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# Remarks on Some Higher Dimensional Hardy Inequalities

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Abstract. In this note, we give an elementary proof of Hardy inequality in higher dimensions introduced by Christ and Grafakos. The advantage of our approach is that it uses the one-dimensional Hardy inequality to obtain higher dimensional versions. We go further and get some well-known weighted estimates using the same approach.

## 1. Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. Let $\mathcal{H}(f)(x)$ be the average of $|f| \in L^p(\mathbb{R}^n)$ over the Euclidean ball $B(0,|x|)$, that is,

$$\mathcal{H}(f)(x) = \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} |f(x)|\,dx.$$

Christ and Grafakos [1] introduced the operator $\mathcal{H}$ in order to get a higher dimensional version of the classical Hardy inequality in one dimension [2]. In fact, they proved the following sharp estimate:

$$||\mathcal{H}(f)||_{L^p(\mathbb{R}^n)} \leq \frac{p}{p-1}||f||_{L^p(\mathbb{R}^n)}, \tag{1.1}$$

for $f \in L^p(\mathbb{R}^n)$ and $1 < p < \infty$. The estimate 1.1 was obtained by using Minkowski's convolution inequality over the space $L^p(\mathbb{R}^+, \frac{dt}{t})$, where $\mathbb{R}^+$ is the multiplicative topological group $(0,\infty)$. It

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should be mentioned that the idea of their proof was very elegent, although they did not use the classical one dimensional estimate

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x |F(t)|\,dy\right)^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty |F(t)|^p\,dt\right)^{\frac{1}{p}} \tag{1.2}$$

or any of its variants at any stage of the proof. So it is natural to ask whether we can attain the sharp estimate 1.1 by invoking the old bottles of Hardy inequalities in [2].

A humble attempt towards answering this question can be done by re-examining the left-hand side of 1.1. More precisely, using polar coordinates and applying Hölder's inequality give

$$\begin{aligned} ||\mathcal{H}(f)||_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}|f(y)|\,dy\right)^p dx\right)^{\frac{1}{p}} \\ &\leq \frac{\omega_{n-1}}{\nu_n}\left(\int_0^\infty\left(\frac{1}{r}\int_0^r\left(\int_{\mathbb{S}^{n-1}}|f(t\theta)|^p t^{n-1}\,d\theta\right)^{\frac{1}{p}} dt\right)^p dr\right)^{\frac{1}{p}}, \end{aligned} \tag{1.3}$$

where $\omega_{n-1}$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ and $\nu_n$ is the volume of the unit ball $B(0,1)$.

Now, let $F(t) := \left(\int_{\mathbb{S}^{n-1}}|f(t\theta)|^p t^{n-1}\,d\theta\right)^{\frac{1}{p}}$ and use 1.2 with the fact that $\omega_{n-1} = n\nu_n$ to obtain

$$||\mathcal{H}(f)||_{L^p(\mathbb{R}^n)} \leq \frac{np}{p-1}||f||_{L^p(\mathbb{R}^n)}. \tag{1.4}$$

The dependence on the dimension ($n$) in estimate 1.4 is natural and indicates that our inquiry makes sense. Based upon this observation, we give a simple and direct proof of 1.1 which reveals that this inequality is one dimensional in spirit.

Let $W$ and $Z$ be two weight functions on $\mathbb{R}^n$, that is, nonnegative and locally integrable on $\mathbb{R}^n$, and denote the conjugate exponent of $p > 1$ by $p' = \dfrac{p}{p-1}$. In [3], the authors introduced a weighted version of Hardy inequality in Higher dimensions which generalizes 1.1. More precisely, they obtained the following result.

**Theorem 1.1.** *Let $W$ and $Z$ be weight functions on $\mathbb{R}^n$. For $1 < p \leq q < \infty$, the inequality*

$$\left(\int_{\mathbb{R}^n} W(x)\left(\int_{B(0,|x|)}|f(y)|\,dy\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} Z(x)|f(x)|^p\,dx\right)^{\frac{1}{p}} \tag{1.5}$$

*holds if and only if*

$$A := \sup_{\alpha>0}\left(\int_{|x|\geq\alpha} W(x)\,dx\right)^{\frac{1}{q}}\left(\int_{|x|\leq\alpha} Z^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} < \infty. \tag{1.6}$$

