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Lattice Valued Fuzzy Sets in UP (BCC)-Algebras

Aiyared Iampan[1,*], Akarachai Satirad[1], Ronnason Chinram[2], Rukchart Prasertpong[3], Pongpun Julatha[4]

[1]*Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[2]*Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*

[3]*Division of Mathematics and Statistics, Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Nakhon Sawan 60000, Thailand*

[4]*Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand*

[*]*Corresponding author:* aiyared.ia@up.ac.th

Abstract. The aim of this paper is to apply the concept of $\mathcal{L}$-fuzzy sets (LFSs) to UP (BCC)-algebras and introduce five types of LFSs in UP (BCC)-algebras: $\mathcal{L}$-fuzzy UP (BCC)-subalgebras, $\mathcal{L}$-fuzzy near UP (BCC)-filters, $\mathcal{L}$-fuzzy UP (BCC)-filters, $\mathcal{L}$-fuzzy UP (BCC)-ideals, and $\mathcal{L}$-fuzzy strong UP (BCC)-ideals. Also, we study the characteristic LFSs, t-level subsets, and the Cartesian product of LFSs in UP (BCC)-algebras.

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [16], BCI-algebras [17], β-algebras [27], BG-algebras [24], BP-algebras [1], UP-algebras [11], fully UP-semigroups [12], topological UP-algebras [32], UP-hyperalgebras [14], extension of KU/UP-algebras [31] and others. They are strongly connected with logic. In 2022, Jun et al. [19] have shown that the concept of UP-algebras (see [11]) and the concept of BCC-algebras

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(see [25]) are the same concept. Therefore, in this article and future research, our research team will use the name BCC instead of UP in honor of Komori, who first defined it in 1984.

The concept of fuzzy sets was first considered by Zadeh [39] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh, Atanassov [4] defined new concept called intuitionistic fuzzy set which is a generalization of fuzzy set, Goguen [8] generalized the notion of fuzzy sets into the notion of LFSs. Lee [26], introduced an extension of fuzzy sets named bipolar-valued fuzzy sets.

The concept of LFSs was applied to many logical algebras such as: in 2010, Chandramouleeswaran [5] introduced the notions of intuitionistic $\mathcal{L}$-fuzzy subalgebras of a BG-algebras and some of their basic properties. In 2012, Subramanian et al. [38] introduced the notions of interval-valued bipolar fuzzy lattices, Cartesian products, fuzzy fully invariant lattices, characteristic and homomorphic image of bipolar fuzzy lattices, and then they investigated several properties. In 2014, Rajam and Chandramouleeswaran [30] introduced the notion of $\mathcal{L}$-fuzzy β-subalgebras on β-algebras and investigate some of their properties. In 2017, Christopher Jefferson and Chandramouleeswaran [18] studied the notions of fuzzy structures, fuzzy subalgebras, fuzzy ideals, and $\mathcal{L}$-fuzzy subalgebras and T-ideals of BP-algebras.

In this paper, we apply the concept of $\mathcal{L}$-fuzzy sets (LFSs) to BCC-algebras and introduce five types of LFSs in BCC-algebras: $\mathcal{L}$-fuzzy BCC-subalgebras, $\mathcal{L}$-fuzzy near BCC-filters, $\mathcal{L}$-fuzzy BCC-filters, $\mathcal{L}$-fuzzy BCC-ideals, and $\mathcal{L}$-fuzzy strong BCC-ideals. Also, we study the characteristic LFSs, t-level subsets, and the Cartesian product of LFSs in BCC-algebras.

The concept of BCC-algebras (see [25]) can be redefined without the condition (1.1) as follows:

Definition 1.1. *[10] A BCC-algebra is one that has the algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$ of type $(2, 0)$, where $\mathcal{U}$ is a nonempty set, $\star$ is a binary operation on $\mathcal{U}$, and 0 is a fixed element of $\mathcal{U}$ if it meets the following axioms:*

$$(\forall a, b, c \in \mathcal{U})((b \star c) \star ((a \star b) \star (a \star c)) = 0), \tag{BCC-1}$$

$$(\forall a \in \mathcal{U})(0 \star a = a), \tag{BCC-2}$$

$$(\forall a \in \mathcal{U})(a \star 0 = 0), \tag{BCC-3}$$

$$(\forall a, b \in \mathcal{U})((a \star b = 0, b \star a = 0) \Rightarrow a = b). \tag{BCC-4}$$

For more examples of BCC-algebras, see [2, 3, 7, 12, 15, 33–36]. According to [11], we know that the concept of BCC-algebras is a generalization of KU-algebras (see [29]).

From now on, unless otherwise stated, $\mathcal{U} = (\mathcal{U}, \star, 0)$ is a BCC-algebra.

In $\mathcal{U}$, the following assertions are valid (see [11, 12]).

$$(\forall a \in \mathcal{U})(a \star a = 0), \tag{1.1}$$

$$(\forall a, b, c \in \mathcal{U})((a \star b = 0, b \star c = 0) \Rightarrow a \star c = 0), \tag{1.2}$$

$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow (c \star a) \star (c \star b) = 0), \tag{1.3}$$

$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow (b \star c) \star (a \star c) = 0), \tag{1.4}$$

$$(\forall a, b \in \mathcal{U})(a \star (b \star a) = 0), \tag{1.5}$$

$$(\forall a, b \in \mathcal{U})((b \star a) \star a = 0 \Leftrightarrow a = b \star a), \tag{1.6}$$

$$(\forall a, b \in \mathcal{U})(a \star (b \star b) = 0), \tag{1.7}$$

$$(\forall u, a, b, c \in \mathcal{U})((a \star (b \star c)) \star (a \star ((u \star b) \star (u \star c))) = 0), \tag{1.8}$$

$$(\forall u, a, b, c \in \mathcal{U})((((u \star a) \star (u \star b)) \star c) \star ((a \star b) \star c) = 0), \tag{1.9}$$

$$(\forall a, b, c \in \mathcal{U})(((a \star b) \star c) \star (b \star c) = 0), \tag{1.10}$$

$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow a \star (c \star b) = 0), \tag{1.11}$$

$$(\forall a, b, c \in \mathcal{U})(((a \star b) \star c) \star (a \star (b \star c)) = 0), \tag{1.12}$$

$$(\forall u, a, b, c \in \mathcal{U})(((a \star b) \star c) \star (b \star (u \star c)) = 0). \tag{1.13}$$

According to [11], the binary relation $\leq$ on $\mathcal{U}$ is defined as follows:

$$(\forall a, b \in \mathcal{U})(a \leq b \Leftrightarrow a \star b = 0).$$

Definition 1.2. *[9, 11, 13, 20–22, 37] A nonempty subset S of $\mathcal{U}$ is called*

(1) *a BCC-subalgebra of $\mathcal{U}$ if it satisfies the following condition:*

$$(\forall a, b \in S)(a \star b \in S), \tag{1.14}$$

(2) *a near BCC-filter of $\mathcal{U}$ if it satisfies the following condition:*

$$(\forall a, b \in \mathcal{U})(b \in S \Rightarrow a \star b \in S), \tag{1.15}$$

(3) *a BCC-filter of $\mathcal{U}$ if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } \mathcal{U} \text{ is in } S, \tag{1.16}$$

$$(\forall a, b \in \mathcal{U})((a \star b \in S, a \in S) \Rightarrow b \in S), \tag{1.17}$$

(4) *a BCC-ideal of $\mathcal{U}$ if it satisfies the condition (1.16) and the following condition:*

$$(\forall a, b, c \in \mathcal{U})((a \star (b \star c) \in S, b \in S) \Rightarrow a \star c \in S), \tag{1.18}$$

(5) *a strong BCC-ideal of $\mathcal{U}$ if it satisfies the condition (1.16) and the following condition:*

$$(\forall a, b, c \in \mathcal{U})(((c \star b) \star (c \star a)) \in S, b \in S) \Rightarrow a \in S). \tag{1.19}$$

We proved that the concept of BCC-subalgebras is a generalization of near BCC-filters, near BCC-filters is a generalization of BCC-filters, BCC-filters is a generalization of BCC-ideals, BCC-ideals is a generalization of strong BCC-ideals. They also proved that $\mathcal{U}$ is the only strong BCC-ideal.

Definition 1.3. *[39] A fuzzy set (FS) L in a nonempty set $\mathcal{U}$ is described by its membership function $\bar{}_{\mathsf{F}}$. To every point $a \in \mathcal{U}$, this function associates a real number $\bar{}_{\mathsf{F}}(a)$ in the closed interval $[0, 1]$. The real number $\bar{}_{\mathsf{F}}(a)$ is interpreted for the point as a degree of membership of an object $a \in \mathcal{U}$ to the FS L, that is, $\mathsf{L} := \{(a, \bar{}_{\mathsf{F}}(a)) \mid a \in \mathcal{U}\}$. We say that a FS L in $\mathcal{U}$ is constant fuzzy set if its membership function $\bar{}_{\mathsf{F}}$ is constant. If $A \subseteq \mathcal{U}$ and $t \in (0, 1]$, the t-characteristic function [23] χ_A^t of $\mathcal{U}$ is a function of $\mathcal{U}$ into $\{0, t\}$ defined as follows:*

$$\chi_A^t(a) = \begin{cases} t & \textit{if } a \in A, \\ 0 & \textit{otherwise}. \end{cases}$$

By the definition of t-characteristic function, χ_A^t is a function of $\mathcal{U}$ into $\{0, t\} \subset [0, 1]$. We denote the fuzzy set L_A^t in $\mathcal{U}$ is described by its membership function χ_A^t, is called the t-characteristic fuzzy set of A in $\mathcal{U}$.

Definition 1.4. *[6] An ordered set (or partially ordered set) $\mathcal{L} = (\mathcal{L}, \leq)$ equipped with a nonempty set $\mathcal{L}$ and a binary relation $\leq$ on $\mathcal{L}$ if it meet the following axioms:*

$$(\forall u \in \mathcal{L})(u \leq u), \qquad \text{(reflexivity)}$$

$$(\forall u, v \in \mathcal{L})(u \leq v, v \leq u \Rightarrow u = v)), \qquad \text{(anti-symmetry)}$$

$$(\forall u, v, w \in \mathcal{L})(u \leq v, v \leq w \Rightarrow u \leq w). \qquad \text{(transitivity)}$$

Definition 1.5. *[6] An ordered set $\mathcal{L} = (\mathcal{L}, \leq)$ is called a linearly ordered set if it satisfies the following condition:*

$$(\forall u, v \in \mathcal{L})(\textit{either } u \leq v \textit{ or } v \leq u).$$

We call a relation $\leq$ on $\mathcal{L}$ that a linear order.

In this paper, for each elements u, v of an ordered set $\mathcal{L}$, we shall write $u \vee v$ (read as u join v) in place of $\sup\{u, v\}$ and $u \wedge v$ (read as u meet v) in place of $\inf\{u, v\}$ if them exist. Similarly, for subset S of $\mathcal{L}$, we write $\bigvee S$ (read as join of S) in place of $\sup S$ and $\bigwedge S$ (read as meet of S) in place of $\inf S$ if them exist.

Definition 1.6. *[6] Let $\mathcal{L} = (\mathcal{L}, \leq)$ be a nonempty ordered set. Then an ordered set $\mathcal{L}$ with sup operation $\vee$ and inf operation $\wedge$ on $\mathcal{L}$ is called*

(1) *a lattice if*

$$(\forall u, v \in \mathcal{L})(u \vee v \textit{ and } u \wedge v \textit{ exist}),$$

(2) *a complete lattice if*

$$(\forall S \subseteq \mathcal{L})(\bigvee S \textit{ and } \bigwedge S \textit{ exist}).$$

We write $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ to denote a lattice.

For a complete lattice $\mathcal{L}$ is easy to verify that it has the least element $0_{\mathcal{L}}$ and the greatest element $1_{\mathcal{L}}$. So we denote a complete lattice by $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, 0_{\mathcal{L}}, 1_{\mathcal{L}})$.

