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The Continuous Wavelet Transform for a q-Bessel Type Operator

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ABSTRACT. In this paper, we consider a differential operator Λ on $[0,\infty)$ - By accomplishing harmonic analysis tools with respect to the operator Λ we study some definitions and properties of q-Bessel continuous wavelet transform. We also explore generalized q-Bessel Fourier transform and convolution product on $[0,\infty)$ associated with the operator Λ and finally a new continuous wavelet transform associated with q-Bessel operator is constructed and investigated.

1. Introduction

For a function $f \in L^2(R)$, the wavelet transform with respect to the wavelet $\phi \in L^2(R)$ is defined by

$$(W_\varphi f)(\sigma_2,\sigma_1) = \int_{-\infty}^{\infty} f(t)\overline{\varphi_{\sigma_2,\sigma_1}(t)}dt, \sigma_2 \in R, \sigma_1 > 0 \tag{1.1}$$

where,

$$\varphi_{\sigma_2,\sigma_1}(t) = \sigma_1^{-1/2}\varphi\left(\frac{t-\sigma_2}{\sigma_1}\right). \tag{1.2}$$

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Translation τ_{σ_2} is defined by

$$\tau_{\sigma_2}\varphi(t)=\varphi(t-\sigma_2),\sigma_2\in R$$

and dilation D_{σ_1} is defined by

$$D_{\sigma_1}\varphi(t)=\sigma_1^{-1/2}\phi\left(\frac{t}{\sigma_1}\right),\sigma_1>0\,.$$

We can write

$$\phi_{\sigma_2,\sigma_1}(t)=\tau_{\sigma_2}D_{\sigma_1}\phi(t)\,. \tag{1.3}$$

From above equations, we can say that wavelet transform of the function f on R is an integral transform and the dilated translate of ϕ is the kernel.

We can also express wavelet transform as the convolution:

$$\left(W_\varphi f\right)(\sigma_2,\sigma_1)=\left(f*g_{o,\sigma_1}\right)(\sigma_2)\,, \tag{1.4}$$

Where,

$$g(t)=\overline{\varphi(-t)}.$$

Since there is a special type of convolution for every integral transform, therefore one can define wavelet transform with respect to a integral transform using associated convolution.

The concept of wavelet is a collection of function derived from a single function called mother wavelet, after that by applying the two operators known as translation and dilation we get a new type of continuous wavelet transform.

Here presently, we introduce a q-Bessel operator [1] and [2].

$$\Lambda_{q,v}f(t)=\frac{1}{t^2}\left(f\left(q^{-1}t\right)-\left(1+q^{2v}\right)f(t)+q^{2v}f(qt)\right). \tag{1.5}$$

The above q-Bessel operator associated with q-Bessel function by the eigenvalue equation.

$$\Lambda_{q,v}j_v\left(x,q^2\right)=-\lambda^2 j_v\left(x,q^2\right).$$

Unlike the elementary functions such as trigonometric, exponential etc the Bessel wavelets are related to special functions and Jachkson introduced the concept of q-analysis at the beginning of the twentieth century. We have arranged this paper as follows: In section 2, we will review briefly the basics of q-Bessel Fourier transform, here we recall notations, some definitions of q-Bessel Fourier and Inverse Fourier transform and the preposition associated with other operators and convolution product. In section 3, some results of harmonic analysis with respect to q-Bessel operator for the generalized q-Bessel transform is collected and the definition and properties of convolution product is also discussed. To extend the classical theory of wavelets to the differential operator $\Delta_{q,\alpha}$ is the actual aim of this work.

We define a generalized wavelet, which satisfy the below admissibility condition

$$0 < C_{\alpha,q,g} = \int_0^\infty \left| F_{\Delta_{q,\alpha}}(g)(\lambda) \right|^2 \frac{d_q\lambda}{\lambda} < \infty. \tag{1.6}$$

Where $F_{\Delta_{q,\alpha}}$ denotes the generalized q-Bessel Fourier transform related to operator given by

$$F_{\Delta_{q,\alpha}}(g)(\lambda) = c_{q,\alpha} \int_0^\infty g(t)\, j_\alpha\left(\lambda t, q^2\right) t^{2\alpha+1} d_q t \quad \forall g \in L_{q,p,\alpha}\left(\mathbb{R}_q^{\,+}\right).$$

With

$$c_{q,\alpha} = \frac{1}{1-q} \frac{\left(q^{2\alpha+2}; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty},$$

and $j_\alpha(x;q)$ being the normalized Bessel function of index α.

Starting with a single generalized wavelet g, a family of generalized wavelets is constructed by putting

$$g_{a,b}(x) = a^{\frac{1}{2}} T_{q,b}^{\;\alpha}(g_a)(x), \quad \forall a \in \mathbb{R}_q^{\,+}, \forall b \in \mathbb{R}_q^{\,+} \cup \{0\},$$

where $g_a(x) = \frac{1}{a^{2\alpha+2n+2}} g\left(\frac{x}{a}\right)$ and $T_{q,b}^{\;\alpha}$ is generalized translation operators related to the differential operator $\Delta_{q,\alpha}$.

The continuous generalized q-Bessel wavelet transform of a function $f \in L_{q,2,\alpha}\left(\mathbb{R}_q^+ \cup \{0\}\right)$ at the scale $a \in \mathbb{R}_q^+$ and the position $b \in \mathbb{R}_q^+ \cup \{0\}$ is defined by

$$\phi_{q,g}^{\alpha}(f)(a,b) = c_{q,\alpha} \int_0^\infty f(x)\, \overline{g_{(a,b),\alpha}}\, x^{2\alpha+1} d_q x. \tag{1.7}$$

In section 4, we develop a relationship between the generalized wavelet transforms and q-Bessel continuous wavelet transforms. Such a relationship helps us to build certain formulas for the generalized q-Bessel continuous wavelet transform (CWT).

In Section 5, we study the intertwining operator χ_q to establish the continuous generalized q-Bessel wavelet transform in form of classical one. As a result, we got a new inversion formulas for dual operator ${}^t\chi_q$ of χ_q.

2. Preliminaries

In the present section we recapitulate some facts about harmonic analysis related to the q-Bessel operator. We cite here, as briefly as possible, only those properties actually required for the discussion.