*Moreover, if $C$ is the smallest constant for which 1.5 holds, then*

$$A \leq C \leq Ap'^{\frac{1}{p'}}p^{\frac{1}{q}}.$$

We remark here that the condition 1.6 extends the one dimensional condition which can be found in [4]. As a special and important case of Theorem 1.1, we have, for $1 < p = q < \infty$, $s > 0$, $W(x) = |B(0,x)|^{-s-1}$ and $U(x) = |B(0,x)|^{p-s-1}$, that

$$\left(\int_{\mathbb{R}^n} |B(0,|x|)|^{-s-1} \left(\int_{B(0,|x|)} |f(y)|\,dy\right)^p dx\right)^{\frac{1}{p}} \leq \frac{p}{s} \left(\int_{\mathbb{R}^n} |B(0,|x|)|^{p-s-1} |f(x)|^p\,dx\right)^{\frac{1}{p}}, \quad (1.7)$$

and the constant $\dfrac{p}{s}$ being best possible. We notice here that by taking $s = p-1$ in 1.7 one can obtain 1.1. Similarly, it is natural to ask whether we can deduce the estimates 1.5 and 1.7 using their one dimensional versions. In the following, we introduce a new and simple proofs of the higher dimensional Hardy inequalities 1.1, 1.5 and 1.7. We show also in Theorem 2.4 that our technique can be used to produce nice estimates by using the classical Hardy inequalities. It should also mention that many authors worked on the equivalence between the higher dimensional and one-dimensional Hardy's inequalities. For instance, Gord [7] studies Hardy inequalities in higher dimensions where the averages are taken over appropriate dilates of a given star-shaped regions. We refer the readers to ( [5], [6], [8]) for more background information and relevant work.

## 2. Proofs and further results

In this section we introduce new and simple proofs of the estimates 1.1, 1.5 and 1.7 using some elementary tools and based on Hardy's inequality in one dimension. We start by recalling the following result.

**Theorem 2.1.** [2, Theorem 330] *Let $f$ be a measurable function. Then*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x |F(t)|\,dt\right)^p x^\eta\,dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1-\eta} \left(\int_0^\infty |F(t)|^p t^\eta\,dt\right)^{\frac{1}{p}} \quad (2.1)$$

*holds for $1 < p < \infty$ and $\eta < p-1$.*

Now, we present our proofs of 1.1, 1.7.

**Proof of estimate 1.1.** Applying polar coordinates and Höldr's inequality yield

$$\begin{aligned} ||\mathcal{H}(f)||_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} |f(y)|\,dy\right)^p dx\right)^{\frac{1}{p}} \\ &\leq n\left(\int_0^\infty \left(\frac{1}{r}\int_0^r \left(\int_{\mathbb{S}^{n-1}} |f(t\theta)|^p t^{n-1}\,d\theta\right)^{\frac{1}{p}} t^{\frac{n-1}{p'}}\,dt\right)^p r^{\frac{(1-n)p}{p'}}\,dr\right)^{\frac{1}{p}}. \end{aligned} \quad (2.2)$$

Using 2.1 with $\eta = \dfrac{(1-n)p}{p'}$ and $F(t) := \left(\int_{\mathbb{S}^{n-1}} |f(t\theta)|^p t^{n-1}\, d\theta\right)^{\frac{1}{p}} t^{\frac{n-1}{p'}}$, we obtain

$$\begin{aligned}
||\mathcal{H}(f)||_{L^p(\mathbb{R}^n)} &\le n\left(\frac{p}{p-1+\frac{(n-1)p}{p'}}\right)\left(\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |f(t\theta)|^p t^{n-1}\, d\theta\right) t^{\frac{(n-1)p}{p'}}\, t^{\frac{(1-n)p}{p'}}\, dt\right)^{\frac{1}{p}} \\
&= \frac{p}{p-1}||f||_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

**Proof of estimate 1.7.** Let $F(t) := \left(\int_{\mathbb{S}^{n-1}} |f(t\theta)|^p t^{n-1}\, d\theta\right)^{\frac{1}{p}} t^{\frac{n-1}{p'}}\, dt$, $\eta = p - sn - 1$ and proceeding as above, we get