Definition 1.7. *[6] Let $\mathcal{L}$ be a lattice with the least element $0_{\mathcal{L}}$ and the greatest element $1_{\mathcal{L}}$. For $u \in \mathcal{L}$, we say $v \in \mathcal{L}$ is a complement of u if*

$$u \wedge v = 0_{\mathcal{L}}, u \vee v = 1_{\mathcal{L}}.$$

If u has the unique complement, we denote this complement by $u^{'}$.

Definition 1.8. *[6] A lattice $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ is called a Boolean lattice if it satisfies the following conditions:*

(1) *$\mathcal{L}$ is distributive,*
(2) *$\mathcal{L}$ has the least element $0_{\mathcal{L}}$ and the greatest element $1_{\mathcal{L}}$.*

For a Boolean lattice $\mathcal{L}$ is easy to verify that for each $u \in \mathcal{L}$, $u^{'} \in \mathcal{L}$ exists. So we denote a Boolean lattice by $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ^{'}, 0_{\mathcal{L}}, 1_{\mathcal{L}})$.

Lemma 1.1. *[6] Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ^{'}, 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Then the following statements hold:*

(1) $(\forall u, v \in \mathcal{L})((u \vee v)^{'} = u^{'} \wedge v^{'})$,
(2) $(\forall u, v \in \mathcal{L})((u \wedge v)^{'} = u^{'} \vee v^{'})$,
(3) $(\forall u, v \in \mathcal{L})(u \leq v \Leftrightarrow u^{'} \geq v^{'})$,
(4) $(\forall u, v \in \mathcal{L})(u = v \Leftrightarrow u^{'} = v^{'})$,
(5) $(\forall u, v \in \mathcal{L})(u < v \Leftrightarrow u^{'} > v^{'})$.

Definition 1.9. *[8] Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ be a lattice. An $\mathcal{L}$-fuzzy set (LFS) L in a nonempty set $\mathcal{U}$ is described by its membership function $\bar{}_{\mathrm{L}}$. To every point $a \in \mathcal{U}$, this function associates an element $\bar{}_{\mathrm{L}}(a)$ in $\mathcal{L}$. The element $\bar{}_{\mathrm{L}}(a)$ is interpreted for the point as a degree of membership of an object $a \in \mathcal{U}$ to the LFS L, that is, $\mathrm{L} := \{(a, \bar{}_{\mathrm{L}}(a)) \mid a \in \mathcal{U}\}$. We say that an LFS L in $\mathcal{U}$ is a constant LFS if its membership function $\bar{}_{\mathrm{L}}$ is constant.*

2. LFSs in BCC-algebras

In this section, we introduce the new concepts of LFSs in BCC-algebras: $\mathcal{L}$-fuzzy BCC-subalgebras, $\mathcal{L}$-fuzzy near BCC-filters, $\mathcal{L}$-fuzzy BCC-filters, $\mathcal{L}$-fuzzy BCC-ideals, and $\mathcal{L}$-fuzzy strong BCC-ideals, and provide their properties and relationships.

Definition 2.1. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ be a lattice. Then an LFS L in $\mathcal{U}$ is called*

(1) *an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$ if it satisfies the following condition:*

$$(\forall a, b \in \mathcal{U})(\bar{}_{\mathrm{L}}(a \star b) \geq \bar{}_{\mathrm{L}}(a) \wedge \bar{}_{\mathrm{L}}(b)), \tag{2.1}$$

(2) *an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$ if it satisfies the following condition:*

$$(\forall a, b \in \mathcal{U})(\bar{}_{\mathrm{L}}(a \star b) \geq \bar{}_{\mathrm{L}}(b)), \tag{2.2}$$

(3) *an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$ if it satisfies the following conditions:*

$$(\forall a \in \mathcal{U})(\bar{}_{\mathrm{L}}(0) \geq \bar{}_{\mathrm{L}}(a)), \tag{2.3}$$

$$(\forall a, b \in \mathcal{U})(\bar{}_{\mathrm{L}}(b) \geq \bar{}_{\mathrm{L}}(a \star b) \wedge \bar{}_{\mathrm{L}}(a)), \tag{2.4}$$

(4) *an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$ if it satisfies the condition* (2.3) *and the following condition:*

$$(\forall a, b, c \in \mathcal{U})(\bar{}_{\mathrm{L}}(a \star c) \geq \bar{}_{\mathrm{L}}(a \star (b \star c)) \wedge \bar{}_{\mathrm{L}}(b)), \tag{2.5}$$

(5) *an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$ if it satisfies the condition* (2.3) *and the following condition:*

$$(\forall a, b, c \in \mathcal{U})(\bar{}_{\mathrm{L}}(a) \geq \bar{}_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \bar{}_{\mathrm{L}}(b)). \tag{2.6}$$

Theorem 2.1. *An LFS in $\mathcal{U}$ is an $\mathcal{L}$-fuzzy strong BCC-ideal if and only if it is constant.*

Proof. Assume that L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. Then it satisfies (2.3). Thus for all $a \in \mathcal{U}$,

$$\begin{aligned}
\bar{}_{\mathrm{L}}(a) &\geq \bar{}_{\mathrm{L}}((a \star 0) \star (a \star a)) \wedge \bar{}_{\mathrm{L}}(0)\} && ((2.6))\\
&= \bar{}_{\mathrm{L}}(0 \star (a \star a)) \wedge \bar{}_{\mathrm{L}}(0)\} && (\text{BCC-3})\\
&= \bar{}_{\mathrm{L}}(a \star a) \wedge \bar{}_{\mathrm{L}}(0)\} && (\text{BCC-2})\\
&= \bar{}_{\mathrm{L}}(0) \wedge \bar{}_{\mathrm{L}}(0)\} && ((1.1))\\
&= \bar{}_{\mathrm{L}}(0).
\end{aligned}$$

Since $\bar{}_{\mathrm{L}}(0) \geq \bar{}_{\mathrm{L}}(a)$, we have $\bar{}_{\mathrm{L}}(x) = \bar{}_{\mathrm{L}}(0)$ for all $a \in \mathcal{U}$. Hence, $\bar{}_{\mathrm{L}}$ is constant, that is, L is constant.

The converse is obvious because L is constant. □

Theorem 2.2. *Every $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$ is an $\mathcal{L}$-fuzzy BCC-subalgebra.*

Proof. Let L be an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. Then for all $a, b \in \mathcal{U}$,

$$\begin{aligned}
\bar{}_{\mathrm{L}}(a \star b) &\geq \bar{}_{\mathrm{L}}(b) && ((2.2))\\
&\geq \bar{}_{\mathrm{L}}(a) \wedge \bar{}_{\mathrm{L}}(b).
\end{aligned}$$

Therefore, L is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. □

The converse of Theorem 2.2 does not hold in general. This is shown by the following example.

Example 2.1. *Consider a BCC-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

Consider a lattice $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$, where $\mathcal{L} = \{a, b, c, d, e\}$ is drawn in Figure 1.

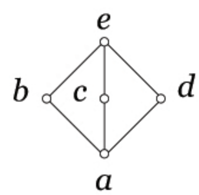

Figure 1. Lattice

We define an LFS L *in $\mathcal{U}$ as follows:*

$$\mu_{\mathrm{L}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ e & a & c & a \end{pmatrix}.$$

Then L *is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. Since $\mu_{\mathrm{L}}(1 \star 2) = \mu_{\mathrm{L}}(1) = a \ngeq c = \mu_{\mathrm{L}}(2)$, we have* L *is not an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$.*

Theorem 2.3. *Every $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$ is an $\mathcal{L}$-fuzzy near BCC-filter.*

Proof. Let L be an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. Then for all $a, b \in \mathcal{U}$,

$$\begin{aligned} \bar{\ }_{\mathrm{L}}(a \star b) &\geq \bar{\ }_{\mathrm{L}}(b \star (a \star b)) \wedge \bar{\ }_{\mathrm{L}}(b) && ((2.4)) \\ &= \bar{\ }_{\mathrm{L}}(0) \wedge \bar{\ }_{\mathrm{L}}(b) && ((1.5)) \\ &= \bar{\ }_{\mathrm{L}}(b). && ((2.3)) \end{aligned}$$

Therefore, L is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. □

The converse of Theorem 2.3 does not hold in general. This is shown by the following example.

Example 2.2. *Consider a BCC-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

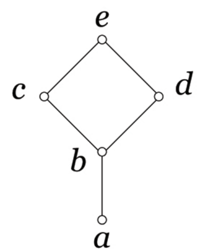

Figure 2. Lattice

Consider a lattice $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$, where $\mathcal{L} = \{a, b, c, d, e\}$ is drawn in Figure 2.
We define an LFS L in $\mathcal{U}$ as follows:

$$\mu_{\mathrm{L}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ e & b & d & b \end{pmatrix}.$$

Then L is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. Since $\mu_{\mathrm{L}}(1) = b \ngeq d = e \wedge d = \mu_{\mathrm{L}}(0) \wedge \mu_{\mathrm{L}}(2) = \mu_{\mathrm{L}}(2 \star 1) \wedge \mu_{\mathrm{L}}(2)$, we have L is not an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$.

Theorem 2.4. *Every $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$ is an $\mathcal{L}$-fuzzy BCC-filter.*

Proof. Let L be an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. It is sufficient to prove the condition (2.3). Then for all $a, b \in \mathcal{U}$,

$$\begin{aligned} {}^{-}{}_{\mathrm{L}}(b) &= {}^{-}{}_{\mathrm{L}}(0 \star b) && \text{(BCC-2)} \\ &\geq {}^{-}{}_{\mathrm{L}}(0 \star (a \star b)) \wedge {}^{-}{}_{\mathrm{L}}(a) && \text{((2.5))} \\ &= {}^{-}{}_{\mathrm{L}}(a \star b) \wedge {}^{-}{}_{\mathrm{L}}(a). && \text{(BCC-2)} \end{aligned}$$

Therefore, L is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. □

The converse of Theorem 2.4 does not hold in general. This is shown by the following example.

Example 2.3. *Consider a BCC-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Consider a lattice $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$, where $\mathcal{L} = \{a, b, c, d, e, f\}$ is drawn in Figure 3.
We define an LFS L in $\mathcal{U}$ as follows:

$$\mu_{\mathrm{L}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ f & d & e & e \end{pmatrix}.$$

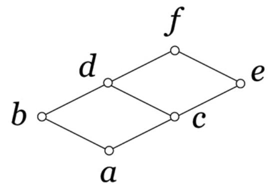

Figure 3. Lattice

Then L *is an* $\mathcal{L}$*-fuzzy BCC-filter of* $\mathcal{U}$*. Since* $\mu_{\mathrm{L}}(2 \star 3) = \mu_{\mathrm{L}}(2) = e \ngeq d = f \wedge d = \mu_{\mathrm{L}}(0) \wedge \mu_{\mathrm{L}}(1) = \mu_{\mathrm{L}}(2 \star (1 \star 3)) \wedge \mu_{\mathrm{L}}(1)$*, we have* L *is not an* $\mathcal{L}$*-fuzzy BCC-ideal of* $\mathcal{U}$*.*

Theorem 2.5. *Every* $\mathcal{L}$*-fuzzy strong BCC-ideal of* $\mathcal{U}$ *is an* $\mathcal{L}$*-fuzzy BCC-ideal.*

Proof. Let L be an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. By Theorem 2.1, we have L is constant. Therefore, it is obvious that L is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. □

The converse of Theorem 2.5 does not hold in general. This is shown by the following example.

Example 2.4. *Consider a BCC-algebra* $\mathcal{U} = (\mathcal{U}, \star, 0)$*, where* $\mathcal{U} = \{0, 1, 2, 3\}$ *is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Consider a lattice $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$*, where* $\mathcal{L} = \{a, b, c, d, e, f, g\}$ *is drawn in Figure 4.*

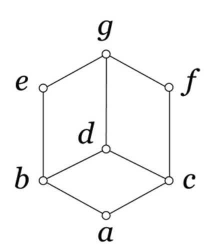

Figure 4. Lattice

We define an LFS L *in* $\mathcal{U}$ *as follows:*

$$\mu_{\mathrm{L}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ g & c & a & e \end{pmatrix}.$$

Then L *is an* $\mathcal{L}$*-fuzzy BCC-ideal of* $\mathcal{U}$*. Since* $\mu_{\mathrm{L}}(2) = a \ngeq c = g \wedge c = \mu_{\mathrm{L}}(0) \wedge \mu_{\mathrm{L}}(1) = \mu_{\mathrm{L}}((2 \star 1) \star (2 \star 2)) \wedge \mu_{\mathrm{L}}(1)$*, we have* L *is not an* $\mathcal{L}$*-fuzzy strong BCC-ideal of* $\mathcal{U}$*.*

3. Characteristic LFSs

Our aim in this section is to study the relation between special subsets and special LFSs in BCC-algebras. In this section only, we shall determine $\mathcal{L}$ is a complete lattice $(\mathcal{L}, \leq, \vee, \wedge, 0_{\mathcal{L}}, 1_{\mathcal{L}})$.