Throughout this section assume $\alpha > -1/2$. Let the space $L_{q,p,\alpha}$, $1 \leq p < \infty$ denote the sets of real functions on $\mathbb{R}_q^+$ for which

$$\| f \|_{q,p,\alpha} = \left[\int_0^\infty \left| f(x) \right|^P x^{2\alpha+1} d_q x \right]^{1/P} < \infty,$$

and $\| f \|_{q,\infty,\alpha} = \underset{x \in \mathbb{R}_q^+}{Sup} \left| f(x) \right| < \infty.$

The q-Bessel Fourier transform $F_{q,\alpha,,n}$ in [3] is defined for $f \in L_{q,1,\alpha}$ by

$$F_{q,\alpha}(f)(\lambda) = c_{q,\alpha} \int_0^\infty f(t)\, j_\alpha\left(\lambda t, q^2\right) t^{2\alpha+1} d_q t, \quad \forall t \in \mathbb{R}_q^+, \tag{2.1}$$

where j_α is normalized q-Bessel function.

$$j_\alpha\left(x, q^2\right) = \sum_{n=0}^\infty (-1)^n \frac{q^{n(n+1)}}{\left(q^{2\alpha+2}; q^2\right)_n \left(q^2, q^2\right)} x^{2n}\ . \tag{2.2}$$

Theorem 2.1 (i) The q-Bessel Fourier transform $F_{q,\alpha} : L_{q,2,\alpha} \to L_{q,2,\alpha}$ defines an isomorphism and for all functions $f \in L_{q,2,\alpha}$,

$$F_{q,\alpha}^2(f) = f, \quad \left\| F_{q,\alpha}(f) \right\|_{q,2,\alpha} = \| f \|_{q,2,\alpha}. \tag{2.3}$$

(ii) If f, $F_{q,\alpha}(f) \in L_{q,1,\alpha}$ then

$$f(x) = \int_0^\infty F_{q,\alpha,n}(f)(\lambda)\, j_\alpha\left(x\lambda, q^2\right) d_q \mu_\alpha(\lambda), \tag{2.4}$$

for almost all $\forall x \in \mathbb{R}_q^+,$ where

$$d_q\mu_{\alpha}(\lambda)=\frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}\lambda^{2\alpha+1}d_q\lambda \tag{2.5}$$

(iii) For all $f\in L_{q,1,\alpha}\cap L_{q,2,\alpha}$ we have

$$\int_0^\infty |F_{q,\alpha}(f)|^2\lambda^{2\alpha+1}d_q\lambda=\int_0^\infty |f(x)|^2 x^{2\alpha+1}d_qx.$$

(iv) The inverse transform is given by

$$F_{q,\alpha,n}^{-1}(g)(x)=\int_0^\infty g(\lambda)\,j_\alpha(\lambda x,q^2)d_q\mu_{\alpha}(\lambda),$$

The q-Bessel translation operators $\tau_{q,x}^{\alpha}, x\geq 0,$ is defined by

$$\tau_{q,x}^{\alpha}(f)(y)=\int_0^\infty f(z)D_{q,\alpha}(x,y,z)z^{2\alpha+1}d_qz, \tag{2.6}$$

where

$$D_{q,\alpha}(x,y,z)=c_{q,\alpha}{}^2\left[\int_0^\infty j_\alpha(xs,q^2)\,j_\alpha(ys,q^2)\,j_\alpha(zs,q^2)s^{2\alpha+1}d_qs\right] \tag{2.7}$$

The convolution product of q-Bessel for two functions f,g is defined as

$$f*_q g(x)=c_{q,\alpha}\int_0^\infty \tau_{q,x}^{\alpha}f(y)g(y)y^{2\alpha+1}d_qy,\ \forall x\geq 0. \tag{2.8}$$

Theorem 2.2 (i) Let $1\leq p<\infty$ and $f\in L_{q,p,\alpha}$. Then $\forall x\geq 0$, $\tau_{q,x}^{\alpha}\in L_{q,p,\alpha}$ and

$$\left\|\tau_{q,x}^{\alpha}f\right\|_{q,p,\alpha}\leq\|f\|_{q,p,\alpha}.$$

(ii) For $f\in L_{q,p,\alpha}$, $1\leq p<\infty$, we have

$$F_{q,\alpha,,n}(\tau_{q,x}^{\alpha}f)(\lambda)=j_\alpha(x,q^2)F_{q,\alpha,,n}(f)(\lambda).$$

(iii) Let $p,r\in[1,\infty)$ such that $\frac{1}{p}+\frac{1}{r}=1$. If $f\in L_{q,p,\alpha}$ and $g\in L_{q,p,\alpha}$, then for every $x\geq 0$ we have

$$\int_0^\infty \tau_{q,x}^{\alpha}f(y)g(y)y^{2\alpha+1}d_qy=\int_0^\infty f(y)\tau_{q,x}^{\alpha}g(y)y^{2\alpha+1}d_qy$$

(iv) For $p,r,s\in[1,\infty)$ such that $\frac{1}{p}+\frac{1}{r}-1=\frac{1}{s}$. If $f\in L_{q,p,\alpha}$ and $g\in L_{q,p,\alpha}$ then

$$\left\| f *_q g \right\|_{q,s,\alpha} \le \left\| f \right\|_{q,p,\alpha} \left\| g \right\|_{q,r,\alpha}$$

(v) For $f \in L_{q,p,\alpha}$ and $g \in L_{q,p,\alpha}$ we have

$$F_{q,\alpha,,n}\left(f *_q g \right) = F_{q,\alpha,,n}\left(f \right) F_{q,\alpha,,n}\left(g \right).$$

Definition 2.1 A function $g \in L_{q,2,\alpha}$ is a q-Bessel wavelet of order α, if it satisfies the admissibility condition.

$$0 < C_{\alpha,g} = \int_0^\infty \left| F_{q,\alpha,,n}\left(g \right)\left(\lambda \right) \right|^2 \frac{d_q \lambda}{\lambda} < \infty. \tag{2.9}$$

Definition 2.2 Let $g \in L_{q,2,\alpha}\left(\mathbb{R}_q^+ \cup \{0\} \right)$ be a q-Bessel wavelet of order α. Then continuous q-Bessel wavelet transform is defined as follows

$$S_{q,g}^{\alpha}\left(f \right)\left(a,b \right) = c_{q,\alpha} \int_0^\infty f\left(x \right) \overline{g_{(a,b)}^{\alpha}} x^{2\alpha+1} d_q x, \quad \forall a \in \mathbb{R}_q{}^+, \forall b \in \mathbb{R}_q{}^+ \cup \{0\}, \tag{2.10}$$

where

$$g_{(a,b)}^{\alpha} = a^{\frac{1}{2}} \tau_{q,b}^{\alpha}\left(g_a \right), \forall a,b \in \mathbb{R}_q{}^+ \tag{2.11}$$

$$g_a = \frac{1}{a^{2\alpha+2}} g\left(\frac{x}{a} \right) \tag{2.12}$$

The q-Bessel continuous wavelet transform has been investigated in detail in [4] from which we see the following basis properties.

