+\xi^2$ for all $\alpha,\xi\in\chi$. Let $p,q:\chi\to\chi$ be defined by*

$$p\alpha=\begin{cases}\frac{\alpha^2}{\alpha+1} & \textit{if } 0\in[0,1],\\ \alpha^2, & \textit{otherwise.}\end{cases},$$

and

$$q\alpha=\begin{cases}\frac{\alpha^3}{\alpha+1} & \textit{if } 0\in[0,1],\\ \alpha^3, & \textit{otherwise.}\end{cases}.$$

We also consider $\zeta(s,t)=\frac{1}{3}s-t$ for all $s,t\geq 0$, $C_F=0$ and $\varphi(t)=t$ for all $\alpha\in\chi$. Note that (χ,σ) is a complete metric-like space.

Without loss of generality we assume that $\alpha,\xi\in\chi$,

$$\begin{aligned}\sigma(p\alpha,q\xi)+\varphi(p\alpha)+\varphi(q\xi) &= \sigma(\frac{\alpha^2}{\alpha+1},\frac{\xi^3}{\xi+1}+\varphi(\frac{\alpha^2}{\alpha+1})+\varphi(\frac{\xi^3}{\xi+1})\\ &= (\frac{\alpha^2}{\alpha+1})^2)+(\frac{\xi^3}{\xi+1})^2)^3+\varphi(\frac{\alpha^2}{\xi+1})+\varphi(\frac{\alpha^2}{\alpha+1})\\ &\leq \frac{1}{6}(\alpha^2+\xi^3)+\frac{1}{3}(\alpha+\xi)\\ &\leq \frac{1}{3}(\alpha^2+\xi^3)+\alpha+\xi)\\ &= \frac{1}{3}(\sigma(\alpha,\xi)+\varphi(\alpha)+\varphi(\xi))\\ &\leq \frac{1}{3}m(\alpha,\xi).\end{aligned}$$

It follows that

$$\zeta(\sigma(p\alpha,q\xi)+\varphi(p\alpha)+\varphi(q\xi),m(\alpha,\xi))=\frac{1}{3}m(\alpha,\xi)-[\sigma(p\alpha,q\xi)+\varphi(p\alpha)+\varphi(q\xi)]\geq 0.$$

Then Theorem 2.1 is applicable to (p,q) and φ on (χ,σ). Moreover, $\alpha=0$ is a common fixed point of (p,q).

3. Application

In this part, we will apply Corollary 2.3 to study the existence and uniqueness of solutions of second type of Fredholm integral equation:

$$\alpha(\vartheta) = \int_0^{\jmath} \pi(\vartheta,\kappa)\varpi(\kappa,\theta(\kappa))d\kappa$$
(3.1)
$$\alpha(\vartheta) = \int_0^{\jmath} \pi(\vartheta,\kappa)\varpi(\kappa,\tau(\kappa))d\kappa.$$

for all $(\vartheta,\kappa) \in [0,\jmath]^2$.

Let $\mathrm{T} = C([0,\jmath],\mathbb{R})$ is the set of real continuous functions on $[0,\jmath]$ for $\jmath > 0$, defined by

$$\sigma(\alpha,\xi) = \parallel \alpha - \xi \parallel_\infty = \sup_{t\in\jmath} |\alpha(t) - \xi(t)|$$

for all $\alpha,\xi \in \mathrm{T}$. Then (T,σ) is a complete metric-like space. We consider the operators

$$p\alpha(\vartheta) = \int_0^{\jmath} \pi(\vartheta,\kappa)\varpi(\kappa,\theta(\kappa))d\kappa,$$

$$q\xi(\vartheta) = \int_0^{\jmath} \pi(\vartheta,\kappa)\varpi(\kappa,\tau(\kappa))d\kappa,$$

Theorem 3.1. *Assume that Equation (3.1) with the following axioms:*

1. $\pi : [0,\jmath]\times[0,\jmath] \to [0,\infty)$ *is a continuous function,*
2. $\varpi : [0,\jmath]\times\mathbb{R} \to \mathbb{R}$ *where* $\varpi(\kappa,.)$ *is monotone nondecreasing mapping for all* $\kappa \in [0,\jmath]$,
3. $\sup_{\vartheta,\kappa\in[0,\jmath]} \int_0^{\jmath} \pi(\vartheta,\kappa)d\kappa \le 1$,
4. *for every* $\delta \in (0,1)$ *such that for all* $(\vartheta,\kappa) \in [0,\jmath]^2$ *and* $\theta,\tau \in \mathbb{R}$,

$$\parallel \varpi(\kappa,\theta(\kappa)) - \varpi(\kappa,\tau(\kappa)) \parallel \le \delta \parallel \alpha(t) - \xi(t) \parallel,$$

Then, the system (3.1)has a unique solution.

Proof. For $\alpha,\xi \in \mathrm{T}$ and from (3) and (4), for all ϑ and κ, we have

$$\begin{aligned}
\sigma(p\alpha(\vartheta),q\xi(\vartheta)) &= \mid p\alpha(\vartheta) - q\xi(\vartheta) \mid \qquad (3.2)\\
&= \mid \int_0^{\jmath} \pi(\vartheta,\kappa)\varpi(\kappa,\theta(\kappa))d\kappa - \int_0^{\jmath} \pi(\vartheta,\kappa)\varpi(\kappa,\tau(\kappa))d\kappa \mid\\
&\le \int_0^{\jmath} \pi(\vartheta,\kappa) \parallel \varpi(\kappa,\theta(\kappa)) - \varpi(\kappa,\tau(\kappa)) \parallel d\kappa\\
&\le \int_0^{\jmath} \pi(\vartheta,\kappa)\delta \parallel \alpha(\vartheta) - \xi(\vartheta) \parallel_\infty d\kappa\\
&\le \pi(\vartheta,\kappa)\delta \parallel \alpha(\vartheta) - \xi(\vartheta) \parallel_\infty\\
&\le \delta\sigma(\alpha,\xi)\\
&\le \delta m(\alpha,\xi). \qquad (3.3)
\end{aligned}$$

Let (ζ_1) and $\zeta(\alpha,\xi)=\delta\alpha-\xi$ for all $\alpha,\zeta\in[0,\infty), C_g=0$. Now

$$\sigma(p\alpha(\vartheta),q\xi(\vartheta))<\delta m(\alpha,\xi). \tag{3.4}$$

Then, from (3.2), we obtain

$$\zeta(\sigma(p\alpha,q\xi),m(\alpha,\xi))\geq C_f.$$

Applying Corollary (3.1), we obtain that (p,q) has a unique common fixed point in $C([0,1])$, say x. Hence, x is a solution of (3.1). □

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