Let A be a subset of $\mathcal{U}$. Then the *characteristic function* χ_A of $\mathcal{U}$ is a function of $\mathcal{U}$ into $\{1_{\mathcal{L}}, 0_{\mathcal{L}}\}$ defined as follows:

$$\chi_A(a) = \begin{cases} 1_{\mathcal{L}} & \text{if } a \in A, \\ 0_{\mathcal{L}} & \text{otherwise.} \end{cases}$$

By the definition of characteristic function, χ_A is a function of $\mathcal{U}$ into $\{1_{\mathcal{L}}, 0_{\mathcal{L}}\} \subset \mathcal{L}$. We denote the LFS L_A in $\mathcal{U}$ is described by its membership function χ_A, is called the *characteristic LFS* of A in $\mathcal{U}$.

Lemma 3.1. *Let the constant* 0 *of* $\mathcal{U}$ *is in* A*. Then* $\chi_A(0) \geq \chi_A(a)$ *for all* $a \in \mathcal{U}$*.*

Proof. Assume that $0 \in A$. Then for all $a \in \mathcal{U}$, $\chi_A(0) = 1_{\mathcal{L}} \geq \chi_A(a)$. □

Lemma 3.2. *Let* A *be a nonempty subset of a BCC-algebra* $\mathcal{U}$*. If* $\chi_A(0) \geq \chi_A(a)$ *for all* $a \in \mathcal{U}$*, then the constant* 0 *of* $\mathcal{U}$ *is in* A*.*

Proof. Assume that $\chi_A(0) \geq \chi_A(a)$ for all $a \in \mathcal{U}$. Since A is a nonempty subset of $\mathcal{U}$, we have an element u in A, that is, $\chi_A(u) = 1_{\mathcal{L}}$. Thus $1_{\mathcal{L}} \geq \chi_A(0) \geq \chi_A(u) = 1_{\mathcal{L}}$. So $\chi_A(0) = 1_{\mathcal{L}}$, that is, $0 \in A$. □

Theorem 3.1. *A nonempty subset* A *of* $\mathcal{U}$ *is a BCC-subalgebra of* $\mathcal{U}$ *if and only if the characteristic LFS* L_A *is an* $\mathcal{L}$*-fuzzy BCC-subalgebra of* $\mathcal{U}$*.*

Proof. Assume that A is a BCC-subalgebra of $\mathcal{U}$. Let $a, b \in \mathcal{U}$.

Case 1: $a, b \in A$. Then $\chi_A(a) = 1_{\mathcal{L}} = \chi_A(b)$, so $\chi_A(a) \wedge \chi_A(b) = 1_{\mathcal{L}}$. Since A is a BCC-subalgebra of $\mathcal{U}$, we have $a \star b \in A$ and so $\chi_A(a \star b) = 1_{\mathcal{L}}$. Therefore, $\chi_A(a \star b) = 1_{\mathcal{L}} \geq 1_{\mathcal{L}} = \chi_A(a) \wedge \chi_A(b)$.

Case 2: $a \notin A$ or $b \notin A$. Then $\chi_A(a) = 0_{\mathcal{L}}$ or $\chi_A(b) = 0_{\mathcal{L}}$, so $\chi_A(a) \wedge \chi_A(b) = 0_{\mathcal{L}}$. Therefore, $\chi_A(a \star b) \geq 0_{\mathcal{L}} = \chi_A(a) \wedge \chi_A(b)$.

Hence, L_A is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$.

Conversely, assume that L_A is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. Let $a, b \in A$. Then $\chi_A(a) = 1_{\mathcal{L}} = \chi_A(b)$, so $\chi_A(a) \wedge \chi_A(b) = 1_{\mathcal{L}}$. Since L_A is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$, we have $1_{\mathcal{L}} \geq \chi_A(a \star b) \geq \chi_A(a) \wedge \chi_A(b) = 1_{\mathcal{L}}$. By anti-symmetry, we have $\chi_A(a \star b) = 1_{\mathcal{L}}$, that is, $a \star b \in A$. Hence, A is a BCC-subalgebra of $\mathcal{U}$.

□

Theorem 3.2. *A nonempty subset* A *of* $\mathcal{U}$ *is a near BCC-filter of* $\mathcal{U}$ *if and only if the characteristic LFS* L_A *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}$*.*

Proof. Assume that A is a near BCC-filter of $\mathcal{U}$. Let $a, b \in \mathcal{U}$.

Case 1: $b \in A$. Then $\chi_A(b) = 1_{\mathcal{L}}$. Since A is a near BCC-filter of $\mathcal{U}$, we have $a \star b \in A$ and so $\chi_A(a \star b) = 1_{\mathcal{L}}$. Therefore, $\chi_A(a \star b) = 1_{\mathcal{L}} \geq 1_{\mathcal{L}} = \chi_A(b)$.

Case 2: $b \notin A$. Then $\chi_A(b) = 0_{\mathcal{L}}$. Therefore, $\chi_A(a \star b) \geq 0_{\mathcal{L}} = \chi_A(b)$.

Hence, L_A is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$.

Conversely, assume that L_A is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. Let $b \in A$. Then $\chi_A(b) = 1_{\mathcal{L}}$. Since L_A is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$, we have $1_{\mathcal{L}} \geq \chi_A(a \star b) \geq \chi_A(b) = 1_{\mathcal{L}}$. By anti-symmetry, we have $\chi_A(a \star b) = 1_{\mathcal{L}}$, that is, $a \star b \in A$. Hence, A is a near BCC-filter of $\mathcal{U}$.

□

Theorem 3.3. *A nonempty subset A of $\mathcal{U}$ is a BCC-filter of $\mathcal{U}$ if and only if the characteristic LFS L_A is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$.*

Proof. Assume that A is a BCC-filter of $\mathcal{U}$. Since $0 \in A$, it follows from Lemma 3.1 that $\chi_A(0) \geq \chi_A(x)$ for all $a \in \mathcal{U}$. Next, let $a, b \in \mathcal{U}$.

Case 1: $a, b \in A$. Then $\chi_A(a) = 1_{\mathcal{L}} = \chi_A(b)$. Thus $\chi_A(b) = 1_{\mathcal{L}} \geq \chi_A(a \star b) = \chi_A(a \star b) \wedge \chi_A(a)$.

Case 2: $a \notin A$ or $b \notin A$. If $a \notin A$, then $\chi_A(a) = 0_{\mathcal{L}}$. Thus $\chi_A(b) \geq 0_{\mathcal{L}} = \chi_A(a \star b) \wedge \chi_A(a)$. If $b \notin A$, then $\chi_A(b) = 0_{\mathcal{L}}$. Since A is a BCC-filter of A, we have $a \star b \notin A$ or $a \notin A$ and so $\chi_A(a \star b) = 0_{\mathcal{L}}$ or $\chi_A(a) = 0_{\mathcal{L}}$. Thus $\chi_A(b) = 0_{\mathcal{L}} \geq 0_{\mathcal{L}} = \chi_A(a \star b) \wedge \chi_A(a)$.

Hence, L_A is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$.

Conversely, assume that L_A is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. Since $\chi_A(0) \geq \chi_A(a)$ for all $a \in \mathcal{U}$, it follows from Lemma 3.1 that $0 \in A$. Next, let $a, b \in \mathcal{U}$ be such that $a \star b, a \in A$. Then $\chi_A(a \star b) = 1_{\mathcal{L}} = \chi_A(a)$, so $\chi_A(a \star b) \wedge \chi_A(a) = 1_{\mathcal{L}}$. Since L_A is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$, we have $1_{\mathcal{L}} \geq \chi_A(b) \geq \chi_A(a \star b) \wedge \chi_A(a) = 1_{\mathcal{L}}$. By anti-symmetry, we have $\chi_A(b) = 1_{\mathcal{L}}$, that is, $y \in A$. Hence, A is a BCC-filter of $\mathcal{U}$. □

Theorem 3.4. *A nonempty subset A of $\mathcal{U}$ is a BCC-ideal of $\mathcal{U}$ if and only if the characteristic LFS L_A is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$.*

Proof. Assume that A is a BCC-ideal of $\mathcal{U}$. Since $0 \in A$, it follows from Lemma 3.1 that $\chi_A(0) \geq \chi_A(x)$ for all $a \in \mathcal{U}$. Next, let $a, b, c \in \mathcal{U}$.

Case 1: $a \star (b \star c), b \in A$. Then $\chi_A(a \star (b \star c)) = 1_{\mathcal{L}} = \chi_A(b)$, so $\chi_A(a \star (b \star c)) \wedge \chi_A(b) = 1_{\mathcal{L}}$. Since A is a BCC-ideal of $\mathcal{U}$, we have $a \star c \in A$ and so $\chi_A(a \star c) = 1_{\mathcal{L}}$. Thus $\chi_A(a \star c) = 1_{\mathcal{L}} \geq 1_{\mathcal{L}} = \chi_A(a \star (b \star c)) \wedge \chi_A(b)$.

Case 2: $a \star (b \star c) \notin A$ or $b \notin A$. Then $\chi_A(a \star (b \star c)) = 0_{\mathcal{L}}$ or $\chi_A(b) = 0_{\mathcal{L}}$, so $\chi_A(a \star (b \star c)) \wedge \chi_A(b) = 0_{\mathcal{L}}$. Thus $\chi_A(a \star c) \geq 0_{\mathcal{L}} = \chi_A(a \star (b \star c)) \wedge \chi_A(b)$.

Hence, L_A is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$.

Conversely, assume that L_A is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. Since $\chi_A(0) \geq \chi_A(a)$ for all $a \in \mathcal{U}$, it follows from Lemma 3.1 that $0 \in A$. Next, let $a, b, c \in \mathcal{U}$ such that $a \star (b \star c), y \in A$. Then $\chi_A(a \star (b \star c)) = 1_{\mathcal{L}} = \chi_A(b)$, so $\chi_A(a \star (b \star c)) \wedge \chi_A(b) = 1_{\mathcal{L}}$. Since L_A is an $\mathcal{L}$-fuzzy BCC-ideal of

$\mathcal{U}$, we have $1_{\mathcal{L}} \geq \chi_A(a\star c) \geq \chi_A(a\star(b\star c))\wedge\chi_A(b) = 1_{\mathcal{L}}$. By anti-symmetry, we have $\chi_A(a\star c) = 1_{\mathcal{L}}$, that is, $a \star c \in A$. Hence, A is a BCC-ideal of $\mathcal{U}$. □

Theorem 3.5. *A nonempty subset A of $\mathcal{U}$ is a strong BCC-ideal of $\mathcal{U}$ if and only if the characteristic LFS L_A is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$.*

Proof. It is straightforward by Theorem 2.1, and $\mathcal{U}$ is the only one strong BCC-ideal of itself. □

4. t-Level subset of an LFS

In this section, we shall discuss the relationships between $\mathcal{L}$-fuzzy BCC-subalgebras (resp., $\mathcal{L}$-fuzzy near BCC-filters, $\mathcal{L}$-fuzzy BCC-filters, $\mathcal{L}$-fuzzy BCC-ideals, and $\mathcal{L}$-fuzzy strong BCC-ideals) of BCC-algebras and their t-level subsets. We shall determine $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ is a lattice.