Theorem 2.3 Let be $g \in L_{q,2,\alpha}\left(\mathbb{R}_q^+ \cup \{0\} \right)$ be a q-Bessel wavelet. Then

(i) For all $f \in L_{q,2,\alpha}\left(\mathbb{R}_q^+ \cup \{0\} \right)$, the Plancherel formula we have

$$\int_0^\infty \left| f\left(x \right) \right|^2 x^{2\alpha+1} d_q x = \frac{1}{C_{\alpha,g}} \int_0^\infty \int_0^\infty \left| S_{q,g}^{\alpha}\left(f \right)\left(a,b \right) \right|^2 b^{2\alpha+1} d_q b \frac{d_q a}{a^2}.$$

(ii) For all $f \in L_{q,2,\alpha}\left(\mathbb{R}_q^+ \cup \{0\} \right)$, we have

$$f\left(x \right) = \frac{c_{q,\alpha}}{C_{\alpha,g}} \int_0^\infty \left(\int_0^\infty S_{q,g}^{\alpha}\left(f \right)\left(a,b \right) g_{(a,b)}^{\alpha} b^{2\alpha+1} d_q b \right) \frac{d_q a}{a^2}, \ \forall x \in \mathbb{R}_q{}^+.$$

3. Harmonic Analysis Associated with Λ and Generalized Fourier Transform

Let M be the map defined by $Mf\left(x \right) = x^{2n} f\left(x \right).$

Let $L_{q,p,\alpha}$, $1 \le p \le \infty$ be the class of measurable functions f on $[0,\infty)$ for which $\|f\|_{q,p,\alpha,n} = \|M^{-1} f\|_{q,p,\alpha+2n} < \infty.$

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, put

$$\varphi_{\alpha,n}\left(x,q^2\right) = x^{2n} j_{\alpha+2n}\left(\lambda x, q^2\right). \tag{3.1}$$

where $j_{\alpha+2n}$ is the normalized Bessel function with index $\alpha+2n$ is given by equation (2.1). From [4] see the following properties.

Theorem 3.1 (i) $\varphi_{\alpha,n}$ possess the Laplace type integral representation

$$\varphi_{\alpha,n}\left(\lambda,q^2\right) = (1+q)C\left(\alpha:q^2\right)x^{2n}\int_0^1 F_\alpha\left(t:q^2\right)\cos\left(xt:q^2\right)d_q t \tag{3.2}$$

when $q \to 1^-$ and $\alpha > \frac{-1}{2}$

where

$$C\left(\alpha:q^2\right) = \frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}\left(\frac{1}{2}\right)\Gamma_{q^2}\left(\alpha+\frac{1}{2}\right)},\ F_\alpha\left(t:q^2\right) = \frac{\left(x^2q^2;q^2\right)_\infty}{\left(x^2q^{2\alpha+1};q^2\right)_\infty},\ \cos\left(x:q^2\right) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}x^{2n}.$$

(ii) $\varphi_{\alpha,n}\left(\lambda,q^2\right)$ satisfies the differential equation

$$\Delta_{q,\alpha,n}\varphi_{\alpha,n}\left(\lambda,q^2\right) = -\lambda^2\varphi_{\alpha,n}\left(\lambda,q^2\right). \tag{3.3}$$

(iii) For all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$

$$\left|\varphi_{\alpha,n}\left(\lambda,q^2\right)\right| \le x^{2n} e^{|\mathrm{Im}\lambda||x|}. \tag{3.4}$$

Definition 3.1 The generalized q-Bessel Fourier transform is defined for a function $f \in L_{q,1,\alpha,n}$ is defined by

$$F_{q,\Lambda}(f)(\lambda) = c_{q,\alpha+2n}\int_0^\infty f(x)\varphi_{\alpha,n}\left(\lambda x,q^2\right)x^{2\alpha+1}d_q x \tag{3.5}$$

By (3.1) and (3.5) we observe that

$$F_{q,\Lambda} = F_{q,\alpha+2n} \circ M^{-1}, \tag{3.6}$$

where $F_{q,\alpha+2n}$ is the Fourier-Bessel transform of order $\alpha+2n$.

(ii) If $f \in L_{q,1,\alpha,n}$ then $F_{q,\Lambda}(f) \in C_0([0,\infty))$ and

$$\left\|F_{q,\alpha,n}(f)\right\|_{q,\alpha,n,\infty} \leq B_{q,\alpha+2n}\left\|f\right\|_{q,1,\alpha,n}$$

where $B_{q,\alpha+2n}$ is given in [3].

Theorem 3.2 Let $f \in L_{q,1,\alpha,n}$ such that $F_{q,\Lambda}(f) \in L_{q,1,\alpha+2n}$. Then for almost all $x \geq 0$,

$$f(x) = c_{q,\alpha+2n}\int_0^\infty F_{q,\Lambda}(f)(\lambda)\varphi_{\alpha,n}\left(\lambda x, q^2\right)\lambda^{2\alpha+1}d_q\lambda.$$

Proof. By (3.1), (3.6) and theorem 2.1(ii) we have

$$\begin{aligned} c_{q,\alpha+2n}\int_0^\infty F_{q,\Lambda}(f)(\lambda)\varphi_{\alpha,n}\left(\lambda x, q^2\right)\lambda^{2\alpha+1}d_q\lambda &= x^{2n}c_{q,\alpha+2n}\int_0^\infty F_{q,\alpha+2n}\left(M^{-1}f\right)(\lambda)\, j_{\alpha+2n}(\lambda x)\lambda^{2\alpha+1}d_q\lambda \\ &= x^{2n}M^{-1}c_{q,\alpha+2n}\int_0^\infty F_{q,\alpha+2n}(f)(\lambda)\, j_{\alpha+2n}(\lambda x)\lambda^{2\alpha+1}d_q\lambda \\ &= x^{2n}M^{-1}f(x) \\ &= f(x), \end{aligned}$$

for all $x \geq 0$.