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |B(0,|x|)|^{-s-1}\left(\int_{B(0,|x|)} f(y)\, dy\right)^p dx\right)^{\frac{1}{p}} \\
&\le \nu_n^{\frac{-s-1}{p}}\, \omega_{n-1}\left(\int_0^\infty \left(\int_0^r F(t)\, dt\right)^p r^{-ns-1}\, dr\right)^{\frac{1}{p}} \\
&\le \nu_n^{\frac{-s+p-1}{p}}\left(\frac{p}{s}\right)\left(\int_0^\infty \int_{\mathbb{S}^{n-1}} |f(t\theta)|^p t^{np-ns-1}\, d\theta\, dt\right)^{\frac{1}{p}} \\
&= \nu_n^{\frac{-s+p-1}{p}}\left(\frac{p}{s}\right)\left(\int_{\mathbb{R}^n} |f(x)|^p\, |x|^{n(p-s-1)}\, dx\right)^{\frac{1}{p}} \\
&= \left(\frac{p}{s}\right)\left(\int_{\mathbb{R}^n} |f(x)|^p\, |B(0,|x|)|^{p-s-1}\, dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Before introducing the proof of Theorem 1.1, we need the following result.

**Theorem 2.2.** *Let $u$ and $v$ be nonnegative measurable functions on $(0,\infty)$. If $f$ is a measurable function, then*

$$\left(\int_0^\infty \left(\int_0^x f(t)\, dt\right)^q u(x)\, dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)\, v(x)\, dx\right)^{\frac{1}{p}} \tag{2.3}$$

*holds for $1 < p \le q < \infty$ if and only if*

$$A := \sup_{x>0}\left(\int_x^\infty u(t)\, dt\right)^{\frac{1}{q}}\left(\int_0^x v^{1-p'}(t)\, dt\right)^{\frac{1}{p'}} < \infty. \tag{2.4}$$

Now, we are ready to prove Theorem 1.1. In fact, our proof is simpler than the proof in [4] and it depends primarily on the appropriate choice of the one dimensional weights.

**Theorem 2.3.** In order to apply Theorem 2.2 we carefully define the weight functions $u$, $v$ and the function $F$. Let

$$\begin{aligned}
u(t) &:= \left(\int_{\mathbb{S}^{n-1}} W(r\varphi)\,d\varphi\right) r^{n-1},\\
v(t) &:= t^{(1-n)(p-1)}\left(\int_{\mathbb{S}^{n-1}} Z^{-\frac{p'}{p}}(t\theta)\,d\theta\right)^{-\frac{p}{p'}},\\
F(t) &:= \left(\int_{\mathbb{S}^{n-1}} |f(t\theta)|^p Z(t\theta)\,d\theta\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^{n-1}} Z^{-\frac{p'}{p}}(t\theta)\,d\theta\right)^{\frac{1}{p'}} t^{n-1}.
\end{aligned}$$

Now, consider

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} W(x)\left(\int_{B(0,|x|)} |f(y)|\,dy\right)^q dx\right)^{\frac{1}{q}}\\
&= \left(\int_0^\infty \left(\int_0^r \int_{\mathbb{S}^{n-1}} |f(t\theta)| t^{n-1}\,d\theta\,dt\right)^q u(r)\,dr\right)^{\frac{1}{q}}\\
&= \left(\int_0^\infty \left(\int_0^r \int_{\mathbb{S}^{n-1}} |f(t\theta)| Z^{\frac{1}{p}}(t\theta) Z^{-\frac{1}{p}}(t\theta) t^{n-1}\,d\theta\,dt\right)^q u(r)\,dr\right)^{\frac{1}{q}}\\
&\le \left(\int_0^\infty \left(\int_0^r F(t)\,dt\right)^q u(r)\,dr\right)^{\frac{1}{q}}. \qquad (2.5)
\end{aligned}$$

Then invoking Theorem 2.2 we have that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} W(x)\left(\int_{B(0,|x|)} |f(y)|\,dy\right)^q dx\right)^{\frac{1}{q}}\\
&\le \left(\int_0^\infty \left(\int_0^r F(t)\,dt\right)^q u(r)\,dr\right)^{\frac{1}{q}}\\
&\le C\left(\int_0^\infty F^p(t) v(t)\,dt\right)^{\frac{1}{p}}\\
&= C\left(\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |f(t\theta)|^p Z(t\theta)\,d\theta\right)\left(\int_{\mathbb{S}^{n-1}} Z^{-\frac{p'}{p}}(t\theta)\,d\theta\right)^{\frac{p}{p'}} t^{(n-1)p} v(t)\,dt\right)^{\frac{1}{p}}\\
&= C\left(\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |f(t\theta)|^p Z(t\theta)\,d\theta\right) t^{n-1}\,dt\right)^{\frac{1}{p}}\\
&= C\left(\int_{\mathbb{R}^n} |f(y)|^p Z(y)\,dy\right)^{\frac{1}{p}} \qquad (2.6)
\end{aligned}$$