Definition 4.1. *Let L be an LFS in $\mathcal{U}$ with the membership function μ_{L}. For any $t \in \mathcal{L}$, the sets*

$$\begin{aligned}
U(\mu_{\mathsf{L}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathsf{L}}(a) \geq t\},\\
U^{+}(\mu_{\mathsf{L}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathsf{L}}(a) > t\},\\
L(\mu_{\mathsf{L}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathsf{L}}(a) \leq t\},\\
L^{-}(\mu_{\mathsf{L}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathsf{L}}(a) < t\},\\
E(\mu_{\mathsf{L}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathsf{L}}(a) = t\}
\end{aligned}$$

are referred to as an upper t-level subset, an upper t-strong level subset, a lower t-level subset, a lower t-strong level subset, and an equal t-level subset of L, respectively.

Theorem 4.1. *An LFS L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$ if and only if $E(\mu_{\mathsf{L}}, \mu_{\mathsf{L}}(0))$ is a strong BCC-ideal of $\mathcal{U}$.*

Proof. Assume that L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. By Theorem 2.1, we have L is constant. Then

$$(\forall a \in \mathcal{U})(\mu_{\mathsf{L}}(a) = \mu_{\mathsf{L}}(0)).$$

Thus $a \in E(\mu_{\mathsf{L}}, \mu_{\mathsf{L}}(0))$ and so $E(\mu_{\mathsf{L}}, \mu_{\mathsf{L}}(0)) = \mathcal{U}$. Hence, $E(\mu_{\mathsf{L}}, \mu_{\mathsf{L}}(0))$ is a strong BCC-ideal of $\mathcal{U}$.

Conversely, assume $E(\mu_{\mathsf{L}}, \mu_{\mathsf{L}}(0))$ is a strong BCC-ideal of $\mathcal{U}$. Then $E(\mu_{\mathsf{L}}, \mu_{\mathsf{L}}(0)) = \mathcal{U}$. We consider

$$(\forall a \in \mathcal{U})(\mu_{\mathsf{L}}(a) = \mu_{\mathsf{L}}(0)).$$

Thus L is constant, that is, L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. □

4.1. **Upper t-level subset of an LFS.**

Theorem 4.2. *An LFS* L *is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a BCC-subalgebra of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned}
a, b \in U(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(a) \geq t, \mu_{\mathrm{L}}(b) \geq t \\
&\Rightarrow \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b) \geq t \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b) \geq \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b) \geq t && ((2.1)) \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b) \geq t && (\leq \text{ is transitive}) \\
&\Rightarrow a \star b \in U(\mu_{\mathrm{L}}, t).
\end{aligned}$$

Hence, $U(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$.

Choose $t = \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a) \geq t$ and $\mu_{\mathrm{L}}(b) \geq t$. Thus $a, b \in U(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $U(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$ and so $a \star b \in U(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star b) \geq t = \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b)$.

Hence, L is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. □

Theorem 4.3. *An LFS* L *is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a near BCC-filter of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned}
b \in U(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(b) \geq t \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b) \geq \mu_{\mathrm{L}}(b) \geq t && ((2.2)) \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b) \geq t && (\leq \text{ is transitive}) \\
&\Rightarrow a \star b \in U(\mu_{\mathrm{L}}, t).
\end{aligned}$$

Hence, $U(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$.

Choose $t = \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(b) \geq t$. Thus $b \in U(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $U(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$ and so $a \star b \in U(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star b) \geq t = \mu_{\mathrm{L}}(b)$.

Hence, L is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. □

Lemma 4.1. *Let* L *be an LFS in $\mathcal{U}$. Then* L *satisfies the condition (2.3) if and only if $U(\mu_{\mathrm{L}}, t)$, if it is nonempty, contains $0 \in \mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Let $t \in \mathcal{L}$ be such that $U(\mu_{\mathsf{L}}, t) \neq \emptyset$. Let $a \in \mathcal{U}$. Then

$$\begin{aligned} a \in U(\mu_{\mathsf{L}}, t) &\Rightarrow \mu_{\mathsf{L}}(a) \geq t \\ &\Rightarrow \mu_{\mathsf{L}}(0) \geq \mu_{\mathsf{L}}(a) \geq t && ((2.3)) \\ &\Rightarrow 0 \in U(\mu_{\mathsf{L}}, t). \end{aligned}$$

Conversely, assume for all $t \in \mathcal{L}, U(\mu_{\mathsf{L}}, t)$ contains $0 \in \mathcal{U}$ if it is nonempty. Choose $t = \mu_{\mathsf{L}}(a) \in \mathcal{L}$. Then $\mu_{\mathsf{L}}(a) \geq t$. Thus $a \in U(\mu_{\mathsf{L}}, t) \neq \emptyset$. As the hypothesis, $0 \in U(\mu_{\mathsf{L}}, t)$. Thus $\mu_{\mathsf{L}}(0) \geq t = \mu_{\mathsf{L}}(a)$. □

Theorem 4.4. *An LFS* L *is an* $\mathcal{L}$*-fuzzy BCC-filter of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathsf{L}}, t)$ *is, if it is nonempty, a BCC-filter of* $\mathcal{U}$ *for every* $t \in \mathcal{L}$.

Proof. Assume L is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U(\mu_{\mathsf{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} a \star b, a \in U(\mu_{\mathsf{L}}, t) &\Rightarrow \mu_{\mathsf{L}}(a \star b) \geq t, \mu_{\mathsf{L}}(a) \geq t \\ &\Rightarrow \mu_{\mathsf{L}}(a \star b) \wedge \mu_{\mathsf{L}}(a) \geq t \\ &\Rightarrow \mu_{\mathsf{L}}(b) \geq \mu_{\mathsf{L}}(a \star b) \wedge \mu_{\mathsf{L}}(a) \geq t && ((2.4)) \\ &\Rightarrow b \in U(\mu_{\mathsf{L}}, t). \end{aligned}$$

By Lemma 4.1, we have $0 \in U(\mu_{\mathsf{L}}, t)$. Hence, $U(\mu_{\mathsf{L}}, t)$ is a BCC-filter of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U(\mu_{\mathsf{L}}, t)$ is a BCC-filter of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$. By Lemma 4.1, we have L satisfies the condition (2.3).

Choose $t = \mu_{\mathsf{L}}(a \star b) \wedge \mu_{\mathsf{L}}(a) \in \mathcal{L}$. Then $\mu_{\mathsf{L}}(a \star b) \geq t$ and $\mu_{\mathsf{L}}(a) \geq t$. Thus $a \star b, a \in U(\mu_{\mathsf{L}}, t) \neq \emptyset$. As the hypothesis, we get $U(\mu_{\mathsf{L}}, t)$ is a BCC-filter of $\mathcal{U}$ and so $b \in U(\mu_{\mathsf{L}}, t)$. Thus $\mu_{\mathsf{L}}(b) \geq t = \mu_{\mathsf{L}}(a \star b) \wedge \mu_{\mathsf{L}}(a)$.

Hence, L is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. □

Theorem 4.5. *An LFS* L *is an* $\mathcal{L}$*-fuzzy BCC-ideal of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathsf{L}}, t)$ *is, if it is nonempty, a BCC-ideal of* $\mathcal{U}$ *for every* $t \in \mathcal{L}$.

Proof. Assume L is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U(\mu_{\mathsf{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} a \star (b \star c), b \in U(\mu_{\mathsf{L}}, t) &\Rightarrow \mu_{\mathsf{L}}(a \star (b \star c)) \geq t, \mu_{\mathsf{L}}(b) \geq t \\ &\Rightarrow \mu_{\mathsf{L}}(a \star (b \star c)) \wedge \mu_{\mathsf{L}}(b) \geq t \\ &\Rightarrow \mu_{\mathsf{L}}(a \star c) \geq \mu_{\mathsf{L}}(a \star (b \star c)) \wedge \mu_{\mathsf{L}}(b) \geq t && ((2.5)) \\ &\Rightarrow a \star c \in U(\mu_{\mathsf{L}}, t). \end{aligned}$$

By Lemma 4.1, we have $0 \in U(\mu_{\mathsf{L}}, t)$. Hence, $U(\mu_{\mathsf{L}}, t)$ is a BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$. By Lemma 4.1, we have L satisfies the condition (2.3).

Choose $t = \mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a \star (b \star c)) \geq t$ and $\mu_{\mathrm{L}}(b) \geq t$. Thus $a \star (b \star c), b \in U(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $U(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$ and so $a \star c \in U(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star c) \geq t = \mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{L}}(b)$.

Hence, L is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. □

Theorem 4.6. *An LFS* L *is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a strong BCC-ideal of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} (c \star b) \star (c \star a), b \in U(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \geq t, \mu_{\mathrm{L}}(b) \geq t \\ &\Rightarrow \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) \geq t \\ &\Rightarrow \mu_{\mathrm{L}}(a) \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) \geq t && ((2.6)) \\ &\Rightarrow a \in U(\mu_{\mathrm{L}}, t). \end{aligned}$$

By Lemma 4.1, we have $0 \in U(\mu_{\mathrm{L}}, t)$. Hence, $U(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$. By Lemma 4.1, we have L satisfies the condition (2.3).

Choose $t = \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \geq t$ and $\mu_{\mathrm{L}}(b) \geq t$. Thus $(c \star b) \star (c \star a), b \in U(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $U(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ and so $a \in U(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a) \geq t = \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b)$.

Hence, L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. □

4.2. **Upper t-strong level subset of an LFS.**

Theorem 4.7. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ be a linearly ordered set. Then* L *is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$ if and only if $U^+(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a BCC-subalgebra of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U^+(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then $\mu_{\mathrm{L}}(a)$ and $\mu_{\mathrm{L}}(b)$ are compatible. Suppose that $\mu_{\mathrm{L}}(a) \geq \mu_{\mathrm{L}}(b)$, that is, $\mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b)$. Then

$$\begin{aligned} a, b \in U^+(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(a) > t, \mu_{\mathrm{L}}(b) > t \\ &\Rightarrow \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b) > t \\ &\Rightarrow \mu_{\mathrm{L}}(a \star b) \geq \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b) > t && ((2.1)) \\ &\Rightarrow a \star b \in U^+(\mu_{\mathrm{L}}, t). \end{aligned}$$

Hence, $U^+(\mu_L, t)$ is a BCC-subalgebra of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U^+(\mu_L, t)$ is a BCC-subalgebra of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_L(a \star b) \ngeq \mu_L(a) \wedge \mu_L(b)$. It means that $\mu_L(a \star b) < \mu_L(a) \wedge \mu_L(b)$. Choose $t = \mu_L(a \star b) \in \mathcal{L}$. Then $\mu_L(a) \wedge \mu_L(b) > t$, and so $\mu_L(a) \geq \mu_L(a) \wedge \mu_L(b) > t$ and $\mu_L(b) \geq \mu_L(a) \wedge \mu_L(b) > t$. Thus $a, b \in U^+(\mu_L, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mu_L, t)$ is a BCC-subalgebra of $\mathcal{U}$ and so $a \star b \in U^+(\mu_L, t)$. Thus $\mu_L(a \star b) > t = \mu_L(a \star b)$, a contradiction. Hence, $\mu_L(a \star b) \geq \mu_L(a) \wedge \mu_L(b)$ for all $a, b \in \mathcal{U}$. Therefore, L is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. ☐

Theorem 4.8. *If* L *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}$*, then* $U^+(\mu_L, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}$ *for every* $t \in \mathcal{L}$*.*

Proof. Assume L is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U^+(\mu_L, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} b \in U^+(\mu_L, t) &\Rightarrow \mu_L(b) > t \\ &\Rightarrow \mu_L(a \star b) \geq \mu_L(b) > t && ((2.2)) \\ &\Rightarrow a \star b \in U^+(\mu_L, t). \end{aligned}$$

Hence, $U^+(\mu_L, t)$ is a near BCC-filter of $\mathcal{U}$. ☐

Theorem 4.9. *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ *be a linearly ordered set. If* $U^+(\mu_L, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}$ *for every* $t \in \mathcal{L}$*, then* L *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}$*.*

Proof. Assume for all $t \in \mathcal{L}, U^+(\mu_L, t)$ is a near BCC-filter of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_L(a \star b) \ngeq \mu_L(b)$. It means that $\mu_L(a \star b) < \mu_L(b)$. Choose $t = \mu_L(a \star b) \in \mathcal{L}$. Then $\mu_L(b) > t$. Thus $b \in U^+(\mu_L, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mu_L, t)$ is a near BCC-filter of $\mathcal{U}$ and so $a \star b \in U^+(\mu_L, t)$. Thus $\mu_L(a \star b) > t = \mu_L(a \star b)$, a contradiction. Hence, $\mu_L(a \star b) \geq \mu_L(b)$ for all $a, b \in \mathcal{U}$. Therefore, L is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. ☐

Lemma 4.2. *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ *be a linearly ordered set and* L *an LFS in* $\mathcal{U}$*. Then* L *satisfies the condition* (2.3) *if and only if* $U^+(\mu_L, t)$*, if it is nonempty, contains* $0 \in \mathcal{U}$ *for every* $t \in \mathcal{L}$*.*

Proof. Let $t \in \mathcal{L}$ be such that $U^+(\mu_L, t) \neq \emptyset$. Let $a \in \mathcal{U}$. Then

$$\begin{aligned} a \in U^+(\mu_L, t) &\Rightarrow \mu_L(a) > t \\ &\Rightarrow \mu_L(0) \geq \mu_L(a) > t && ((2.3)) \\ &\Rightarrow 0 \in U^+(\mu_L, t). \end{aligned}$$

Conversely, assume for all $t \in \mathcal{L}, U^+(\mu_L, t)$ contains $0 \in \mathcal{U}$ if it is nonempty. Suppose there exists $a \in \mathcal{U}$ such that $\mu_L(0) \ngeq \mu_L(a)$. It means that $\mu_L(0) < \mu_L(a)$. Choose $t = \mu_L(0) \in \mathcal{L}$.