Theorem 3.3 (i) For every $f \in L_{q,1,\alpha,n} \cap L_{q,p,\alpha,n}$ space where $p > 2$ we have the Plancherel formula

$$\int_0^\infty \left(f(t)\right)^2 t^{2\alpha+1}d_qt = \int_0^\infty \left(F_{q,\Lambda}(f)\right)^2 d_q\mu_{\alpha+2n}(\lambda).$$

(ii) The inverse of this transform is given by

$$F_{q,\Lambda}{}^{-1}(g)(x) = \int_0^\infty g(\lambda)\varphi_{\alpha,n}\left(\lambda x, q^2\right)d_q\mu_{\alpha+2n}(\lambda).$$

Proof. (i) Let $f \in L_{q,1,\alpha,n} \cap L_{q,p,\alpha,n}$. By (3.6) and theorem 2.1 (iii) we have

$$\begin{aligned} \int_0^\infty \left(F_{q,\Lambda}(f)\right)^2 d_q\mu_{\alpha+2n}(\lambda) &= \int_0^\infty \left(F_{q,\alpha+2n}\left(M^{-1}f\right)(\lambda)\right)^2 d_q\mu_{\alpha+2n}(\lambda) \\ &= \int_0^\infty \left(\left(M^{-1}f(x)\right)\right)^2 x^{2\alpha+4n+1}d_qx \\ &= \int_0^\infty \left(f(x)\right)^2 x^{2\alpha+1}d_qx, \end{aligned}$$

The proof of (ii) is standard.

4. Generalized Convolution Product

Definition 4.1 The generalized translation operator $T^{\alpha}_{q,x,n}$ is define by the relation

$$T^{\alpha}_{q,x,n} = x^{2n} M \circ \tau^{\alpha+2n}_{q,x} \circ M^{-1} \tag{4.1}$$

where $\tau^{\alpha+2n}_{q,x}$ are the Bessel translation operators of order $\alpha+2n$.

Definition 4.2 Define the generalized convolution product of two functions f and g on $[0,\infty)$ by

$$f \#_q g(x) = c_{q,\alpha+2n} \int_0^{\infty} T^{\alpha}_{q,x,n} f(y) g(y) y^{2\alpha+1} d_q y \tag{4.2}$$

where $c_{q,\alpha+2n}$ is given by (1.6).

From by (4.1) we have

$$f \#_q g = M\left[\left(M^{-1} f\right) *_{q,\alpha+2n} \left(M^{-1} g\right)\right], \tag{4.3}$$

where $*_{q,\alpha+2n}$ is the Bessel convolution.

Theorem 4.1 (i) Let f be in $L_{q,1,\alpha,n}$, $1 \le p \le \infty$. Then

$$\left\| T^{\alpha}_{q,x,n} \right\|_{q,1,\alpha,n} \le x^{2n} \| f \|_{q,1,\alpha,n} .$$

(ii) For $f \in L_{q,2,\alpha,n}$, we have

$$F_{q,\Lambda}\left(T^{\alpha}_{q,x,n} f\right)(\lambda) = \varphi_{\alpha,n}\left(\lambda x, q^2\right) F_{q,\alpha,n}(f)(\lambda) .$$

(iii) If $f \in L_{q,1,\alpha,n}$ and $g \in L_{q,1,\alpha,n}$ then

$$\int_0^{\infty} T^{\alpha}_{q,x,n} f(y) g(y) y^{2\alpha+1} d_q y = \int_0^{\infty} f(y) T^{\alpha}_{q,x,n} g(y) y^{2\alpha+1} d_q y.$$

(iv) For $f, g \in L_{q,1,\alpha,n}$ then $f \#_q g \in L_{q,1,\alpha,n}$ and

$$\left\| f \#_q g \right\|_{q,1,\alpha,n} \le \| f \|_{q,1,\alpha,n} \| g \|_{q,1,\alpha,n} .$$

(v) For $f \in L_{q,1,\alpha,n}$ and $g \in L_{q,1,\alpha,n}$ we have

$$F_{q,\Lambda}\left(f \#_q g\right)(\lambda) = F_{q,\Lambda}(f)(\lambda) F_{q,\Lambda}(g)(\lambda).$$

Proof. (i) By (4.1) and Theorem 2.2(i) we have

$$\begin{aligned}
\left\| T^{\alpha}_{q,x,n} f \right\|_{q,1,\alpha,n} &= x^{2n} \left\| M \tau^{\alpha+2n}_{q,x} \circ M^{-1} f \right\|_{q,1,\alpha,n} \\
&= x^{2n} \left\| \tau^{\alpha+2n}_{q,x} \circ M^{-1} f \right\|_{q,1,\alpha+2n} \\
&\le x^{2n} \left\| M^{-1} f \right\|_{q,1,\alpha+2n} \\
&= x^{2n} \| f \|_{q,1,\alpha,n} .
\end{aligned}$$

(ii) By (3.1), (3.6), (4.1) and Theorem 2.2(ii) we have

$$\begin{aligned} F_{q,\Lambda}\left(T^{\alpha}_{q,x,n} f\right)(\lambda) &= F_{q,\alpha+2n} \circ M^{-1}\left(x^{2n}\lambda^{2n}\tau^{\alpha+2n}_{q,x} M^{-1} f\right)(\lambda) \\ &= x^{2n}\lambda^{2n} M^{-1} F_{q,\alpha+2n}\tau^{\alpha+2n}_{q,x}\left(M^{-1} f\right)(\lambda) \\ &= x^{2n}\lambda^{2n} M^{-1} j_{\alpha+2n} F_{q,\alpha+2n} M^{-1} f(\lambda) \\ &= \varphi_{\alpha,n}\left(\lambda x, q^2\right) F_{q,\alpha+2n}\left(M^{-1} f\right)(\lambda) \\ &= \varphi_{\alpha,n}\left(\lambda x, q^2\right) F_{q,\Lambda}(f)(\lambda). \end{aligned}$$

(iii) By (4.1) and Theorem 2.2(iii) we have

$$\begin{aligned} \int_0^\infty T^{\alpha}_{q,x,n} f(y) g(y) y^{2\alpha+1} d_q y &= x^{2n}\int_0^\infty y^{4n}\tau^{\alpha+2n}_{q,x}\left(M^{-1} f\right)(y) M^{-1} g(y) y^{2\alpha+1} d_q y \\ &= x^{2n}\int_0^\infty y^{4n} M^{-1} f(y)\tau^{\alpha+2n}_{q,x} M^{-1} g(y) y^{2\alpha+1} d_q y \\ &= \int_0^\infty y^{2n} M^{-1} f(y)(xy)^{2n}\tau^{\alpha+2n}_{q,x} M^{-1} g(y) y^{2\alpha+1} d_q y \\ &= \int_0^\infty y^{2n} M^{-1} f(y) T^{\alpha}_{q,x,n} g(y) y^{2\alpha+1} d_q y \\ &= \int_0^\infty f(y) T^{\alpha}_{q,x,n} g(y) y^{2\alpha+1} d_q y \ . \end{aligned}$$