if and only if

$$\begin{aligned}
A &:= \sup_{\alpha>0}\left(\int_\alpha^\infty u(t)\,dt\right)^{\frac{1}{q}}\left(\int_0^\alpha v^{1-p'}(t)\,dt\right)^{\frac{1}{p'}}\\
&= \sup_{\alpha>0}\left(\int_\alpha^\infty \left(\int_{\mathbb{S}^{n-1}} W(t\varphi)\,d\varphi\right) t^{n-1}\,dt\right)^{\frac{1}{q}}\left(\int_0^\alpha \left(\int_{\mathbb{S}^{n-1}} Z^{1-p'}(t\theta)\right) t^{n-1}\,dt\right)^{\frac{1}{p'}}
\end{aligned}$$

$$= \sup_{\alpha>0} \left( \int_{|x|\geq\alpha} W(x)\, dx \right)^{\frac{1}{q}} \left( \int_{|x|\leq\alpha} Z^{1-p'}(x)\, dx \right)^{\frac{1}{p'}} < \infty. \tag{2.7}$$

Next, we use the main scheme of the previous proofs to introduce the following result.

**Theorem 2.4.** *Let $f$ be a measurable function and $-\frac{n}{p'} < \alpha < \frac{1}{p'}$. Then*

$$\left( \int_{\mathbb{R}^n} \left( \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} |f(y)||x \cdot y|^{-\alpha}\, dy \right)^p dx \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n} |f(y)\, |y|^{-2\alpha}|^p \right)^{\frac{1}{p}}$$

*for $1 < p < \infty$, where $C = \left( \frac{C_{\alpha,p,n}\, \omega_{n-1}^{\frac{1}{p}}}{\nu_n} \right) \left( \frac{p}{np + \alpha n - n} \right)$.*

*Proof.* Let

$$G(\varphi, t) = \int_{\mathbb{S}^{n-1}} |f(t\theta)| t^{\frac{n-1}{p}} |\varphi \cdot \theta|^{-\alpha}\, d\theta.$$

Then applying Holder's inequality and using the fact that

$$\int_{\mathbb{S}^{n-1}} |\varphi \cdot \theta|^{-\alpha p'}\, d\theta = \omega_{n-2}\, B\left( \frac{1-\alpha p'}{2}, \frac{n-1}{2} \right) := C_{\alpha,p,n}^{p'}$$

for $\alpha < \frac{1}{p'}$, we get

$$G(\varphi, t) \leq \left( \int_{\mathbb{S}^{n-1}} |f(t\theta)| t^{n-1}\, d\theta \right)^{\frac{1}{p}} C_{\alpha,p,n} \tag{2.8}$$

Now set $D_{\alpha,p,n} = n - 1 - (\alpha + n - 1)p$, $F(t) = \left( \int_{\mathbb{S}^{n-1}} |f(t\theta)| t^{n-1}\, d\theta \right)^{\frac{1}{p}}\, t^{\frac{n-1}{p'} - \alpha}$ and use 2.8 to obtain

$$\left( \int_{\mathbb{R}^n} \left( \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} |f(y)||x \cdot y|^{-\alpha}\, dy \right)^p dx \right)^{\frac{1}{p}}$$

$$= \frac{1}{\nu_n} \left( \int_0^\infty \int_{\mathbb{S}^{n-1}} \left( \frac{1}{r} \int_0^r G(\varphi, t)\, t^{\frac{n-1}{p'} - \alpha}\, dt \right)^p r^{D_{\alpha,p,n}}\, d\varphi\, dr \right)^{\frac{1}{p}}$$

$$\leq \frac{C_{\alpha,p,n}\, \omega_{n-1}^{\frac{1}{p}}}{\nu_n} \left( \int_0^\infty \left( \frac{1}{r} \int_0^r F(t)\, dt \right)^p r^{D_{\alpha,p,n}}\, dr \right)^{\frac{1}{p}}. \tag{2.9}$$

Finally, Applying Theorem 2.1 with $\eta = D_{\alpha,p,n}$ and $\alpha > -\frac{n}{p'}$ we get the desired result. $\square$

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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