Then $\mu_L(a) > t$. Thus $a \in U^+(\mu_L, t) \neq \emptyset$. As the hypothesis, we get $0 \in U^+(\mu_L, t)$. Thus $\mu_L(0) > t = \mu_L(0)$, a contradiction. Hence, $\mu_L(0) \geq \mu_L(a)$ for all $a \in \mathcal{U}$. □

Theorem 4.10. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ be a linearly ordered set. Then* L *is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$ if and only if $U^+(\mu_L, t)$ is, if it is nonempty, a BCC-filter of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U^+(\mu_L, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then $\mu_L(a \star b)$ and $\mu_L(a)$ are compatible. Suppose that $\mu_L(a \star b) \geq \mu_L(a)$, that is, $\mu_L(a \star b) \wedge \mu_L(a) = \mu_L(a)$. Then

$$\begin{aligned} a \star b, a \in U^+(\mu_L, t) &\Rightarrow \mu_L(a \star b) > t, \mu_L(a) > t \\ &\Rightarrow \mu_L(a \star b) \wedge \mu_L(a) = \mu_L(a) > t \\ &\Rightarrow \mu_L(b) \geq \mu_L(a \star b) \wedge \mu_L(a) > t && ((2.4)) \\ &\Rightarrow b \in U^+(\mu_L, t). \end{aligned}$$

By Lemma 4.2, we have $0 \in U^+(\mu_L, t)$. Hence, $U^+(\mu_L, t)$ is a BCC-filter of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U^+(\mu_L, t)$ is a BCC-filter of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_L(b) \not\geq \mu_L(a \star b) \wedge \mu_L(a)$. It means that $\mu_L(b) < \mu_L(a \star b) \wedge \mu_L(a)$. By Lemma 4.2, we have L satisfies the condition (2.3). Choose $t = \mu_L(b) \in \mathcal{L}$. Then $\mu_L(a \star b) \wedge \mu_L(a) > t$, and so $\mu_L(a \star b) \geq \mu_L(a \star b) \wedge \mu_L(a) > t$ and $\mu_L(a) \geq \mu_L(a \star b) \wedge \mu_L(a) > t$. Thus $a \star b, a \in U^+(\mu_L, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mu_L, t)$ is a BCC-filter of $\mathcal{U}$ and so $b \in U^+(\mu_L, t)$. Thus $\mu_L(b) > t = \mu_L(b)$, a contradiction. Hence, $\mu_L(b) \geq \mu_L(a \star b) \wedge \mu_L(a)$ for all $a, b \in \mathcal{U}$. Therefore, L is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. □

Theorem 4.11. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ be a linearly ordered set. Then* L *is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$ if and only if $U^+(\mu_L, t)$ is, if it is nonempty, a BCC-ideal of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U^+(\mu_L, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then $\mu_L(a \star (b \star c))$ and $\mu_L(b)$ are compatible. Suppose that $\mu_L(a \star (b \star c)) \geq \mu_L(b)$, that is, $\mu_L(a \star (b \star c)) \wedge \mu_L(b) = \mu_L(b)$. Then

$$\begin{aligned} a \star (b \star c), b \in U^+(\mu_L, t) &\Rightarrow \mu_L(a \star (b \star c)) \geq t, \mu_L(b) > t \\ &\Rightarrow \mu_L(a \star (b \star c)) \wedge \mu_L(b) = \mu_L(b) > t \\ &\Rightarrow \mu_L(a \star c) \geq \mu_L(a \star (b \star c)) \wedge \mu_L(b) > t && ((2.5)) \\ &\Rightarrow a \star c \in U^+(\mu_L, t). \end{aligned}$$

By Lemma 4.2, we have $0 \in U^+(\mu_L, t)$. Hence, $U^+(\mu_L, t)$ is a BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U^+(\mu_L, t)$ is a BCC-ideal of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b, c \in \mathcal{U}$ such that $\mu_L(a \star c) \not\geq \mu_L(a \star (b \star c)) \wedge \mu_L(b)$. It means that $\mu_L(a \star c) < \mu_L(a \star (b \star c)) \wedge \mu_L(b)$. By Lemma 4.2, we have L satisfies the condition (2.3). Choose $t = \mu_L(a \star c) \in \mathcal{L}$.

Then $\mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{L}}(b) > t$, and so $\mu_{\mathrm{L}}(a \star (b \star c)) \geq \mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{L}}(b) > t$ and $\mu_{\mathrm{L}}(b) \geq \mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{L}}(b) > t$. Thus $a \star (b \star c), b \in U^+(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$ and so $a \star c \in U^+(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star c) > t = \mu_{\mathrm{L}}(a \star c)$, a contradiction. Hence, $\mu_{\mathrm{L}}(a \star c) \geq \mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{L}}(b)$ for all $a, b.c \in \mathcal{U}$. Therefore, L is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. □

Theorem 4.12. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ be a linearly ordered set. Then* L *is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$ if and only if $U^+(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a strong BCC-ideal of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $U^+(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then $\mu_{\mathrm{L}}((c \star b) \star (c \star a))$ and $\mu_{\mathrm{L}}(b)$ are compatible. Suppose that $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \geq \mu_{\mathrm{L}}(b)$, that is, $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b)$. Then

$$
\begin{aligned}
(c \star b) \star (c \star a), b \in U^+(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}((c \star b) \star (c \star a)) > t, \mu_{\mathrm{L}}(b) > t \\
&\Rightarrow \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b) > t \\
&\Rightarrow \mu_{\mathrm{L}}(a) \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) > t \qquad ((2.6)) \\
&\Rightarrow a \in U^+(\mu_{\mathrm{L}}, t).
\end{aligned}
$$

By Lemma 4.2, we have $0 \in U^+(\mu_{\mathrm{L}}, t)$. Hence, $U^+(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, U^+(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b, c \in \mathcal{U}$ such that $\mu_{\mathrm{L}}(a) \ngeq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b)$. It means that $\mu_{\mathrm{L}}(a) < \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b)$. By Lemma 4.2, we have L satisfies the condition (2.3). Choose $t = \mu_{\mathrm{L}}(a) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) > t$, and so $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) > t$ and $\mu_{\mathrm{L}}(b) \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b) > t$. Thus $(c \star b) \star (c \star a), b \in U^+(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ and so $a \in U^+(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a) > t = \mu_{\mathrm{L}}(a)$, a contradiction. Hence, $\mu_{\mathrm{L}}(a) \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b)$ for all $a, b, c \in \mathcal{U}$. Therefore, L is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. □

4.3. **Lower t-level subset of an LFS.**

Definition 4.2. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Let* L *be an LFS in $\mathcal{U}$. The LFS* L' *defined by*

$$(\forall a \in \mathcal{U})(\mu_{\mathrm{L}'}(a) = (\mu_{\mathrm{L}}(a))' = \mu_{\mathrm{L}}(a)')$$

is called the complement of L *in $\mathcal{U}$.*

Theorem 4.13. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Then* L' *is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$ if and only if $L(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a BCC-subalgebra of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned}
a, b \in L(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(a) \leq t, \mu_{\mathrm{L}}(b) \leq t \\
&\Rightarrow \mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) \leq t \\
&\Rightarrow ((\mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b))^{'} \geq t^{'} && \text{(Lemma 1.1 (3))} \\
&\Rightarrow \mu_{\mathrm{L}}(a)^{'} \wedge \mu_{\mathrm{L}}(b)^{'} = (\mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b))^{'} \geq t^{'} && \text{(Lemma 1.1 (1))} \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b)^{'} \geq \mu_{\mathrm{L}}(a)^{'} \wedge \mu_{\mathrm{L}}(b)^{'} \geq t^{'} && \text{((2.1))} \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b)^{'} \geq t^{'} && (\leq \text{ is transitive}) \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b) \leq t && \text{(Lemma 1.1 (3))} \\
&\Rightarrow a \star b \in L(\mu_{\mathrm{L}}, t).
\end{aligned}$$

Hence, $L(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$. Choose $t = \mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a) \leq t$ and $\mu_{\mathrm{L}}(b) \leq t$. Thus $a, b \in L(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$ and so $a \star b \in L(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star b) \leq t = \mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b)$. By Lemma 1.1 (1), we have $\mu_{\mathrm{L}}(a \star b)^{'} \geq \mu_{\mathrm{L}}(a)^{'} \wedge \mu_{\mathrm{L}}(b)^{'}$. Hence, $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. □

Theorem 4.14. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ^{'}, 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Then $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$ if and only if $L(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a near BCC-filter of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned}
b \in L(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(b) \leq t \\
&\Rightarrow \mu_{\mathrm{L}}(b))^{'} \geq t^{'} && \text{(Lemma 1.1 (3))} \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b)^{'} \geq \mu_{\mathrm{L}}(b)^{'} \geq t^{'} && \text{((2.1))} \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b)^{'} \geq t^{'} && (\leq \text{ is transitive}) \\
&\Rightarrow \mu_{\mathrm{L}}(a \star b) \leq t && \text{(Lemma 1.1 (3))} \\
&\Rightarrow a \star b \in L(\mu_{\mathrm{L}}, t).
\end{aligned}$$

Hence, $L(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$. Choose $t = \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(b) \leq t$. Thus $b \in L(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$ and so $a \star b \in L(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star b) \leq t = \mu_{\mathrm{L}}(b)$. By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(a \star b)^{'} \geq \mu_{\mathrm{L}}(b)^{'}$. Hence, $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. □

Lemma 4.3. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice and* L *an LFS in* $\mathcal{U}$*. Then* L' *satisfies the condition* (2.3) *if and only if* $L(\mu_{\mathrm{L}}, t)$*, if it is nonempty, contains* $0 \in \mathcal{U}$ *for every* $t \in \mathcal{L}$*.*

Proof. Let $t \in \mathcal{L}$ be such that $L(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a \in \mathcal{U}$. Then

$$\begin{aligned} a \in L(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(a) \leq t \\ &\Rightarrow \mu_{\mathrm{L}}(a)' \geq t' && \text{(Lemma 1.1 (3))} \\ &\Rightarrow \mu_{\mathrm{L}}(0)' \geq \mu_{\mathrm{L}}(a)' \geq t' && \text{((2.3))} \\ &\Rightarrow \mu_{\mathrm{L}}(0)' \geq t' && (\leq \text{ is transitive}) \\ &\Rightarrow \mu_{\mathrm{L}}(0) \leq t && \text{(Lemma 1.1 (3))} \\ &\Rightarrow 0 \in L(\mu_{\mathrm{L}}, t). \end{aligned}$$