(iv) By (4.3) and Theorem 2.2(iv) we have

$$\begin{aligned} \left\| f \#_q g\right\|_{q,1,\alpha,n} &\le \left\| M^{-1}\left(f \#_q g\right)\right\|_{q,1,\alpha+2n} \\ &\le \left\| M^{-1} f\right\|_{q,1,\alpha+2n}\left\| M^{-1} g\right\|_{q,1,\alpha+2n} \\ &= \left\| f\right\|_{q,1,\alpha,n}\left\| g\right\|_{q,1,\alpha,n} \ . \end{aligned}$$

(v) By (3.6), (4.3) and Theorem 2.2(v) we have

$$\begin{aligned} F_{q,\Lambda}\left(f \#_q g\right)(\lambda) &= F_{q,\Lambda}\left(M\left[\left(M^{-1} f\right) *_{q,\alpha+2n}\left(M^{-1} g\right)\right]\right)(\lambda) \\ &= F_{q,\Lambda} \circ M^{-1}\left(M\left[\left(M^{-1} f\right)\#_q\left(M^{-1} g\right)\right]\right)(\lambda) \\ &= F_{q,\Lambda}\left(M^{-1} f\right)(\lambda) F_{q,\Lambda}\left(M^{-1} g\right)(\lambda) \\ &= F_{q,\Lambda}(f)(\lambda) F_{q,\Lambda}(g)(\lambda) \ . \end{aligned}$$

This concludes the proof.

5. Transmutation Operators

Definition 5.1 For a bounded function f on $[0,\infty)$, define the integral transform χ_q by

$$\chi_q f(x) = (1+q) C(\alpha : q^2) x^{2n} \int_0^1 F_\alpha(t : q^2) f(xt) d_q t \;, \tag{5.1}$$

where $C(\alpha : q^2)$ and $F_\alpha(t : q^2)$ is given Theorem 3.1(i).

Remark 5.1 (i) For n=0, χ_q reduces to q-Riemann Liouville integral transform of order α given by

$$R_{\alpha,q}(f)(x) = \begin{cases} (1+q) C(\alpha : q^2) x^{2n} \int_0^1 F_\alpha(t : q^2) f(xt) d_q t \;, if \;\; x > 0 \\ f(0) \;,\;\; x = 0 \;. \end{cases}$$

(ii) It is checked that

$$\chi_q = M \circ R_{\alpha+2n,q} \tag{5.2}$$

(iii) From Theorem 3.1(i) and (5.1) we have

$$\varphi_{\alpha,n}(\lambda x, q^2) = \chi_q\left(\cos(xt, q^2)\right)(x) \tag{5.3}$$

Definition 5.2 Define the integral transform ${}^t\chi_q$ for a differential function f on $[0,\infty)$ by

$${}^t\chi_q f(y) = (1+q) C(\alpha : q^2) \int_{qy}^{\infty} F_\alpha\left(\frac{x}{t} : q^2\right) f(t) \frac{d_q t}{t^{2n-\alpha}}$$

Remark 5.2 (i) For n=0, ${}^t\chi_q$ reduces to q-Weyl integral transform of order α given by

$$W_{\alpha,q}(f)(y) = (1+q) C(\alpha : q^2) \int_0^1 F_\alpha\left(\frac{y}{t} : q^2\right) f(t) t^\alpha d_q t \;,\;\; y \ge 0 \;.$$

(ii) It is seen that

$${}^t\chi_q = W_{\alpha+2n,q} \circ M^{-1} \tag{5.4}$$

Theorem 5.1 (i) If $f \in L_{q,\infty}([0,\infty), dx)$ then $\chi_q f \in L_{q,\infty\alpha,n}$ and $\|\chi_q f\|_{q,\infty,\alpha,n} \le \|f\|_{q,\infty}$.

(ii) If $f \in L_{q,1,\alpha,n}$ then ${}^t\chi_q f \in L_{q,1}([0,\infty), dx)$ and $\|{}^t\chi_q f\|_{q,1} \le \|f\|_{q,1,\alpha,n}$.

(iii) For any $f \in L_{q,1}([0,\infty), dx)$ and $g \in L_{q,1,\alpha,n}$ we have the duality relation

$$\int_0^\infty \chi_q f(x) g(x) x^{2\alpha+1} d_q x = \int_0^\infty f(y)\, {}^t\chi_q g(y) d_q y.$$

(iv) For all $f \in L_{q,1,\alpha,n}$ we have

$$F_{q,\Lambda}(f) = F_{q,C} \circ {}^t\chi_q(f), \tag{5.5}$$

where $F_{q,C}$ is the q-cosine Fourier transform given by

$$F_{q,C}(f)(\lambda) = \int_0^\infty f(x) \cos(\lambda x; q^2) d_q x, \ \ \lambda \geq 0.$$

(v) Let $f, g \in L_{q,1,\alpha,n}$. Then

$${}^t\chi_q(f \#_q g) = {}^t\chi_q f * {}^t\chi_q g,$$

where * is the convolution product defined by

$$f_1 * f_2(x) = \frac{(1+q)^{-1}}{\Gamma_{q^2}(\alpha+1)} \int_0^\infty \sigma_x^\alpha f_1(y) f_2(y) y^{2\alpha+1} d_q y\ ,$$

with σ_x^α is a q-generalized translation given in details in [5].