Conversely, assume for all $t \in \mathcal{L}, L(\mu_{\mathrm{L}}, t)$ contains $0 \in \mathcal{U}$ if it is nonempty. Choose $t = \mu_{\mathrm{L}}(a) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a) \leq t$. Thus $a \in L(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, $0 \in L(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(0) \leq t = \mu_{\mathrm{L}}(a)$. By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(0)' \geq \mu_{\mathrm{L}}(a)'$. □

Theorem 4.15. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Then* L' *is an* $\mathcal{L}$*-fuzzy BCC-filter of* $\mathcal{U}$ *if and only if* $L(\mu_{\mathrm{L}}, t)$ *is, if it is nonempty, a BCC-filter of* $\mathcal{U}$ *for every* $t \in \mathcal{L}$*.*

Proof. Assume L' is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} a \star b, a \in L(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(a \star b) \leq t, \mu_{\mathrm{L}}(a) \leq t \\ &\Rightarrow \mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) \leq t \\ &\Rightarrow ((\mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a))' \geq t' && \text{(Lemma 1.1 (3))} \\ &\Rightarrow \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)' = (\mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a))' \geq t' && \text{(Lemma 1.1 (1))} \\ &\Rightarrow \mu_{\mathrm{L}}(b)' \geq \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)' \geq t' && \text{((2.4))} \\ &\Rightarrow \mu_{\mathrm{L}}(b)' \geq t' && (\leq \text{ is transitive}) \\ &\Rightarrow \mu_{\mathrm{L}}(b) \leq t && \text{(Lemma 1.1 (3))} \\ &\Rightarrow b \in L(\mu_{\mathrm{L}}, t). \end{aligned}$$

By Lemma 4.3, we have $0 \in L(\mu_{\mathrm{L}}, t)$. Hence, $L(\mu_{\mathrm{L}}, t)$ is a BCC-filter of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L(\mu_{\mathrm{L}}, t)$ is a BCC-filter of $\mathcal{U}$ if it is nonempty. Let $a, b \in \mathcal{U}$. By Lemma 4.3, we have L' satisfies the condition (2.3). Choose $t = \mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a \star b) \leq t$ and $\mu_{\mathrm{L}}(a) \leq t$. Thus $a \star b, a \in L(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L(\mu_{\mathrm{L}}, t)$ is a BCC-filter of $\mathcal{U}$ and so $b \in L(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(b) \leq t = \mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a)$. By Lemma 1.1 (1), we have $\mu_{\mathrm{L}}(b)' \geq \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)'$. Hence, L' is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. □

Theorem 4.16. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Then L' is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$ if and only if $L(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a BCC-ideal of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L' is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then

$$
\begin{aligned}
a \star (b \star c), b \in L(\mu_{\mathrm{L}}, t) &\Rightarrow \mu_{\mathrm{L}}(a \star (b \star c)) \leq t, \mu_{\mathrm{L}}(b) \leq t \\
&\Rightarrow \mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) \leq t \\
&\Rightarrow ((\mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b))' \geq t' && \text{(Lemma 1.1 (3))} \\
&\Rightarrow \mu_{\mathrm{L}}(a \star (b \star c))' \wedge \mu_{\mathrm{L}}(b)' \\
&\qquad = (\mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b))' \geq t' && \text{(Lemma 1.1 (1))} \\
&\Rightarrow \mu_{\mathrm{L}}(a \star c)' \geq \mu_{\mathrm{L}}(a \star (b \star c))' \wedge \mu_{\mathrm{L}}(b)' \geq t' && \text{((2.5))} \\
&\Rightarrow \mu_{\mathrm{L}}(a \star c)' \geq t' && (\leq \text{ is transitive}) \\
&\Rightarrow \mu_{\mathrm{L}}(a \star c) \leq t && \text{(Lemma 1.1 (3))} \\
&\Rightarrow a \star c \in L(\mu_{\mathrm{L}}, t).
\end{aligned}
$$

By Lemma 4.3, we have $0 \in L(\mu_{\mathrm{L}}, t)$. Hence, $L(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$ if it is nonempty. Let $a, b, c \in \mathcal{U}$. By Lemma 4.3, we have L' satisfies the condition (2.3). Choose $t = \mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a \star (b \star c)) \leq t$ and $\mu_{\mathrm{L}}(b) \leq t$. Thus $a \star (b \star c), b \in L(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$ and so $a \star c \in L(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star c) \leq t = \mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b)$. By Lemma 1.1 (1), we have $\mu_{\mathrm{L}}(a \star c)' \geq \mu_{\mathrm{L}}(a \star (b \star c))' \wedge \mu_{\mathrm{L}}(b)'$. Hence, L' is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. □

Theorem 4.17. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Then L' is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$ if and only if $L(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a strong BCC-ideal of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L' is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then

$$
\begin{aligned}
&(c \star b) \star (c \star a), b \in L(\mu_{\mathrm{L}}, t) \\
&\Rightarrow \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \leq t, \mu_{\mathrm{L}}(b) \leq t \\
&\Rightarrow \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b) \leq t \\
&\Rightarrow ((\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b))' \geq t' && \text{(Lemma 1.1 (3))} \\
&\Rightarrow \mu_{\mathrm{L}}((c \star b) \star (c \star a))' \wedge \mu_{\mathrm{L}}(b)' \\
&\qquad = (\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b))' \geq t' && \text{(Lemma 1.1 (1))}
\end{aligned}
$$

$$\Rightarrow \mu_{\mathrm{L}}(a)' \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a))' \wedge \mu_{\mathrm{L}}(b)' \geq t' \qquad ((2.6))$$

$$\Rightarrow \mu_{\mathrm{L}}(a)' \geq t' \qquad (\leq \text{ is transitive})$$

$$\Rightarrow \mu_{\mathrm{L}}(a) \leq t \qquad (\text{Lemma 1.1 (3)})$$

$$\Rightarrow a \in L(\mu_{\mathrm{L}}, t).$$

By Lemma 4.3, we have $0 \in L(\mu_{\mathrm{L}}, t)$. Hence, $L(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ if it is nonempty. Let $a, b, c \in \mathcal{U}$. By Lemma 4.3, we have L' satisfies the condition (2.3). Choose $t = \mu_{\mathrm{L}}((c\star b)\star(c\star a)) \vee \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \leq t$ and $\mu_{\mathrm{L}}(b) \leq t$. Thus $(c \star b) \star (c \star a), b \in L(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ and so $a \in L(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a) \leq t = \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b)$. By Lemma 1.1 (1), we have $\mu_{\mathrm{L}}(a)' \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a))' \wedge \mu_{\mathrm{L}}(b)'$. Hence, L' is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. □

4.4. **Lower t-strong level subset of an LFS.**

Theorem 4.18. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ is a linear order. Then L' is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$ if and only if $L^{-}(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a BCC-subalgebra of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L' is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in L^{-}(\mu_{\mathrm{L}}, t)$. Then $\mu_{\mathrm{L}}(a)$ and $\mu_{\mathrm{L}}(b)$ are compatible. Suppose that $\mu_{\mathrm{L}}(a) \leq \mu_{\mathrm{L}}(b)$, that is, $\mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b)$. Then $\mu_{\mathrm{L}}(a) < t$ and $\mu_{\mathrm{L}}(b) < t$ and so $\mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b) < t$. Since L' is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$, we have

$$\mu_{\mathrm{L}}(a \star b)' \geq \mu_{\mathrm{L}}(a)' \wedge \mu_{\mathrm{L}}(b)' = (\mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b))'. \qquad (\text{Lemma 1.1 (1)})$$

By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(a\star b) \leq \mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) < t$. Thus $a \star b \in L^{-}(\mu_{\mathrm{L}}, t)$. Hence, $L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_{\mathrm{L}}(a\star b)' \ngeq \mu_{\mathrm{L}}(a)' \wedge \mu_{\mathrm{L}}(b)'$. It means that $\mu_{\mathrm{L}}(a\star b)' < \mu_{\mathrm{L}}(a)' \wedge \mu_{\mathrm{L}}(b)'$. By Lemma 1.1 (1), we have

$$\mu_{\mathrm{L}}(a \star b)' < \mu_{\mathrm{L}}(a)' \wedge \mu_{\mathrm{L}}(b)' = (\mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b))'.$$

By Lemma 1.1 (5), we have $\mu_{\mathrm{L}}(a \star b) > \mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b)$. Choose $t = \mu_{\mathrm{L}}(a \star b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) < t$, and so $\mu_{\mathrm{L}}(a) \leq \mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) < t$ and $\mu_{\mathrm{L}}(b) \leq \mu_{\mathrm{L}}(a) \vee \mu_{\mathrm{L}}(b) < t$. Thus $a, b \in L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-subalgebra of $\mathcal{U}$ and so $a \star b \in L^{-}(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star b) < t = \mu_{\mathrm{L}}(a \star b)$, a contradiction. Hence, $\mu_{\mathrm{L}}(a \star b)' \geq \mu_{\mathrm{L}}(a)' \wedge \mu_{\mathrm{L}}(b)'$ for all $a, b \in \mathcal{U}$. Hence, L' is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}$. □

Theorem 4.19. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. If L' is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$, then $L^{-}(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a near BCC-filter of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$ and $b \in L^{-}(\mu_{\mathrm{L}}, t)$. Then $\mu_{\mathrm{L}}(b) < t$. Since $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$, we have

$$\mu_{\mathrm{L}}(a \star b)^{'} \geq \mu_{\mathrm{L}}(b)^{'}.$$

By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(a \star b) \leq \mu_{\mathrm{L}}(b) < t$. Thus $a \star b \in L^{-}(\mu_{\mathrm{L}}, t)$. Hence, $L^{-}(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$. □

Theorem 4.20. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ^{'}, 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ is a linear order. If $L^{-}(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a near BCC-filter of $\mathcal{U}$ for every $t \in \mathcal{L}$, then $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$.*

Proof. Assume for all $t \in \mathcal{L}, L^{-}(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_{\mathrm{L}}(a \star b)^{'} \ngeq \mu_{\mathrm{L}}(b)^{'}$. It means that $\mu_{\mathrm{L}}(a \star b)^{'} < \mu_{\mathrm{L}}(b)^{'}$. By Lemma 1.1 (5), we have $\mu_{\mathrm{L}}(a \star b) > \mu_{\mathrm{L}}(b)$. Choose $t = \mu_{\mathrm{L}}(a \star b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(b) < t$. Thus $b \in L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mu_{\mathrm{L}}, t)$ is a near BCC-filter of $\mathcal{U}$ and so $a \star b \in L^{-}(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a \star b) < t = \mu_{\mathrm{L}}(a \star b)$, a contradiction. Hence, $\mu_{\mathrm{L}}(a \star b)^{'} \geq \mu_{\mathrm{L}}(b)^{'}$ for all $a, b \in \mathcal{U}$. Hence, $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy near BCC-filter of $\mathcal{U}$. □

Lemma 4.4. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ^{'}, 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ is a linear order and L an LFS in $\mathcal{U}$. Then $\mathrm{L}^{'}$ satisfies the condition (2.3) if and only if $L^{-}(\mu_{\mathrm{L}}, t)$, if it is nonempty, contains $0 \in \mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Let $t \in \mathcal{L}$ be such that $L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a \in \mathcal{U}$ and $a \in L^{-}(\mu_{\mathrm{L}}, t)$. Then $\mu_{\mathrm{L}}(a) < t$. Since $\mathrm{L}^{'}$ satisfies the condition (2.3), we have

$$\mu_{\mathrm{L}}(0)^{'} \geq \mu_{\mathrm{L}}(a)^{'}.$$

By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(0) \leq \mu_{\mathrm{L}}(a) < t$. Thus $0 \in L^{-}(\mu_{\mathrm{L}}, t)$.

Conversely, assume for all $t \in \mathcal{L}, L^{-}(\mu_{\mathrm{L}}, t)$ contains $0 \in \mathcal{U}$ if it is nonempty. Suppose there exist $a \in \mathcal{U}$ such that $\mu_{\mathrm{L}}(0)^{'} \ngeq \mu_{\mathrm{L}}(a)^{'}$. It means that $\mu_{\mathrm{L}}(0)^{'} < \mu_{\mathrm{L}}(a)^{'}$. By Lemma 1.1 (5), we have $\mu_{\mathrm{L}}(0) > \mu_{\mathrm{L}}(a)$. Choose $t = \mu_{\mathrm{L}}(0) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a) < t$. Thus $a \in L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $0 \in L^{-}(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(0) < t = \mu_{\mathrm{L}}(0)$, a contradiction. Hence, $\mu_{\mathrm{L}}(0)^{'} \geq \mu_{\mathrm{L}}(a)^{'}$ for all $a \in \mathcal{U}$. □

Theorem 4.21. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ^{'}, 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ is a linear order. Then $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$ if and only if $L^{-}(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a BCC-filter of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then $\mu_{\mathrm{L}}(a \star b)$ and $\mu_{\mathrm{L}}(a)$ are compatible. Suppose that $\mu_{\mathrm{L}}(a \star b) \leq \mu_{\mathrm{L}}(a)$, that is,

$\mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) = \mu_{\mathrm{L}}(b)$. Let $a \star b, a \in L^{-}(\mu_{\mathrm{L}}, t)$. Then $\mu_{\mathrm{L}}(a \star b) < t$ and $\mu_{\mathrm{L}}(a) < t$ and so $\mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) = \mu_{\mathrm{L}}(a) < t$. Since L' is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$, we have

$$\mu_{\mathrm{L}}(b)' \geq \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)' = (\mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a))'. \qquad \text{(Lemma 1.1 (1))}$$

By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(b) \leq \mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) < t$. Thus $b \in L^{-}(\mu_{\mathrm{L}}, t)$. By Lemma 4.4, we have $0 \in L^{-}(\mu_{\mathrm{L}}, t)$. Hence, $L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-filter of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-filter of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_{\mathrm{L}}(b)' \ngeq \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)'$. It means that $\mu_{\mathrm{L}}(b)' < \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)'$. By Lemma 1.1 (1), we have

$$\mu_{\mathrm{L}}(b)' < \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)' = (\mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a))'.$$

By Lemma 1.1 (5), we have $\mu_{\mathrm{L}}(b) > \mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a)$. By Lemma 4.4, we have L' satisfies the condition (2.3). Choose $t = \mu_{\mathrm{L}}(b) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) < t$, and so $\mu_{\mathrm{L}}(a \star b) \leq \mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) < t$ and $\mu_{\mathrm{L}}(a) \leq \mu_{\mathrm{L}}(a \star b) \vee \mu_{\mathrm{L}}(a) < t$. Thus $a \star b, a \in L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-filter of $\mathcal{U}$ and so $b \in L^{-}(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(b) < t = \mu_{\mathrm{L}}(b)$, a contradiction. Hence, $\mu_{\mathrm{L}}(b)' \geq \mu_{\mathrm{L}}(a \star b)' \wedge \mu_{\mathrm{L}}(a)'$ for all $a, b \in \mathcal{U}$. Hence, L' is an $\mathcal{L}$-fuzzy BCC-filter of $\mathcal{U}$. □

Theorem 4.22. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ is a linear order. Then L' is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$ if and only if $L^{-}(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a BCC-ideal of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume L' is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then $\mu_{\mathrm{L}}(a \star (b \star c))$ and $\mu_{\mathrm{L}}(b)$ are compatible. Suppose that $\mu_{\mathrm{L}}(a \star (b \star c)) \leq \mu_{\mathrm{L}}(b)$, that is, $\mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b)$. Let $a \star (b \star c), b \in L^{-}(\mu_{\mathrm{L}}, t)$. Then $\mu_{\mathrm{L}}(a \star (b \star c)) < t$ and $\mu_{\mathrm{L}}(b) < t$ and so $\mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b) < t$. Since L' is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$, we have

$$\mu_{\mathrm{L}}(a \star c)' \geq \mu_{\mathrm{L}}(a \star (b \star c))' \wedge \mu_{\mathrm{L}}(b)' = (\mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b))'. \qquad \text{(Lemma 1.1 (1))}$$

By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(a \star c) \leq \mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) < t$. Thus $a \star c \in L^{-}(\mu_{\mathrm{L}}, t)$. By Lemma 4.4, we have $0 \in L^{-}(\mu_{\mathrm{L}}, t)$. Hence, $L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b, c \in \mathcal{U}$ such that $\mu_{\mathrm{L}}(a \star c)' \ngeq \mu_{\mathrm{L}}(a \star (b \star c))' \wedge \mu_{\mathrm{L}}(b)'$. It means that $\mu_{\mathrm{L}}(a \star c)' < \mu_{\mathrm{L}}(a \star (b \star c))' \wedge \mu_{\mathrm{L}}(b)'$. By Lemma 1.1 (1), we have

$$\mu_{\mathrm{L}}(a \star c)' < \mu_{\mathrm{L}}(a \star (b \star c))' \wedge \mu_{\mathrm{L}}(b)' = (\mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b))'.$$

By Lemma 1.1 (5), we have $\mu_{\mathrm{L}}(a \star c) > \mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b)$. By Lemma 4.4, we have L' satisfies the condition (2.3). Choose $t = \mu_{\mathrm{L}}(a \star c) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) < t$, and so $\mu_{\mathrm{L}}(a \star (b \star c)) \leq \mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) < t$ and $\mu_{\mathrm{L}}(b) \leq \mu_{\mathrm{L}}(a \star (b \star c)) \vee \mu_{\mathrm{L}}(b) < t$. Thus $a \star (b \star c), b \in L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mu_{\mathrm{L}}, t)$ is a BCC-ideal of $\mathcal{U}$ and so $a \star c \in L^{-}(\mu_{\mathrm{L}}, t)$.

Thus $\mu_{\mathrm{L}}(a \star c) < t = \mu_{\mathrm{L}}(a \star c)$, a contradiction. Hence, $\mu_{\mathrm{L}}(a \star c)^{'} \geq \mu_{\mathrm{L}}(a \star (b \star c))^{'} \wedge \mu_{\mathrm{L}}(b)^{'}$ for all $a, b \in \mathcal{U}$. Hence, $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy BCC-ideal of $\mathcal{U}$. □

Theorem 4.23. *Let $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ^{'}, 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ is a linear order. Then $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$ if and only if $L^{-}(\mu_{\mathrm{L}}, t)$ is, if it is nonempty, a strong BCC-ideal of $\mathcal{U}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. Let $t \in \mathcal{L}$ be such that $L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then $\mu_{\mathrm{L}}((c \star b) \star (c \star a))$ and $\mu_{\mathrm{L}}(b)$ are compatible. Suppose that $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \leq \mu_{\mathrm{L}}(b)$, that is, $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b)$. Let $(c \star b) \star (c \star a), b \in L^{-}(\mu_{\mathrm{L}}, t)$. Then $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) < t$ and $\mu_{\mathrm{L}}(b) < t$ and so $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b) = \mu_{\mathrm{L}}(b) < t$. Since $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$, we have

$$\mu_{\mathrm{L}}(a)^{'} \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a))^{'} \wedge \mu_{\mathrm{L}}(b)^{'} = (\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b))^{'}. \qquad \text{(Lemma 1.1 (1))}$$

By Lemma 1.1 (3), we have $\mu_{\mathrm{L}}(a) \leq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b) < t$. Thus $a \in L^{-}(\mu_{\mathrm{L}}, t)$. By Lemma 4.4, we have $0 \in L^{-}(\mu_{\mathrm{L}}, t)$. Hence, $L^{-}(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$.

Conversely, assume for all $t \in \mathcal{L}, L^{-}(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ if it is nonempty. Suppose there exist $a, b, c \in \mathcal{U}$ such that $\mu_{\mathrm{L}}(a)^{'} \ngeq \mu_{\mathrm{L}}((c \star b) \star (c \star a))^{'} \wedge \mu_{\mathrm{L}}(b)^{'}$. It means that $\mu_{\mathrm{L}}(a)^{'} < \mu_{\mathrm{L}}((c \star b) \star (c \star a))^{'} \wedge \mu_{\mathrm{L}}(b)^{'}$. By Lemma 1.1 (1), we have

$$\mu_{\mathrm{L}}(a)^{'} < \mu_{\mathrm{L}}((c \star b) \star (c \star a))^{'} \wedge \mu_{\mathrm{L}}(b)^{'} = (\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b))^{'}.$$

By Lemma 1.1 (5), we have $\mu_{\mathrm{L}}(a) > \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b)$. By Lemma 4.4, we have $\mathrm{L}^{'}$ satisfies the condition (2.3). Choose $t = \mu_{\mathrm{L}}(a) \in \mathcal{L}$. Then $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b) < t$, and so $\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \leq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b) < t$ and $\mu_{\mathrm{L}}(b) \leq \mu_{\mathrm{L}}((c \star b) \star (c \star a)) \vee \mu_{\mathrm{L}}(b) < t$. Thus $(c \star b) \star (c \star a), b \in L^{-}(\mu_{\mathrm{L}}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mu_{\mathrm{L}}, t)$ is a strong BCC-ideal of $\mathcal{U}$ and so $a \in L^{-}(\mu_{\mathrm{L}}, t)$. Thus $\mu_{\mathrm{L}}(a) < t = \mu_{\mathrm{L}}(a)$, a contradiction. Hence, $\mu_{\mathrm{L}}(a)^{'} \geq \mu_{\mathrm{L}}((c \star b) \star (c \star a))^{'} \wedge \mu_{\mathrm{L}}(b)^{'}$ for all $a, b \in \mathcal{U}$. Hence, $\mathrm{L}^{'}$ is an $\mathcal{L}$-fuzzy strong BCC-ideal of $\mathcal{U}$. □

5. Cartesian product of LFSs

Definition 5.1. *Let L and M be LFSs in nonempty sets $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. The Cartesian product of L and M is $\mathrm{L} \times \mathrm{M} : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{L}$ described by its membership function $\bar{}_{\mathrm{L} \times \mathrm{M}}$ such that*

$$(\forall a \in \mathcal{U}_1, b \in \mathcal{U}_2)(\mu_{\mathrm{L} \times \mathrm{M}}(a, b) = \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{M}}(b)).$$

It is clearly that $\mathrm{L} \times \mathrm{M}$ is an LFS in $\mathcal{U}_1 \times \mathcal{U}_2$.

Remark 5.1. *[28] Let $\mathcal{U}_1 = (\mathcal{U}_1, \star, 0_1)$ and $\mathcal{U}_2 = (\mathcal{U}_2, \circ, 0_2)$ be BCC-algebras. We can easily prove that $\mathcal{U}_1 \times \mathcal{U}_2$ is a BCC-algebra defined by*

$$(\forall a, b \in \mathcal{U}_1, u, v \in \mathcal{U}_2)((a, u) \circledast (b, v) = (a \star b, u \circ v)).$$

Theorem 5.1. *Let* L *and* M *be* $\mathcal{L}$*-fuzzy BCC-subalgebras of BCC-algebras* $\mathcal{U}_1 = (\mathcal{U}_1, \star, 0_1)$ *and* $\mathcal{U}_2 = (\mathcal{U}_2, \circ, 0_2)$*, respectively. Then* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-subalgebra of a BCC-algebra* $\mathcal{U}_1 \times \mathcal{U}_2$*.*

Proof. Let $a, b \in \mathcal{U}_1$, $u, v \in \mathcal{U}_2$. Then

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}((a,u) \circledast (b,v)) &= \mu_{\mathrm{L}\times\mathrm{M}}(a \star b, u \circ v) \\
&= \mu_{\mathrm{L}}(a \star b) \wedge \mu_{\mathrm{M}}(u \circ v) \\
&\geq (\mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{L}}(b)) \wedge (\mu_{\mathrm{M}}(u) \wedge \mu_{\mathrm{M}}(v)) && ((2.1)) \\
&= (\mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{M}}(u)) \wedge (\mu_{\mathrm{L}}(b) \wedge \mu_{\mathrm{M}}(v)) && (\wedge \text{ is associative and commutative}) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(a,u) \wedge \mu_{\mathrm{L}\times\mathrm{M}}(b,v).
\end{aligned}$$