(vi) Let $f \in L_{q,1,\alpha,n}$ and $g \in L_{q,\infty}([0,\infty), dx)$. Then

$$\chi_q({}^t\chi_q f * g) = f \#_q (\chi_q g). \tag{5.6}$$

Proof. (i) By (5.1) and [5.2] we have

$$\|\chi_q f\|_{q,\infty,\alpha,n} = \|M \circ R_{\alpha,q}\|_{q,\infty,\alpha,n} = \|R_{\alpha+2n} f\|_{q,\infty} \leq \|f\|_{q,\infty}$$

(ii) By (5.1) and [5.4] we have

$$\|{}^t\chi_q f\|_{q,1} \leq \|M^{-1} \circ R_{\alpha,q}\|_{q,1,\alpha+2n} = \|f\|_{q,1,\alpha,n}$$

(iii) By (4.3), (5.2) we have

$$\begin{aligned} \int_0^\infty \chi_q f(x) g(x) x^{2\alpha+1} d_q x &= \int_0^\infty R_{\alpha+2n,q}(f)(x) M^{-1} g(x) x^{2\alpha+4n+1} d_q x \\ &= \int_0^\infty f(y) W_{\alpha+2n,q}(M^{-1} g)(y) d_q y \\ &= \int_0^\infty f(y)\, {}^t\chi_q g(y) d_q y. \end{aligned}$$

(iv) By (3.6), (5.4) we have

$$\begin{aligned} F_{q,C} \circ {}^t\chi_q(f) &= F_{q,C} \circ W_{\alpha+2n,q} \circ M^{-1}(f) \\ &= F_{q,\alpha+2n} \circ M^{-1}(f) \\ &= F_{q,\Lambda}(f). \end{aligned}$$

(v) By (4.3), (5.4) we have

$$\begin{aligned} {}^t\chi_q(f \#_q g) &= W_{\alpha+2n,q}\left[\left(M^{-1}f\right) *_{q,\alpha+2n} \left(M^{-1}g\right)\right] \\ &= \left(W_{\alpha+2n,q}M^{-1}f\right) * \left(W_{\alpha+2n,q}M^{-1}g\right) \\ &= {}^t\chi_q f * {}^t\chi_q g. \end{aligned}$$

(vi) By (3.6), (4.3) ,(5.4) we have

$$\begin{aligned} f \#_q \left(\chi_q g\right) &= M\left[\left(M^{-1}f\right) *_{q,\alpha+2n} \left(M^{-1}\chi_q g\right)\right] \\ &= M\left[\left(M^{-1}f\right) *_{q,\alpha+2n} \left(R_{\alpha+2n,q} g\right)\right] \\ &= MR_{\alpha+2n,q}\left[\left(W_{\alpha+2n,q}M^{-1}f\right) * g\right] \\ &= \chi_q\left({}^t\chi_q f * g\right). \end{aligned}$$

This achieves the proof.

6. Generalized Wavelets

Definition 6.1 A generalized q-Bessel wavelet is a function $g \in L_{q,2,\alpha,n}$ satisfying the admissibility condition

$$0 < C_g = \int_0^\infty \left|F_{q,\Lambda}(g)(\lambda)\right|^2 \frac{d_q\lambda}{\lambda} < \infty. \tag{6.1}$$

Remark 6.1 By (3.6) and (6.1), $g \in L_{q,2,\alpha,n}$ is a generalized q-Bessel wavelet if and only if, $M^{-1}g$ is a q-Bessel wavelet of order $\alpha + 2n$, and we have

$$C_g = \int_0^\infty \left|F_{q,\alpha+2n} \circ M^{-1}(g)(\lambda)\right|^2 \frac{d_q\lambda}{\lambda} = C_{M^{-1}g}^{\alpha+2n}. \tag{6.2}$$

Note 6.1 For $g \in L_{q,2,\alpha,n}$ where $a \in \mathbb{R}_q^+$ and $b \in \mathbb{R}_q^+ \cup \{0\}$ we have

$$g_{a,b,\alpha,n}(x) = a^{1/2} T_{q,b,n}^{\alpha}(g_a)(x) \tag{6.3}$$

where g_a is given in (2.12) and $T_{q,b}^{\alpha}$ are the generalized translation operators defined by (4.1).

Theorem 6.1 For all $a \in \mathbb{R}_q^+$ and $b \in \mathbb{R}_q^+ \cup \{0\}$ we have

$$g_{a,b,\alpha,n}(x) = (bx)^{2n} \left(M^{-1}g\right)_{a,b}^{\alpha+2n}(x) \tag{6.4}$$

Proof. Using (2.11), (4.1) and (6.3) we have

$$\begin{aligned} g_{a,b,\alpha,n}(x) &= a^{1/2} T_{q,b,n}^{\alpha}(g_a)(x) \\ &= (bx)^{2n} a^{1/2} \tau_{a,b}^{\alpha+2n}\left(M^{-1}g_a\right)(x) \\ &= (bx)^{2n} a^{1/2} \tau_{q,b}^{\alpha+2n}\left(M^{-1}g\right)_a(x) \\ &= (bx)^{2n} \left(M^{-1}g\right)_{q,b}^{\alpha+2n}(x), \end{aligned}$$

which ends the proof.

Definition 6.2 Let $g \in L_{q,2,\alpha,n}\left(\mathbb{R}_q^+ \cup \{0\}\right)$ be a generalized a q-Bessel wavelet. Then for a function $f \in L_{q,2,\alpha,n}\left(\mathbb{R}_q^+ \cup \{0\}\right)$, the continuous generalized a q-Bessel wavelet transform by

$$\phi_{q,g,n}^{\alpha}(f)(a,b) = c_{q,\alpha+2n} \int_0^{\infty} f(x) \overline{g_{a,b,\alpha,n}(x)} x^{2\alpha+1} d_q x \ \forall a \in \mathbb{R}_q^+, \forall b \in \mathbb{R}_q^+ \cup \{0\}, \tag{6.5}$$

where $g_{a,b,\alpha,n}(x) = a^{1/2} T_{q,b,n}^{\alpha}(g_a)$ and $g_a = \frac{1}{a^{2\alpha+2}} g(x/a)$.

It can also be written in the form

$$\phi_{q,g,n}^{\alpha}(f)(a,b) = a^{1/2} f \#_q \overline{g_a}(b), \tag{6.6}$$

where $\#_q$ is the generalized convolution product given by (4.2).

Theorem 6.2 We have

$$\phi_{q,g,n}^{\alpha}(f)(a,b) = (b)^{2n} S_{q,M^{-1}g}^{\alpha+2n}\left(M^{-1}f\right)(a,b). \tag{6.7}$$

Proof. From (2.10), (6.4) and (6.5) we deduce that

$$\begin{aligned} \phi_{q,g,n}^{\alpha}(f)(a,b) &= c_{q,\alpha+2n} \int_0^{\infty} f(x) \overline{g_{a,b,\alpha,n}(x)} x^{2\alpha+1} d_q x \\ &= c_{q,\alpha+2n} \int_0^{\infty} f(x) (b)^{2n} \overline{\left(M^{-1}g\right)_{a,b}^{\alpha+2n}(x)} x^{2n} x^{2\alpha+1} d_q x \\ &= c_{q,\alpha+2n} (b)^{2n} \int_0^{\infty} \left(M^{-1}f\right)(x) \overline{\left(M^{-1}g\right)_{a,b}^{\alpha+2n}(x)} x^{2\alpha+4n+1} d_q x \\ &= (b)^{2n} S_{q,M^{-1}g}^{\alpha+2n}\left(M^{-1}f\right)(a,b), \end{aligned}$$

which concludes the proof.

Theorem 6.3 (Plancherel formula) Let $g \in L_{q,2,\alpha,n}\left(\mathbb{R}_q^+ \cup \{0\}\right)$ be a generalized wavelet. For every $f \in L_{q,2,\alpha,n}\left(\mathbb{R}_q^+ \cup \{0\}\right)$ we have the Plancherel formula

$$\int_0^\infty \left| f(x) \right|^2 x^{2\alpha+1} d_q x = \frac{1}{C_g} \int_0^\infty \int_0^\infty \left| \phi_{q,g,n}^{\alpha}(f)(a,b) \right|^2 b^{2\alpha+1} d_q b \frac{d_q a}{a^2}.$$

Proof. By (6.2) and Theorem 2.1(i) we have

$$\begin{aligned}\int_0^\infty \int_0^\infty \left| \phi_{q,g,n}^{\alpha}(f)(a,b) \right|^2 b^{2\alpha+1} d_q b \frac{d_q a}{a^2} &= \int_0^\infty \int_0^\infty \left| S_{q,M^{-1}g}^{\alpha+2n}\left(M^{-1} f\right)(a,b) \right|^2 b^{2\alpha+4n+1} d_q b \frac{d_q a}{a^2} \\ &= C_{M^{-1}g}^{\alpha+2n} \int_0^\infty \left| M^{-1} f(x) \right|^2 x^{2\alpha+4n+1} d_q x \\ &= C_g \int_0^\infty \left| f(x) \right|^2 x^{2\alpha+1} d_q x.\end{aligned}$$

Theorem 6.4 (Calderon's formula) Let $g \in L_{q,2,\alpha,n}$ be a generalized wavelet, such that $\left\| F_{q,\Lambda}(g) \right\|_{q,\alpha} < \infty$. Then for $f \in L_{q,2,\alpha,n}$ and $0 < \in < \delta < \infty$, the function

$$f^{\in,\delta}(x) = \frac{1}{C_g} \int_{\in}^{\delta} \int_0^\infty \phi_{q,g,n}^{\alpha}(f)(a,b) g_{a,b,\alpha,n}(x) b^{2\alpha+1} d_q b \frac{d_q a}{a^2}$$

belongs to $L_{q,2,\alpha,n}$.

Proof. By (6.2), (6.4), (6.7) and theorem 2.1(ii) we have

$$\begin{aligned}\frac{1}{C_g} \int_{\in}^{\delta} \int_0^\infty \phi_{q,g,n}^{\alpha}(f)(a,b) g_{a,b,\alpha,n}(x) b^{2\alpha+1} d_q b \frac{d_q a}{a^2} &= \frac{x^{2n}}{C_{M^{-1}g}^{\alpha+2n}} \int_{\in}^{\delta} \int_0^\infty S_{q,M^{-1}g}^{\alpha+2n}\left(M^{-1} f\right)(a,b)\left(M^{-1} g\right) b^{2\alpha+4n+1} d_q b \frac{d_q a}{a^2} \\ &= f^{\in,\delta}(x).\end{aligned}$$

Theorem 6.5 (Inversion formula) Let $g \in L_{q,2,\alpha,n}$ be a generalized wavelet. If $f \in L_{q,1,\alpha,n}$ and $F_{q,\Lambda}(f) \in L_{q,1,\alpha+2n}$ then we have

$$f(x) = \frac{1}{C_g} \int_0^{\delta} \left(\int_0^\infty \phi_{q,g,n}^{\alpha}(f)(a,b) g_{a,b,\alpha,n}(x) b^{2\alpha+1} d_q b \right) \frac{d_q a}{a^2}$$

for $x \geq 0$.

Proof. By (6.2), (6.4), (6.7 we have

$$\frac{1}{C_g}\int_0^{\infty}\left(\int_0^{\infty}\phi_{q,g,n}^{\alpha}\left(f\right)\left(a,b\right)g_{a,b,\alpha,n}\left(x\right)b^{2\alpha+1}d_qb\right)\frac{d_qa}{a^2}=\frac{x^{2n}}{C_{M^{-1}g}^{\alpha+2n}}\int_0^{\infty}\left(\int_0^{\infty}S_{q,M^{-1}g}^{\alpha+2n}\left(M^{-1}f\right)\left(a,b\right)\left(M^{-1}g\right)b^{2\alpha+4n+1}d_qb\right)\frac{d_qa}{a^2},$$

the result shows from theorem 2.1(iii).

7. Inversion of the Intertwining Operator ${}^t\chi_q$ Through the Generalized Wavelet Transform

To obtain inversion formulas or ${}^t\chi_q$ involving generalized wavelets, we have to establish some preliminary lemmas.

Lemma 7.1 Let $0\neq g\in L_{q,1,\alpha,n}\cap L_{q,2,\alpha,n}\left([0,\infty[,dx\right)$ such that $F_c\left(g\right)\in L_{q,1,\alpha,n}\left([0,\infty[,dx\right)$ and satisfying

$$\exists\eta>\alpha+2n \text{ such that } F_c\left(g\right)\left(\lambda\right)=O\left(\lambda^{n}\right) \tag{7.1}$$

as $\lambda\to 0$. Then $\chi_{q,g}\in L_{q,2,\alpha,n}$ and

$$F_c\left(\chi_{q,g}\right)\left(\lambda\right)=\frac{2^{2\alpha+4n+1}\left(\Gamma\left(\alpha+2n+1\right)\right)^2}{\pi\lambda^{2\alpha+4n+1}}F_c\left(g\right)\left(\lambda\right). \tag{7.2}$$

Proof. We have

$$g\left(x\right)=\frac{2}{\pi}\int_0^{\infty}F_c\left(g\right)\left(\lambda\right)\cos\left(\lambda x\right)d\lambda.$$

So by (5.3),

$$\chi_q g\left(x\right)=\int_0^{\infty}h\left(\lambda\right)\varphi_{\lambda}\left(x\right)d\mu_{\alpha+2n}\left(\lambda\right),$$

where

$$h\left(\lambda\right)=\frac{2^{2\alpha+4n+1}\left(\Gamma\left(\alpha+2n+1\right)\right)^2}{\pi\lambda^{2\alpha+4n+1}}F_c\left(g\right)\left(\lambda\right)$$