Hence, $\mathrm{L} \times \mathrm{M}$ is an $\mathcal{L}$-fuzzy BCC-subalgebra of $\mathcal{U}_1 \times \mathcal{U}_2$. □

Theorem 5.2. *Let* L *and* M *be* $\mathcal{L}$*-fuzzy near BCC-filters of BCC-algebras* $\mathcal{U}_1 = (\mathcal{U}_1, \star, 0_1)$ *and* $\mathcal{U}_2 = (\mathcal{U}_2, \circ, 0_2)$*, respectively. Then* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy near BCC-filter of a BCC-algebra* $\mathcal{U}_1 \times \mathcal{U}_2$*.*

Proof. Let $a, b \in \mathcal{U}_1$, $u, v \in \mathcal{U}_2$. Then

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}((a,u) \circledast (b,v)) &= \mu_{\mathrm{L}\times\mathrm{M}}(a \star b, u \circ v) \\
&= \mu_{\mathrm{L}}(a \star b) \wedge \mu_{\mathrm{M}}(u \circ v) \\
&\geq \mu_{\mathrm{L}}(b) \wedge \mu_{\mathrm{M}}(v) && ((2.2)) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(b,v).
\end{aligned}$$

□

Theorem 5.3. *Let* L *and* M *be* $\mathcal{L}$*-fuzzy BCC-filters of BCC-algebras* $\mathcal{U}_1 = (\mathcal{U}_1, \star, 0_1)$ *and* $\mathcal{U}_2 = (\mathcal{U}_2, \circ, 0_2)$*, respectively. Then* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-filter of a BCC-algebra* $\mathcal{U}_1 \times \mathcal{U}_2$*.*

Proof. Let $a, b \in \mathcal{U}_1$, $u, v \in \mathcal{U}_2$. Then

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}(0_1, 0_2) &= \mu_{\mathrm{L}}(0_1) \wedge \mu_{\mathrm{M}}(0_2) \\
&\geq \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{M}}(u) && ((2.3)) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(a,u)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}(b,v) &= \mu_{\mathrm{L}}(b) \wedge \mu_{\mathrm{M}}(v) \\
&\geq (\mu_{\mathrm{L}}(a \star b) \wedge \mu_{b}(a)) \wedge (\mu_{\mathrm{M}}(u \circ v) \wedge \mu_{\mathrm{M}}(u)) && ((2.4)) \\
&= (\mu_{\mathrm{L}}(a \star b) \wedge \mu_{\mathrm{M}}(u \circ v)) \wedge (\mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{M}}(u)) && (\wedge \text{ is associative and commutative}) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(a \star b, u \circ v) \wedge \mu_{\mathrm{L}\times\mathrm{M}}(a,u) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}((a,u) \circledast u(b,v)) \wedge \mu_{\mathrm{L}\times\mathrm{M}}(a,u).
\end{aligned}$$

□

Theorem 5.4. *Let* L *and* M *be* $\mathcal{L}$*-fuzzy BCC-ideals of BCC-algebras* $\mathcal{U}_1 = (\mathcal{U}_1, \star, 0_1)$ *and* $\mathcal{U}_2 = (\mathcal{U}_2, \circ, 0_2)$*, respectively. Then* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-ideal of a BCC-algebra* $\mathcal{U}_1 \times \mathcal{U}_2$*.*

Proof. Let $a, b, c \in \mathcal{U}_1$, $u, v, w \in \mathcal{U}_2$. Then

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}(0_1, 0_2) &= \mu_{\mathrm{L}}(0_1) \wedge \mu_{\mathrm{M}}(0_2) \\
&\geq \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{M}}(u) && ((2.3)) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(a, u)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}((a, u) \circledast (c, w)) &= \mu_{\mathrm{L}\times\mathrm{M}}(a \star c, u \circ w) \\
&= \mu_{\mathrm{L}}(a \star c) \wedge \mu_{\mathrm{M}}(u \circ w) \\
&\geq (\mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{L}}(b)) \wedge (\mu_{\mathrm{M}}(u \circ (v \circ w)) \wedge \mu_{\mathrm{M}}(v)) && ((2.5)) \\
&= (\mu_{\mathrm{L}}(a \star (b \star c)) \wedge \mu_{\mathrm{M}}(u \circ (v \circ w))) \wedge (\mu_{\mathrm{L}}(b) \wedge \mu_{\mathrm{M}}(v)) \\
& && (\wedge \text{ is associative and commutative}) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(a \star (b \star c), u \circ (v \circ w)) \wedge \mu_{\mathrm{L}\times\mathrm{M}}(b, v) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}((a, u) \circledast ((b, v) \circledast (c, w))) \wedge \mu_{\mathrm{L}\times\mathrm{M}}(b, v).
\end{aligned}$$

□

Theorem 5.5. *Let* L *and* M *be* $\mathcal{L}$*-fuzzy strong BCC-ideals of BCC-algebras* $\mathcal{U}_1 = (\mathcal{U}_1, \star, 0_1)$ *and* $\mathcal{U}_2 = (\mathcal{U}_2, \circ, 0_2)$*, respectively. Then* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy strong BCC-ideal of a BCC-algebra* $\mathcal{U}_1 \times \mathcal{U}_2$*.*

Proof. Let $a, b, c \in \mathcal{U}_1$, $u, v, w \in \mathcal{U}_2$. Then

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}(0_1, 0_2) &= \mu_{\mathrm{L}}(0_1) \wedge \mu_{\mathrm{M}}(0_2) \\
&\geq \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{M}}(u) && ((2.3)) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(a, u)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\mathrm{L}\times\mathrm{M}}(a, u) &= \mu_{\mathrm{L}}(a) \wedge \mu_{\mathrm{M}}(u) \\
&\geq (\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{L}}(b)) \wedge (\mu_{\mathrm{M}}((w \circ v) \circ (w \circ u)) \wedge \mu_{\mathrm{M}}(v)) && ((2.6)) \\
&= (\mu_{\mathrm{L}}((c \star b) \star (c \star a)) \wedge \mu_{\mathrm{M}}((w \circ v) \circ (w \circ u))) \wedge (\mu_{\mathrm{L}}(b) \wedge \mu_{\mathrm{M}}(v)) \\
& && (\wedge \text{ is associative and commutative}) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}((c \star b) \star (c \star a), (w \circ v) \circ (w \circ u)) \wedge \mu_{\mathrm{L}\times\mathrm{M}}(b, v) \\
&= \mu_{\mathrm{L}\times\mathrm{M}}(((c, w) \circledast (b, v)) \circledast ((c, w) \circledast (a, u))) \wedge \mu_{\mathrm{L}\times\mathrm{M}}(b, v).
\end{aligned}$$

□

Finally, we shall discuss the relationships between the Cartesian product of two LFSs and their t-level subsets. After this, $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ refers to a lattice until otherwise defined.

The following theorem is a straightforward result of Theorems 4.2, 4.3, 4.4, 4.5, and 4.6.

Theorem 5.6. (1) *An LFS* $\mathrm{L}\times\mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-subalgebra of* $\mathcal{U}_1\times\mathcal{U}_2$ *if and only if* $U(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-subalgebra of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(2) *An LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $U(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(3) *An LFS* $\mathrm{L}\times\mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-filter of* $\mathcal{U}_1\times\mathcal{U}_2$ *if and only if* $U(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(4) *An LFS* $\mathrm{L}\times\mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-ideal of* $\mathcal{U}_1\times\mathcal{U}_2$ *if and only if* $U(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(5) *An LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $U(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

The following theorem is a straightforward result of Theorems 4.8 and 4.9.

Theorem 5.7. (1) *If an LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$*, then* $U^+(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(2) *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ *be a linearly ordered set. If* $U^+(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$*, then* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$.

The following theorem is a straightforward result of Theorems 4.7, 4.10, 4.11, and 4.12.

Theorem 5.8. *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge)$ *be a linearly ordered set. Then the following statements are true.*

(1) *An LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-subalgebra of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $U^+(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-subalgebra of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(2) *An LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $U^+(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(3) *An LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $U^+(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(4) *An LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $U^+(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

The following theorem is a straightforward result of Theorems 4.13, 4.14, 4.15, 4.16, and 4.17.

Theorem 5.9. *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ *be a Boolean lattice. Then the following statements are true.*

(1) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy BCC-subalgebra of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-subalgebra of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(2) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(3) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(4) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(5) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

The following theorem is a straightforward result of Theorems 4.19 and 4.20.

Theorem 5.10. (1) *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ *be a Boolean lattice. If an LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$*, then* $L^-(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(2) *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ *be a Boolean lattice with* $\leq$ *is a linear order. If* $L^-(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$*, then* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy near BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$.

The following theorem is a straightforward result of Theorems 4.18, 4.21, 4.22, and 4.23.

Theorem 5.11. *Let* $\mathcal{L} = (\mathcal{L}, \leq, \vee, \wedge, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ *be a Boolean lattice with* $\leq$ *is a linear order. Then the following statements are true.*

(1) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy BCC-subalgebra of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L^-(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-subalgebra of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(2) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L^-(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-filter of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(3) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L^-(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

(4) *An LFS* $(\mathrm{L} \times \mathrm{M})'$ *is an* $\mathcal{L}$*-fuzzy strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $L^-(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is, if it is nonempty, a strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

The following theorem is a straightforward result of Theorem 4.1.

Theorem 5.12. *An LFS* $\mathrm{L} \times \mathrm{M}$ *is an* $\mathcal{L}$*-fuzzy strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *if and only if* $E(\mu_{\mathrm{L}\times\mathrm{M}}, t)$ *is a strong BCC-ideal of* $\mathcal{U}_1 \times \mathcal{U}_2$ *for every* $t \in \mathcal{L}$.

6. Conclusions and future works

In this paper, we have introduced the concept of LFSs in BCC-algebras, and then we have introduced five types of LFSs in BCC-algebras, namely $\mathcal{L}$-fuzzy BCC-subalgebras, $\mathcal{L}$-fuzzy near BCC-filters, $\mathcal{L}$-fuzzy BCC-filters, $\mathcal{L}$-fuzzy BCC-ideals, and $\mathcal{L}$-fuzzy strong BCC-ideals. Further, we have discussed the relationship between $\mathcal{L}$-fuzzy BCC-subalgebras (resp., $\mathcal{L}$-fuzzy near BCC-filters, $\mathcal{L}$-fuzzy BCC-filters, $\mathcal{L}$-fuzzy BCC-ideals, $\mathcal{L}$-fuzzy strong BCC-ideals) and BCC-subalgebras (resp., near BCC-filters, BCC-filters, BCC-ideals, strong BCC-ideals) with characteristic functions and t-level subsets of LFSs. In addition, we proved that the Cartesian product of two $\mathcal{L}$-fuzzy BCC-subalgebras (resp., $\mathcal{L}$-fuzzy near BCC-filters, $\mathcal{L}$-fuzzy BCC-filters, $\mathcal{L}$-fuzzy BCC-ideals, $\mathcal{L}$-fuzzy strong BCC-ideals) is also a $\mathcal{L}$-fuzzy BCC-subalgebra (resp., $\mathcal{L}$-fuzzy near BCC-filter, $\mathcal{L}$-fuzzy BCC-filter, $\mathcal{L}$-fuzzy BCC-ideal, $\mathcal{L}$-fuzzy strong BCC-ideal). After, we studies the relationship between above results and BCC-subalgebras (resp., near BCC-filters, BCC-filters, BCC-ideals, strong BCC-ideals) with t-level subsets of the Cartesian product of LFSs. Finally, we get the diagram of generalization of LFSs in BCC-algebras, which is shown with Figure 5.

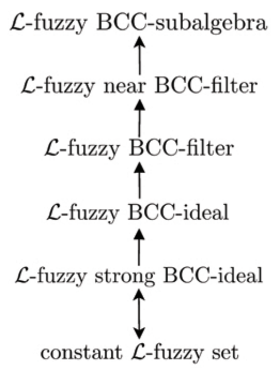

Figure 5. LFSs in BCC-algebras

Some important topics for our future study of BCC-algebras are as follows:

(1) to define new types of LFSs,
(2) to apply the concept of LFSs to intuitionistic fuzzy sets, and
(3) to apply the concept of LFSs to hesitant fuzzy sets.

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