Clearly, $h\in L_{q,1,\alpha,n}\left([0,\infty[,dx\right)$. So by (7.2) and Theorem 6.3 we have

$$\begin{aligned}\int_0^{\infty}\left|h\left(\lambda\right)\right|^2d\mu_{\alpha+2n}\left(\lambda\right)&=m\left(\alpha,n\right)\int_0^{\infty}\lambda^{-2\alpha-4n-1}\left|F_c\left(g\right)\left(\lambda\right)\right|^2d\lambda\\&=m\left(\alpha,n\right)\left(\int_0^{1}+\int_1^{\infty}\right)\lambda^{-2\alpha-4n-1}\left|F_c\left(g\right)\left(\lambda\right)\right|^2d\lambda\\&=m\left(\alpha,n\right)\left(I_1+I_2\right),\end{aligned}$$

where $m(\alpha,n)=4^{\alpha+4n+1}\pi^{-2}\left(\Gamma(\alpha+4n+1)\right)^2$. By (7.1) there is a positive constant k such that

$$I_1 \le k\int_0^\infty \lambda^{2\eta-2\alpha-4n-1}d\lambda=\frac{k}{2(\eta-\alpha-2n)}<\infty.$$

From the Plancherel theorem for the cosine transform, it follows that

$$I_2=\int_1^\infty \lambda^{-2\alpha-4n-1}\left|F_c(g)(\lambda)\right|^2 d\lambda \le \int_0^\infty \left|F_c(g)(\lambda)\right|^2 d\lambda=\frac{\pi}{2}\int_0^\infty \left|g(x)\right|^2 dx<\infty,$$

which achieves the proof.

Lemma 7.2 Let $0\ne g\in L_{q,1,\alpha,n}\cap L_{q,2,\alpha,n}([0,\infty[,dx)$ such that $F_c(g)\in L_{q,1,\alpha,n}([0,\infty[,dx)$ and satisfying

$\eta>2\alpha+4n+1$ such that

$$F_c(g)(\lambda)=O\left(\lambda^\eta\right) \tag{7.3}$$

as $\lambda\to 0$. Then $\chi_q g\in L_{q,2,\alpha,n}$ is a generalized wavelet and $F_c\left(\chi_{q,g}\right)\in L_{q,\infty,\alpha,n}([0,\infty[,dx)$.

Proof. By (7.3) and Lemma 7.1 , $\chi_q g\in L_{q,2,\alpha,n}$, $F_\Lambda\left(\chi_{q,g}\right)$ is bounded and

$$F_\Lambda\left(\chi_{q,g}\right)(\lambda)=O\left(\lambda^{\eta-2\alpha-4n-1}\right)\ as\ \lambda\to 0.$$

Hence $\chi_q g$ satisfies the admissibility condition (6.1).

The continuous wavelet transform on $[0,\infty)$ is defined by

$$W_{q,g}(f)(a,b)=\frac{1}{a}\int_0^\infty f(x)\overline{\sigma_b(g_a)(x)}dx, \tag{7.4}$$

where $a>0, b\ge 0$ and $g\in L_{q,2,\alpha,n}([0,\infty[,dx)$ is a classical wavelet on $[0,\infty)$, i.e., satisfies the admissibility condition

$$0<C_q(g)=\int_0^\infty \left|F_c(g)(\lambda)\right|^2\frac{d\lambda}{\lambda}<\infty. \tag{7.5}$$

Remark 7.1 (ii) By (5.5), (6.1) and (7.5), $g\in D(\mathbb{R})$ is a generalized wavelet, if and only if ${}^t\chi_{q,g}$ is a wavelet and

$$C\left({}^t\chi_{q,g}\right)=C_g\ .$$

Lemma 7.3 Let g be as in Lemma 7.2. Then $\forall f\in L_{q,p,\alpha,n}$, p=1 or 2, we have

$$\phi_{\chi_{q,g}}(f)(a,b)=\frac{1}{a^{2\alpha+4n+1}}\chi_q\left[W_{q,g}\left({}^t\chi_q f\right)(a,\cdot)\right](b).$$

Proof. By (6.6) we have

$$\phi_{\chi_{q,g}}(f)(a,b)=\frac{1}{a^{2\alpha+4n+1}}f\neq_q\overline{\left(\chi_{q,g}\right)_a}(b).$$

But

$$\overline{\left(\chi_{q,g}\right)_a}=\frac{1}{a^{2n}}\chi_q\left(\overline{g_a}\right)$$

by (2.12) and (5.1). So by (5.6) and (7.4) we get

$$\begin{aligned}\phi_{\chi_{q,g}}(f)(a,b)&=\frac{1}{a^{2\alpha+4n+1}}f\neq_q\left[\chi_q\left(\overline{g_a}\right)\right](b)\\&=\frac{1}{a^{2\alpha+4n+1}}\chi_q\left[{}^t\chi_q f*\overline{g_a}\right](b)\\&=\frac{1}{a^{2\alpha+4n+1}}\chi_q\left[W_{q,g}\left({}^t\chi_q\right)(a,\cdot)\right](b),\end{aligned}$$

which completes the proof.

Theorem 7.1 Let g be as in Lemma 7.2. Then we have the following inversion formulas for ${}^t\chi_q$:

(i) If $f\in L_{q,1,\alpha,n}$ and $F_{q,\Lambda}(f)\in L_{q,1,\alpha+2n}$ then for almost all $x\geq 0$ we have

$$f(x)=\frac{1}{C_{\chi_{q,g}}}\int_0^\infty\left(\int_0^\infty\chi_q\left[W_{q,g}\left({}^t\chi_q f\right)(a,\cdot)\right](b)\times\left(\chi_{q,g}\right)_{a,b}(x)b^{2\alpha+1}db\right)\frac{da}{a^{2\alpha+4n+2}}.$$

(ii) For $f\in L_{q,1,\alpha,n}\cap L_{q,2,\alpha,n}$ and $0<\in<\delta<\infty$, the function

$$f^{\in,\delta}(x)=\frac{1}{C_{\chi_{q,g}}}\int_\in^\delta\int_0^\infty\chi_q\left[W_{q,g}\left({}^t\chi_q f\right)(a,\cdot)\right](b)\times\left(\chi_{q,g}\right)_{a,b}(x)b^{2\alpha+1}db\frac{da}{a^{2\alpha+4n+2}}$$

satisfies

$$\lim_{\in\to 0,\delta\to\infty}\left\|f^{\in,\delta}-f\right\|_{q,2,\alpha,n}=0.$$

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